On characterization of horizontal biharmonic curves in $\mathbb{H}^2 \times \mathbb{R}$

Talat Körpinar and Essin Turhan

ABSTRACT. In this paper, we study biharmonic curves in $\mathbb{H}^2 \times \mathbb{R}$. We show that all of them are helices. By using the curvature and torsion of the curves, we give some characterizations of horizontal biharmonic curves in $\mathbb{H}^2 \times \mathbb{R}$.

1. Introduction

Let $f : (M, g) \to (N, h)$ be a smooth function between two Riemannian manifolds. The bienergy $E_2(f)$ of f over compact domain $\Omega \subset M$ is defined by

$$E_2(f) = \int_\Omega h\left(\tau(f), \tau(f)\right) dv_g,$$

where $\tau(f) = \operatorname{trace}_g \nabla df$ is the tension field of f and dv_g is the volume form of M. Using the first variational formula one sees that f is a biharmonic function if and only if its bitension field vanishes identically, i.e.,

$$\widetilde{\tau}(f) := -\triangle^f(\tau(f)) - \operatorname{trace}_g R^N(df, \tau(f))df = 0, \qquad (1.1)$$

where

$$\triangle^f = -\operatorname{trace}_g(\nabla^f)^2 = -\operatorname{trace}_g\left(\nabla^f \nabla^f - \nabla^f_{\nabla^M}\right)$$

is the Laplacian on sections of the pull-back bundle $f^{-1}(TN)$ and R^N is the curvature operator of (N, h) defined by

$$R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z.$$

The theory of biharmonic functions is an old and rich subject. The biharmonic functions were first studied by Maxwell and Airy to describe a mathematical model of elasticity in 1862. The theory of polyharmonic functions was later on developed, for example, by E. Almansi, T. Levi-Civita

and M. Nicolescu. Recently, biharmonic functions on Riemannian manifolds have been studied by R. Caddeo and L. Vanhecke [2, 3], L. Sario et al. [7], and others.

Recently, there have been a growing interest in the theory of biharmonic functions which can be divided into two main research directions. On the one side, the differential geometric aspect has driven attention to the construction of examples and classification results. The other side is the analytic aspect from the point of view of PDE: biharmonic functions are solutions of a fourth order strongly elliptic semilinear PDE.

Chen and Ishikawa [4] classified biharmonic curves in semi-Euclidean 3-spaces. In particular, they showed that in Euclidean 3-spaces there are no proper biharmonic curves (i.e., biharmonic curves which are not harmonic). On the other hand, in an indefinite semi-Euclidean 3-space, there exist proper biharmonic curves.

Recently, some work has been done in the study of non-geodesic biharmonic curves in some model spaces. For example, the study of biharmonic curves in Berger's spheres, in contact and Sasakian manifolds and in the Minkowski 3-space, see [1], [5] and [6], respectively.

In this paper, we first write down the conditions that any non-harmonic (non-geodesic) biharmonic curve in $\mathbb{H}^2 \times \mathbb{R}$ must satisfy. Then we prove that the non-geodesic biharmonic curves in $\mathbb{H}^2 \times \mathbb{R}$ are helices. Finally we deduce the explicit parametric equations of the non-geodesic horizontal biharmonic curves in $\mathbb{H}^2 \times \mathbb{R}$.

2. Left invariant metric in $\mathbb{H}^2 \times \mathbb{R}$

Let $\mathbb{H}^2$ be the upper half-plane model $\{(x, y) \in \mathbb{R}^2 : y > 0\}$ of the hyperbolic plane endowed with the metric

$$g_{\mathbb{H}} = \frac{(dx^2 + dy^2)}{y^2}$$

of constant Gauss curvature -1. The space $\mathbb{H}^2$, with the group structure derived by the composition of proper affine maps, is a Lie group and the metric $g_{\mathbb{H}}$ is left invariant. Therefore the product $\mathbb{H}^2 \times \mathbb{R}$ is a Lie group with the left invariant product metric

$$g_{\mathbb{H}^2 \times \mathbb{R}} = \frac{(dx^2 + dy^2)}{y^2} + dz^2. \tag{2.1}$$

The left-invariant orthonormal frame is

$$e_1 = y\frac{\partial}{\partial x}, \quad e_2 = y\frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}.$$

The corresponding Lie brackets are

$$[e_1, e_2] = -e_1, \qquad [e_1, e_3] = [e_2, e_3] = 0.$$

In order to calculate the Christoffel symbols L_{ij}^k we can use

$$2g(\nabla_{e_i} e_j, e_k) = L_{ij}^k,$$

where the non zero symbols L_{ij}^k are

$$L_{12}^1 = -2, \quad L_{11}^2 = 2.$$

The Riemannian curvature tensor is given by

$$R_{121} = e_2, \quad R_{1212} = 1. \tag{2.2}$$

3. Biharmonic curves in $\mathbb{H}^2 \times \mathbb{R}$

Let $I \subset \mathbb{R}$ be an open interval and $\gamma : I \longrightarrow \mathbb{H}^2 \times \mathbb{R}$ be a curve on a Riemannian manifold parametrized by arc length. Let κ be the curvature of γ and τ its torsion. Putting $T = \gamma'$, we can write the tension field of γ as $\tau(\gamma) = \nabla_{\gamma'}\gamma'$ and the biharmonic map equation (1.1) reduces to

$$\nabla_T^3 T + R(T, \nabla_T T)T = 0. \tag{3.1}$$

A successful key to study the geometry of a curve is to use the Frenet frames along the curve.

With respect to the orthonormal basis $\{e_1, e_2, e_3\}$ we can write

$$\begin{aligned} T &= T_1 e_1 + T_2 e_2 + T_3 e_3, \\ N &= N_1 e_1 + N_2 e_2 + N_3 e_3, \\ B &= T \times N = B_1 e_1 + B_2 e_2 + B_3 e_3. \end{aligned} \tag{3.2}$$

Theorem 3.1. *A curve* $\gamma : I \longrightarrow \mathbb{H}^2 \times \mathbb{R}$ *is biharmonic if and only if*

$$\begin{aligned} \kappa &= constant \neq 0, \\ \kappa^2 + \tau^2 &= B_3^2, \\ \tau' &= -N_3 B_3. \end{aligned} \tag{3.3}$$

Proof. From (3.1) we obtain

$$\begin{aligned} \widetilde{\tau}(\gamma) &= \nabla_T^3 T + R(T, \nabla_T T)T \\ &= (-3\kappa\kappa')T + (\kappa'' - \kappa^3 - \kappa\tau^2 + \kappa R(T, N, T, N))N \\ &\quad + (-2\kappa'\tau - \kappa\tau' + \kappa R(T, N, T, B))B \\ &= 0. \end{aligned}$$

We see that γ is a biharmonic curve if and only if

$$\begin{aligned} \kappa\kappa' &= 0, \\ \kappa'' - \kappa^3 - \kappa\tau^2 + \kappa R(T, N, T, N) &= 0, \\ -2\kappa'\tau - \kappa\tau' + \kappa R(T, N, T, B) &= 0, \end{aligned}$$

which is equivalent to

$$\kappa = \text{constant} \neq 0,$$
$$\kappa^2 + \tau^2 = R(T, N, T, N), \tag{3.4}$$
$$\tau' = R(T, N, T, B).$$

A direct computation using (2.2) yields

$$R(T, N, T, N) = B_3^2, \quad R(T, N, T, B) = -N_3 B_3.$$

These, together with (3.4), complete the proof of the theorem. $\square$

Corollary 3.2. *Let* $\gamma : I \longrightarrow \mathbb{H}^2 \times \mathbb{R}$ *be a curve with constant curvature and* $N_3 B_3 \neq 0$. *Then* γ *is not biharmonic.*

Proof. We use the covariant derivatives of the vector fields. From Frenet formulae it follows that

$$T_3' = \kappa N_3,$$
$$N_3' = -\kappa T_3 - \tau B_3, \tag{3.5}$$
$$B_3' = \tau N_3.$$

Assume now that γ is biharmonic. Then $\tau' = -N_3 B_3 \neq 0$ and from (3.3) we obtain

$$\tau\tau' = B_3 B_3'.$$

Since $\tau' = -N_3 B_3$, this is rewritten as

$$\tau = -\frac{B_3'}{N_3}.$$

From (3.5) we have

$$\tau = \frac{B_3'}{N_3}.$$

Hence

$$\tau = 0.$$

Therefore $\tau' = 0$ and we have a contradiction. This completes the proof of the corollary. $\square$

By using Theorem 3.1 and Corollary 3.2, we have the following corollary.

Corollary 3.3. *A curve* $\gamma : I \longrightarrow \mathbb{H}^2 \times \mathbb{R}$ *is biharmonic if and only if*

$$\begin{aligned} \kappa &= constant \neq 0, \\ \tau &= constant, \\ N_3 B_3 &= 0, \\ \kappa^2 + \tau^2 &= B_3^2. \end{aligned}$$

Corollary 3.4. (i) *If $N_3 \neq 0$, then γ is not biharmonic.*
(ii) *If $N_3 = 0$, then*

$$T(s) = \sin \alpha_0 \cos \beta(s) e_1 + \sin \alpha_0 \sin \beta(s) e_2 + \cos \alpha_0 e_3, \qquad (3.6)$$

where $\alpha_0 \in \mathbb{R}$ and β is a differentiable function of s.

Proof. (i) Using the above Corollary 3.3, it is easy to see that γ is not biharmonic.

(ii) Since γ is parametrized by arc length s, we can write

$$T(s) = \sin \alpha \cos \beta(s) e_1 + \sin \alpha \sin \beta(s) e_2 + \cos \alpha e_3. \qquad (3.7)$$

From (3.5) we obtain

$$T_3' = \kappa N_3.$$

Since $N_3 = 0$,

$$T_3' = 0.$$

Then T_3 is constant. From (3.7) we have

$$T_3 = \cos \alpha_0 = \text{constant}.$$

$\square$

Theorem 3.5. *The parametric equations of all biharmonic curves in $\mathbb{H}^2 \times \mathbb{R}$ are*

$$x(s) = c_2 \sin \alpha_0 \int \sin \beta(s) e^{\sin \alpha_0 \int \cos \beta(s) ds} ds + c_1,$$

$$y(s) = c_2 e^{\sin \alpha_0 \int \cos \beta(s) ds}, \qquad (3.8)$$

$$z(s) = s \cos \alpha_0 + c_3,$$

where c_1, c_2, c_3 are arbitrary constants.

Proof. We note that

$$\frac{d\gamma}{ds} = T(s) = T_1 e_1 + T_2 e_2 + T_3 e_3,$$

and our left-invariant vector fields are

$$e_1 = y \frac{\partial}{\partial x}, \quad e_2 = y \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}.$$

Hence

$$\frac{d\gamma}{ds} = (y(s) T_1, y(s) T_2, T_3).$$

By using equation (3.6), we have

$$\frac{d\gamma}{ds} = T(s) = (y(s) \sin \alpha_0 \cos \beta(s), y(s) \sin \alpha_0 \sin \beta(s), \cos \alpha_0).$$

In order to find the explicit equations for $\gamma(s) = (x(s), y(s), z(s))$, we must integrate the system $\frac{d\gamma}{ds} = T(s)$, that in our case is

$$
\begin{aligned}
\frac{dx}{ds} &= y(s) \sin \alpha_0 \cos \beta(s), \\
\frac{dy}{ds} &= y(s) \sin \alpha_0 \sin \beta(s), \\
\frac{dz}{ds} &= \cos \alpha_0.
\end{aligned}
$$

The integration is immediate and yields (3.8). $\square$

Corollary 3.6. *If $\beta(s) = s$, then the parametric equations of all biharmonic curves are*

$$
\begin{aligned}
x(s) &= e^{-\sin \alpha_0 \cos s}(\sin \alpha_0 \cos s + 1) + c_1, \\
y(s) &= c_2 e^{-\sin \alpha_0 \cos s}, \\
z(s) &= s \cos \alpha_0 + c_3,
\end{aligned}
$$

where c_1, c_2, c_3 are arbitrary constants.

Example 3.7. Let us consider a biharmonic curve γ with $c_1 = c_2 = c_3 = 1$. Then γ is given by

$$
\gamma(s) = \left(e^{-\sin \alpha_0 \cos s}(\sin \alpha_0 \cos s + 1) + 1, e^{-\sin \alpha_0 \cos s}, s \cos \alpha_0 + 1 \right).
$$

4. Horizontal biharmonic curves in $\mathbb{H}^2 \times \mathbb{R}$

Consider a 2-dimensional distribution $(x, y) \longrightarrow \mathcal{H}_{(x,y)}$ in $\mathbb{H}^2 \times \mathbb{R}$ defined as $\mathcal{H} = \ker \omega$, where ω is a 1-form on $\mathbb{H}^2 \times \mathbb{R}$. The distribution $\mathcal{H}$ is called the horizontal distribution.

A curve $s \longrightarrow \gamma(s) = (x(s), y(s), z(s))$ is called horizontal curve if $\gamma'(s) \in \mathcal{H}_{\gamma(s)}$, for every s. Since

$$
\begin{aligned}
\gamma'(s) &= x'(s)\frac{\partial}{\partial x} + y'(s)\frac{\partial}{\partial y} + z'(s)\frac{\partial}{\partial z} \\
&= \frac{x'(s)}{y(s)}e_1 + \frac{y'(s)}{y(s)}e_2 + \omega(\gamma'(s))e_3,
\end{aligned}
$$

$\gamma(s)$ is a horizontal curve if and only if

$$
\gamma'(s) = \frac{x'(s)}{y(s)}e_1 + \frac{y'(s)}{y(s)}e_2, \quad \omega(\gamma'(s)) = z'(s) = 0.
$$

If $\gamma(s)$ is horizontal curve, then we have

$$
\gamma'(s) = \frac{x'(s)}{y(s)}e_1 + \frac{y'(s)}{y(s)}e_2 = x'(s)\frac{\partial}{\partial x} + y'(s)\frac{\partial}{\partial y}. \tag{4.1}
$$

Using (2.1) and (4.1) we obtain

$$T = T_1 y(s) \frac{\partial}{\partial x} + T_2 y(s) \frac{\partial}{\partial y} + T_3 \frac{\partial}{\partial z}. \tag{4.2}$$

Theorem 4.1. *The parametric equations of all horizontal biharmonic curves in $\mathbb{H}^2 \times \mathbb{R}$ are*

$$
\begin{aligned}
x(s) &= c_2 \int \sin \beta(s) e^{\int \cos \beta(s) ds} ds + c_1, \\
y(s) &= c_2 e^{\int \cos \beta(s) ds}, \\
z(s) &= c_3,
\end{aligned} \tag{4.3}
$$

where c_1, c_2, c_3 are arbitrary constants.

Proof. Using (4.1) and (4.2) we have

$$T_3 = \cos \alpha_0 = 0. \tag{4.4}$$

Substituting (4.4) into (3.8), we get (4.3). $\square$

Corollary 4.2. *If $\beta(s) = s$, then the parametric equations of all horizontal biharmonic curves in $\mathbb{H}^2 \times \mathbb{R}$ are*

$$
\begin{aligned}
x(s) &= e^{-\cos s}(\cos s + 1) + c_1, \\
y(s) &= c_2 e^{-\cos s}, \\
z(s) &= c_3,
\end{aligned}
$$

where c_1, c_2, c_3 are arbitrary constants.

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FIRAT UNIVERSITY, DEPARTMENT OF MATHEMATICS, 23119 ELAZIĞ, TURKEY
E-mail address: essin.turhan@gmail.com