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Congruences on bicyclic extensions of a linearly ordered group

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Abstract. In the paper we study inverse semigroups $\mathscr{B}(G)$, $\mathscr{B}^+(G)$, $\overline{\mathscr{B}}(G)$ and $\overline{\mathscr{B}}^+(G)$ which are generated by partial monotone injective translations of a positive cone of a linearly ordered group G. We describe Green's relations on the semigroups $\mathscr{B}(G)$, $\mathscr{B}^+(G)$, $\overline{\mathscr{B}}(G)$ and $\overline{\mathscr{B}}^+(G)$, their bands and show that they are simple, and moreover, the semigroups $\mathscr{B}(G)$ and $\mathscr{B}^+(G)$ are bisimple. We show that for a commutative linearly ordered group G all non-trivial congruences on the semigroup $\mathscr{B}(G)$ (and $\mathscr{B}^+(G)$) are group congruences if and only if the group G is archimedean. Also we describe the structure of group congruences on the semigroups $\mathscr{B}(G)$, $\mathscr{B}^+(G)$, $\overline{\mathscr{B}}(G)$ and $\overline{\mathscr{B}}^+(G)$.

1. Introduction and main definitions

In this article we shall follow the terminology of [7, 8, 14, 16, 20].

A *semigroup* is a non-empty set with a binary associative operation. A semigroup S is called *inverse* if for any $x \in S$ there exists a unique $y \in S$ such that $x \cdot y \cdot x = x$ and $y \cdot x \cdot y = y$. Such an element y in S is called the *inverse* of x and denoted by x^{-1}. The map defined on an inverse semigroup S which maps every element x of S to its inverse x^{-1} is called the *inversion*.

If S is a semigroup, then we shall denote the subset of idempotents in S by $E(S)$. If S is an inverse semigroup, then $E(S)$ is closed under multiplication and we shall refer to $E(S)$ as the *band of* S. If the band $E(S)$ is a non-empty subset of S, then the semigroup operation on S determines the following partial order $\preccurlyeq$ on $E(S)$: $e \preccurlyeq f$ if and only if $ef = fe = e$. This order is called the *natural partial order* on $E(S)$. A *semilattice* is a commutative semigroup of idempotents. A semilattice E is called *linearly ordered* or a *chain* if its natural order is a linear order.

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If $\mathfrak{C}$ is an arbitrary congruence on a semigroup S, then we denote by $\Phi_{\mathfrak{C}}\colon S\to S/\mathfrak{C}$ the natural homomorphisms from S onto the quotient semigroup $S/\mathfrak{C}$. Also we denote by Ω_S and Δ_S the *universal* and the *identity* congruences, respectively, on the semigroup S, i.e., $\Omega(S)=S\times S$ and $\Delta(S)=\{(s,s)\mid s\in S\}$. A congruence $\mathfrak{C}$ on a semigroup S is called *non-trivial* if $\mathfrak{C}$ is distinct from the universal and the identity congruences on S, and a *group congruence* if the quotient semigroup $S/\mathfrak{C}$ is a group. Every inverse semigroup S admits a least group congruence $\mathfrak{C}_{mg}$:

$$a\mathfrak{C}_{mg}b \text{ if and only if there exists } e\in E(S) \text{ such that } ae=be$$

(see [20, Lemma III.5.2]).

A map $h\colon S\to T$ from a semigroup S to a semigroup T is said to be an *antihomomorphism* if $(a\cdot b)h=(b)h\cdot(a)h$. A bijective antihomomorphism is called an *antiisomorphism*.

If S is a semigroup, then we shall denote by $\mathscr{R}$, $\mathscr{L}$, $\mathscr{J}$, $\mathscr{D}$ and $\mathscr{H}$ Green's relations on S (see [8]):

$$\begin{aligned}
&a\mathscr{R}b \text{ if and only if } aS^1=bS^1;\\
&a\mathscr{L}b \text{ if and only if } S^1a=S^1b;\\
&a\mathscr{J}b \text{ if and only if } S^1aS^1=S^1bS^1;\\
&\mathscr{D}=\mathscr{L}\circ\mathscr{R}=\mathscr{R}\circ\mathscr{L};\\
&\mathscr{H}=\mathscr{L}\cap\mathscr{R}.
\end{aligned}$$

Let $\mathscr{I}_X$ denote the set of all partial one-to-one transformations of an infinite set X together with the following semigroup operation: $x(\alpha\beta)=(x\alpha)\beta$ if $x\in\operatorname{dom}(\alpha\beta)=\{y\in\operatorname{dom}\alpha\mid y\alpha\in\operatorname{dom}\beta\}$, for $\alpha,\beta\in\mathscr{I}_X$. The semigroup $\mathscr{I}_X$ is called the *symmetric inverse semigroup* over the set X (see [8]). The symmetric inverse semigroup was introduced by Wagner [21] and it plays a major role in the theory of semigroups.

The bicyclic semigroup $\mathscr{C}(p,q)$ is the semigroup with the identity 1 generated by two elements p and q subjected only to the condition $pq=1$. The distinct elements of $\mathscr{C}(p,q)$ are exhibited in the following useful array:

$$\begin{array}{ccccc}
1 & p & p^2 & p^3 & \cdots\\
q & qp & qp^2 & qp^3 & \cdots\\
q^2 & q^2p & q^2p^2 & q^2p^3 & \cdots\\
q^3 & q^3p & q^3p^2 & q^3p^3 & \cdots\\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

and the semigroup operation on $\mathscr{C}(p,q)$ is determined as follows:

$$q^kp^l\cdot q^mp^n=q^{k+m-\min\{l,m\}}p^{l+n-\min\{l,m\}}.$$

The bicyclic semigroup plays an important role in the algebraic theory of semigroups and in the theory of topological semigroups. For example a well-known O. Andersen's result [1] states that a (0–) simple semigroup is completely (0–) simple if and only if it does not contain the bicyclic semigroup. The bicyclic semigroup does not embed into stable semigroups [15].

Remark 1.1. We observe that the bicyclic semigroup is isomorphic to the semigroup $\mathscr{C}_{\mathbb{N}}(\alpha,\beta)$ which is generated by injective partial transformations α and β of the set of positive integers $\mathbb{N}$, defined as follows:

$$(n)\alpha = n+1 \quad \text{if } n \geqslant 1;$$
$$(n)\beta = n-1 \quad \text{if } n > 1$$

(see Exercise IV.1.11(ii) in [20]).

Recall from [11] that a *partially-ordered group* is a group $(G,\cdot)$ equipped with a partial order $\leqslant$ that is translation-invariant; in other words, $\leqslant$ has the property that, for all $a,b,g\in G$, if $a\leqslant b$ then $a\cdot g\leqslant b\cdot g$ and $g\cdot a\leqslant g\cdot b$.

By e we denote the identity of a group G. The set $G^+=\{x\in G\mid e\leqslant x\}$ in a partially ordered group G is called the *positive cone* or the *integral part* of G and it satisfies the properties

1) $G^+\cdot G^+\subseteq G^+$;
2) $G^+\cap (G^+)^{-1}=\{e\}$;
3) $x^{-1}\cdot G^+\cdot x\subseteq G^+$ for all $x\in G$.

Any subset P of a group G that satisfies conditions 1)−3) induces a partial order on G ($x\leqslant y$ if and only if $x^{-1}\cdot y\in P$) for which P is the positive cone.

A *linearly ordered* or *totally ordered group* is an ordered group G such that the order relation $\leqslant$ is total [7].

In the remainder we shall assume that G is a linearly ordered group.

For every $g\in G$ we denote

$$G^+(g)=\{x\in G\mid g\leqslant x\}.$$

The set $G^+(g)$ is called a *positive cone on element* g in G.

For arbitrary elements $g,h\in G$ we consider a partial map $\alpha^g_h\colon G\rightharpoonup G$ defined by the formula

$$(x)\alpha^g_h = x\cdot g^{-1}\cdot h \quad \text{for } x\in G^+(g).$$

We observe that Lemma XIII.1 from [7] implies that for such a partial map $\alpha^g_h\colon G\rightharpoonup G$ the restriction $\alpha^g_h\colon G^+(g)\to G^+(h)$ is a bijective map.

We denote

$$\mathscr{B}(G)=\{\alpha^g_h\colon G\rightharpoonup G\mid g,h\in G\} \text{ and } \mathscr{B}^+(G)=\{\alpha^g_h\colon G\rightharpoonup G\mid g,h\in G^+\},$$

and consider, on the sets $\mathscr{B}(G)$ and $\mathscr{B}^+(G)$, the operation of the composition of partial maps. Simple verifications show that

$$\alpha^g_h\cdot\alpha^k_l=\alpha^a_b, \quad \text{where } a=(h\vee k)\cdot h^{-1}\cdot g \text{ and } b=(h\vee k)\cdot k^{-1}\cdot l, \qquad (1)$$

for $g, h, k, l \in G$. Therefore, property 1) of the positive cone and condition (1) imply that $\mathscr{B}(G)$ and $\mathscr{B}^+(G)$ are subsemigroups of $\mathscr{I}_G$.

Proposition 1.2. *Let G be a linearly ordered group. Then the following assertions hold:*

(*i*) *elements α_h^g and α_g^h are inverses of each other in $\mathscr{B}(G)$ for all $g, h \in G$ (resp., in $\mathscr{B}^+(G)$ for all $g, h \in G^+$);*

(*ii*) *an element α_h^g of the semigroup $\mathscr{B}(G)$ (resp., $\mathscr{B}^+(G)$) is an idempotent if and only if $g = h$;*

(*iii*) *$\mathscr{B}(G)$ and $\mathscr{B}^+(G)$ are inverse subsemigroups of $\mathscr{I}_G$;*

(*iv*) *the semigroup $\mathscr{B}(G)$ (resp., $\mathscr{B}^+(G)$) is isomorphic to $S_G = G \times G$ (resp., $S_G^+ = G^+ \times G^+$) with the semigroup operation*

$$(a, b) \cdot (c, d) = \begin{cases} (c \cdot b^{-1} \cdot a, d), & \text{if } b < c; \\ (a, d), & \text{if } b = c; \\ (a, b \cdot c^{-1} \cdot d), & \text{if } b > c, \end{cases}$$

where $a, b, c, d \in G$ (resp., $a, b, c, d \in G^+$).

Proof. (*i*) Condition (1) implies that

$$\alpha_h^g \cdot \alpha_g^h \cdot \alpha_h^g = \alpha_h^g \quad \text{and} \quad \alpha_g^h \cdot \alpha_h^g \cdot \alpha_g^h = \alpha_g^h,$$

and hence α_h^g and α_g^h are inverse elements for each other in $\mathscr{B}(G)$ (resp., in $\mathscr{B}^+(G)$).

Statement (*ii*) follows from the property of the semigroup $\mathscr{I}_G$ that $\alpha \in \mathscr{I}_G$ is an idempotent if and only if $\alpha\colon \operatorname{dom}\alpha \to \operatorname{ran}\alpha$ is an identity map.

Statements (*i*), (*ii*) and Theorem 1.17 from [8] imply statement (*iii*).

Statement (*iv*) is a corollary of condition (1). □

Remark 1.3. We observe that Proposition 1.2 implies that

(1) if G is the additive group of integers $(\mathbb{Z}, +)$ with usual linear order $\leqslant$ then the semigroup $\mathscr{B}^+(G)$ is isomorphic to the bicyclic semigroup $\mathscr{C}(p, q)$;

(2) if G is the additive group of real numbers $(\mathbb{R}, +)$ with usual linear order $\leqslant$ then the semigroup $\mathscr{B}(G)$ is isomorphic to $B_{(-\infty,\infty)}$ (see [17, 18]) and the semigroup $\mathscr{B}^+(G)$ is isomorphic to $B_{[0,\infty)}$ (see [2, 3, 4, 5, 6]) and

(3) the semigroup $\mathscr{B}^+(G)$ is isomorphic to the semigroup $S(G)$ which is defined in [9, 10].

We shall say that a linearly ordered group G is a *d-group* if for every element $g \in G^+ \setminus \{e\}$ there exists $x \in G^+ \setminus \{e\}$ such that $x < g$. We observe that a linearly ordered group G is a d-group if and only if the set $G^+ \setminus \{e\}$ does not contain a minimal element.

Definition 1.4. Suppose that G is a linearly ordered d-group. For every $g \in G$ we denote

$$\mathring{G}^{+}(g) = \{x \in G \mid g < x\}.$$

The set $\mathring{G}^{+}(g)$ is called a *$\circ$-positive cone on element g in G.*

For arbitrary elements $g, h \in G$ we consider a partial map $\mathring{\alpha}_h^g \colon G \rightharpoonup G$ defined by the formula

$$(x)\mathring{\alpha}_h^g = x \cdot g^{-1} \cdot h \quad \text{ for } \ x \in \mathring{G}^{+}(g).$$

We observe that Lemma XIII.1 from [7] implies that for such a partial map $\mathring{\alpha}_h^g \colon G \rightharpoonup G$ the restriction $\mathring{\alpha}_h^g \colon \mathring{G}^{+}(g) \to \mathring{G}^{+}(h)$ is a bijective map.

We denote

$$\mathring{\mathscr{B}}(G) = \left\{\mathring{\alpha}_h^g \colon G \rightharpoonup G \mid g, h \in G\right\} \ \text{ and } \ \mathring{\mathscr{B}}^{+}(G) = \left\{\mathring{\alpha}_h^g \colon G \rightharpoonup G \mid g, h \in G^{+}\right\},$$

and consider, on the sets $\mathring{\mathscr{B}}(G)$ and $\mathring{\mathscr{B}}^{+}(G)$, the operation of the composition of partial maps. Simple verifications show that

$$\mathring{\alpha}_h^g \cdot \mathring{\alpha}_l^k = \mathring{\alpha}_b^a, \ \text{ where } \ a = (h \vee k) \cdot h^{-1} \cdot g \ \text{ and } \ b = (h \vee k) \cdot k^{-1} \cdot l, \tag{2}$$

for $g, h, k, l \in G$. Therefore, property 1) of the positive cone and condition (2) imply that $\mathring{\mathscr{B}}(G)$ and $\mathring{\mathscr{B}}^{+}(G)$ are subsemigroups of the symmetric inverse semigroup $\mathscr{I}_G$.

Proposition 1.5. *If G is a linearly ordered d-group then the semigroups $\mathring{\mathscr{B}}(G)$ and $\mathring{\mathscr{B}}^{+}(G)$ are isomorphic to $\mathscr{B}(G)$ and $\mathscr{B}^{+}(G)$, respectively.*

Proof. Define a map $\mathfrak{h} \colon \mathscr{B}(G) \to \mathring{\mathscr{B}}(G)$ (resp., $\mathfrak{h} \colon \mathscr{B}^{+}(G) \to \mathring{\mathscr{B}}^{+}(G)$) by the formula

$$(\alpha_h^g)\mathfrak{h} = \mathring{\alpha}_h^g \quad \text{ for } \ g, h \in G \text{ (resp., } g, h \in G^{+}).$$

Simple verifications show that $\mathfrak{h}$ is an isomorphism of the semigroups $\mathring{\mathscr{B}}(G)$ and $\mathscr{B}(G)$ (resp., $\mathring{\mathscr{B}}^{+}(G)$ and $\mathscr{B}^{+}(G)$). □

Suppose that G is a linearly ordered d-group. Then obviously $\mathring{\mathscr{B}}(G) \cap \mathscr{B}(G) = \varnothing$ and $\mathring{\mathscr{B}}^{+}(G) \cap \mathscr{B}^{+}(G) = \varnothing$. We define

$$\overline{\mathscr{B}}(G) = \mathring{\mathscr{B}}(G) \cup \mathscr{B}(G) \qquad \text{and} \qquad \overline{\mathscr{B}}^{+}(G) = \mathring{\mathscr{B}}^{+}(G) \cup \mathscr{B}^{+}(G).$$

Proposition 1.6. *If G is a linearly ordered d-group then $\overline{\mathscr{B}}(G)$ and $\overline{\mathscr{B}}^{+}(G)$ are inverse semigroups.*

Proof. Since $\mathring{\mathscr{B}}(G)$, $\mathscr{B}(G)$, $\mathring{\mathscr{B}}^{+}(G)$ and $\mathscr{B}^{+}(G)$ are inverse subsemigroups of the symmetric inverse semigroup $\mathscr{I}_G$ over the group G we conclude that it is sufficient to show that $\overline{\mathscr{B}}(G)$ and $\overline{\mathscr{B}}^{+}(G)$ are subsemigroups of $\mathscr{I}_G$.

We fix arbitrary elements $g,h,k,l \in G$. Since α^g_h, α^k_l, $\mathring{\alpha}^g_h$ and $\mathring{\alpha}^k_l$ are partial injective maps from G into G we have

$$\alpha^g_h \cdot \mathring{\alpha}^k_l = \begin{cases} \mathring{\alpha}^{k\cdot h^{-1}\cdot g}_l, & \text{if } h<k;\\ \mathring{\alpha}^g_l, & \text{if } h=k;\\ \alpha^g_{h\cdot k^{-1}\cdot l}, & \text{if } h>k \end{cases} \quad \text{and } \mathring{\alpha}^g_h \cdot \alpha^k_l = \begin{cases} \alpha^{k\cdot h^{-1}\cdot g}_l, & \text{if } h<k;\\ \mathring{\alpha}^g_l, & \text{if } h=k;\\ \mathring{\alpha}^g_{h\cdot k^{-1}\cdot l}, & \text{if } h>k. \end{cases}$$

Hence $\overline{\mathscr{B}}(G)$ is a subsemigroup of $\mathscr{I}_G$.

Similar arguments and property 1) of the positive cone imply that $\overline{\mathscr{B}}^+(G)$ is a subsemigroup of $\mathscr{I}_G$. This completes the proof of our proposition. □

In our paper we study semigroups $\mathscr{B}(G)$ and $\mathscr{B}^+(G)$ for a linearly ordered group G, and semigroups $\overline{\mathscr{B}}(G)$ and $\overline{\mathscr{B}}^+(G)$ for a linearly ordered d-group G. We describe Green's relations on the semigroups $\mathscr{B}(G)$, $\mathscr{B}^+(G)$, $\overline{\mathscr{B}}(G)$ and $\overline{\mathscr{B}}^+(G)$, their bands and show that they are simple, and moreover, the semigroups $\mathscr{B}(G)$ and $\mathscr{B}^+(G)$ are bisimple. We show that for a commutative linearly ordered group G all non-trivial congruences on the semigroup $\mathscr{B}(G)$ (and $\mathscr{B}^+(G)$) are group congruences if and only if the group G is archimedean. Also, we describe the structure of group congruences on the semigroups $\mathscr{B}(G)$, $\mathscr{B}^+(G)$, $\overline{\mathscr{B}}(G)$ and $\overline{\mathscr{B}}^+(G)$.

2. Algebraic properties of the semigroups $\mathscr{B}(G)$ and $\mathscr{B}^+(G)$

Proposition 2.1. *Let G be a linearly ordered group. Then the following assertions hold:*

(i) if $\alpha^g_g, \alpha^h_h \in E(\mathscr{B}(G))$ $\big($resp., $\alpha^g_g, \alpha^h_h \in E(\mathscr{B}^+(G))\big)$ then $\alpha^g_g \preccurlyeq \alpha^h_h$ if and only if $g \geqslant h$ in G $\big($resp., in $G^+\big)$;

(ii) the semilattice $E(\mathscr{B}(G))$ $\big($resp., $E(\mathscr{B}^+(G))\big)$ is isomorphic to G $\big($resp., $G^+\big)$, considered as a $\vee$-semilattice, under the mapping $(\alpha^g_g)\mathfrak{i} = g$;

(iii) $\alpha^g_h \mathscr{R} \alpha^k_l$ in $\mathscr{B}(G)$ $\big($resp., in $\mathscr{B}^+(G)\big)$ if and only if $g=k$ in G $\big($resp., in $G^+\big)$;

(iv) $\alpha^g_h \mathscr{L} \alpha^k_l$ in $\mathscr{B}(G)$ $\big($resp., in $\mathscr{B}^+(G)\big)$ if and only if $h=l$ in G $\big($resp., in $G^+\big)$;

(v) $\alpha^g_h \mathscr{H} \alpha^k_l$ in $\mathscr{B}(G)$ $\big($resp., in $\mathscr{B}^+(G)\big)$ if and only if $g=k$ and $h=l$ in G $\big($resp., in $G^+\big)$, and hence every $\mathscr{H}$-class in $\mathscr{B}(G)$ $\big($resp., in $\mathscr{B}^+(G)\big)$ is a singleton set;

(vi) $\alpha^g_h \mathscr{D} \alpha^k_l$ in $\mathscr{B}(G)$ $\big($resp., in $\mathscr{B}^+(G)\big)$ for all $g,h,k,l \in G$, and hence $\mathscr{B}(G)$ $\big($resp., $\mathscr{B}^+(G)\big)$ is a bisimple semigroup;

(vii) $\mathscr{B}(G)$ $\big($resp., $\mathscr{B}^+(G)\big)$ is a simple semigroup.

Proof. Statements (i) and (ii) are trivial and follow from the definition of the semigroup $\mathscr{B}(G)$.

(iii) Let $\alpha^g_h, \alpha^k_l \in \mathscr{B}(G)$ be such that $\alpha^g_h \mathscr{R} \alpha^k_l$. Since $\alpha^g_h\mathscr{B}(G) = \alpha^k_l\mathscr{B}(G)$ and $\mathscr{B}(G)$ is an inverse semigroup, Theorem 1.17 from [8] implies that

$\alpha_h^g\mathscr{B}(G) = \alpha_h^g(\alpha_h^g)^{-1}\mathscr{B}(G)$ and $\alpha_l^k\mathscr{B}(G) = \alpha_l^k(\alpha_l^k)^{-1}\mathscr{B}(G)$, and hence $\alpha_g^g = \alpha_h^g(\alpha_h^g)^{-1} = \alpha_l^k(\alpha_l^k)^{-1} = \alpha_k^k$. Therefore we get that $g = k$.

Conversely, let $\alpha_h^g, \alpha_l^k \in \mathscr{B}(G)$ be such that $g = k$. Then $\alpha_h^g(\alpha_h^g)^{-1} = \alpha_l^k(\alpha_l^k)^{-1}$. Since $\mathscr{B}(G)$ is an inverse semigroup, Theorem 1.17 from [8] implies that $\alpha_h^g\mathscr{B}(G) = \alpha_h^g(\alpha_h^g)^{-1}\mathscr{B}(G) = \alpha_l^k\mathscr{B}(G)$ and hence $\alpha_h^g\mathscr{R}\alpha_l^k$ in $\mathscr{B}(G)$.

The proof of statement (iv) is similar to (iii).

Statement (v) follows from statements (iii) and (iv).

(vi) For every $g, h \in \mathscr{B}(G)$ we have $\alpha_h^g(\alpha_h^g)^{-1} = \alpha_g^g$ and $(\alpha_h^g)^{-1}\alpha_h^g = \alpha_h^h$, and hence by statement (ii), Proposition 1.2 and Lemma 1.1 from [19] we get that $\mathscr{B}(G)$ is a bisimple semigroup.

(vii) Since every two $\mathscr{D}$-equivalent elements of an arbitrary semigroup S are $\mathscr{J}$-equivalent (see [8, Section 2.1]) we have that $\mathscr{B}(G)$ is a simple semigroup.

The proof of the proposition for the semigroup $\mathscr{B}^+(G)$ is similar. □

Given two partially ordered sets $(A, \leqslant_A)$ and $(B, \leqslant_B)$, the *lexicographical order* $\leqslant_{\mathbf{lex}}$ on the Cartesian product $A \times B$ is defined as follows:

$$(a, b) \leqslant_{\mathbf{lex}} (a', b') \quad \text{if and only if} \quad a <_A a' \quad \text{or} \quad (a = a' \text{ and } b \leqslant_B b').$$

In this case we shall say that the partially ordered set $(A \times B, \leqslant_{\mathbf{lex}})$ is the *lexicographic product* of partially ordered sets $(A, \leqslant_A)$ and $(B, \leqslant_B)$ and it is denoted by $A \times_{\mathbf{lex}} B$. We observe that the lexicographic product of two linearly ordered sets is a linearly ordered set.

Proposition 2.2. *Let G be a linearly ordered d-group. Then the following assertions hold:*

(i) $E\big(\overline{\mathscr{B}}(G)\big) = E(\mathscr{B}(G)) \cup E\big(\mathring{\mathscr{B}}(G)\big)$ *and* $E\big(\overline{\mathscr{B}}^{+}(G)\big) = E(\mathscr{B}^{+}(G)) \cup E\big(\mathring{\mathscr{B}}^{+}(G)\big)$.

(ii) *If* $\alpha_g^g, \mathring{\alpha}_g^g, \alpha_h^h, \mathring{\alpha}_h^h \in E\big(\overline{\mathscr{B}}(G)\big)$ $\big($*resp.,* $\alpha_g^g, \mathring{\alpha}_g^g, \alpha_h^h, \mathring{\alpha}_h^h \in E\big(\overline{\mathscr{B}}^{+}(G)\big)\big)$ *then:*

- (a) $\alpha_g^g \preccurlyeq \alpha_h^h$ *if and only if* $g \geqslant h$ *in* G $\big($*resp., in* $G^+\big)$;
- (b) $\mathring{\alpha}_g^g \preccurlyeq \mathring{\alpha}_h^h$ *if and only if* $g \geqslant h$ *in* G $\big($*resp., in* $G^+\big)$;
- (c) $\alpha_g^g \preccurlyeq \mathring{\alpha}_h^h$ *if and only if* $g > h$ *in* G $\big($*resp., in* $G^+\big)$;
- (d) $\mathring{\alpha}_g^g \preccurlyeq \alpha_h^h$ *if and only if* $g \geqslant h$ *in* G $\big($*resp., in* $G^+\big)$.

(iii) *The semilattice* $E\big(\overline{\mathscr{B}}(G)\big)$ $\big($*resp.,* $E\big(\overline{\mathscr{B}}^{+}(G)\big)\big)$ *is isomorphic to the lexicographic product* $G \times_{\boldsymbol{lex}} \{0, 1\}$ $\big($*resp.,* $G^+ \times_{\boldsymbol{lex}} \{0, 1\}\big)$ *of semilattices* $(G, \vee)$ $\big($*resp.,* $(G^+, \vee)\big)$ *and* $(\{0, 1\}, \min)$ *under the mapping* $(\alpha_g^g)\mathfrak{i} = (g, 1)$ *and* $(\mathring{\alpha}_g^g)\mathfrak{i} = (g, 0)$, *and hence* $E\big(\overline{\mathscr{B}}(G)\big)$ $\big($*resp.,* $E\big(\overline{\mathscr{B}}^{+}(G)\big)\big)$ *is a linearly ordered semilattice.*

(iv) *The elements* α *and* β *of the semigroup* $\overline{\mathscr{B}}(G)$ $\big($*resp.,* $\overline{\mathscr{B}}^{+}(G)\big)$ *are* $\mathscr{R}$*-equivalent in* $\overline{\mathscr{B}}(G)$ $\big($*resp., in* $\overline{\mathscr{B}}^{+}(G)\big)$ *provided either* $\alpha, \beta \in \mathscr{B}(G)$

$\big($resp., $\alpha,\beta \in \overline{\mathscr{B}}^{\,+}(G)\big)$ or $\alpha,\beta \in \mathring{\mathscr{B}}(G)$ $\big($resp., $\alpha,\beta \in \mathring{\mathscr{B}}^{\,+}(G)\big)$; and moreover, we have that

(a) *$\alpha^g_h \mathscr{R} \alpha^k_l$ in $\overline{\mathscr{B}}(G)$ $\big($resp., in $\overline{\mathscr{B}}^{\,+}(G)\big)$ if and only if $g=k$; and*

(b) *$\mathring{\alpha}^g_h \mathscr{R} \mathring{\alpha}^k_l$ in $\overline{\mathscr{B}}(G)$ $\big($resp., in $\overline{\mathscr{B}}^{\,+}(G)\big)$ if and only if $g=k$.*

(v) *The elements α and β of the semigroup $\overline{\mathscr{B}}(G)$ $\big($resp., $\overline{\mathscr{B}}^{\,+}(G)\big)$ are $\mathscr{L}$-equivalent in $\overline{\mathscr{B}}(G)$ $\big($resp., in $\overline{\mathscr{B}}^{\,+}(G)\big)$ provided either $\alpha,\beta \in \mathscr{B}(G)$ $\big($resp., $\alpha,\beta \in \overline{\mathscr{B}}^{\,+}(G)\big)$ or $\alpha,\beta \in \mathring{\mathscr{B}}(G)$ $\big($resp., $\alpha,\beta \in \mathring{\mathscr{B}}^{\,+}(G)\big)$; and moreover, we have that*

(a) *$\alpha^g_h \mathscr{L} \alpha^k_l$ in $\overline{\mathscr{B}}(G)$ $\big($resp., in $\overline{\mathscr{B}}^{\,+}(G)\big)$ if and only if $h=l$; and*

(b) *$\mathring{\alpha}^g_h \mathscr{L} \mathring{\alpha}^k_l$ in $\overline{\mathscr{B}}(G)$ $\big($resp., in $\overline{\mathscr{B}}^{\,+}(G)\big)$ if and only if $h=l$.*

(vi) *The elements α and β of the semigroup $\overline{\mathscr{B}}(G)$ $\big($resp., $\overline{\mathscr{B}}^{\,+}(G)\big)$ are $\mathscr{H}$-equivalent in $\overline{\mathscr{B}}(G)$ $\big($resp., in $\overline{\mathscr{B}}^{\,+}(G)\big)$ provided either $\alpha,\beta \in \mathscr{B}(G)$ $\big($resp., $\alpha,\beta \in \overline{\mathscr{B}}^{\,+}(G)\big)$ or $\alpha,\beta \in \mathring{\mathscr{B}}(G)$ $\big($resp., $\alpha,\beta \in \mathring{\mathscr{B}}^{\,+}(G)\big)$; and moreover, we have that*

(a) *$\alpha^g_h \mathscr{H} \alpha^k_l$ in $\overline{\mathscr{B}}(G)$ $\big($resp., in $\overline{\mathscr{B}}^{\,+}(G)\big)$ if and only if $g=k$ and $h=l$;*

(b) *$\mathring{\alpha}^g_h \mathscr{H} \mathring{\alpha}^k_l$ in $\overline{\mathscr{B}}(G)$ $\big($resp., in $\overline{\mathscr{B}}^{\,+}(G)\big)$ if and only if $g=k$ and $h=l$; and*

(c) *every $\mathscr{H}$-class in $\overline{\mathscr{B}}(G)$ $\big($resp., in $\overline{\mathscr{B}}^{\,+}(G)\big)$ is a singleton set.*

(vii) *$\overline{\mathscr{B}}(G)$ $\big($resp., $\overline{\mathscr{B}}^{\,+}(G)\big)$ is a simple semigroup.*

($viii$) *The semigroup $\overline{\mathscr{B}}(G)$ $\big($resp., $\overline{\mathscr{B}}^{\,+}(G)\big)$ has only two distinct $\mathscr{D}$-classes which are inverse subsemigroups $\mathscr{B}(G)$ and $\mathring{\mathscr{B}}(G)$ $\big($resp., $\mathscr{B}^+(G)$ and $\mathring{\mathscr{B}}^{\,+}(G)\big)$.*

Proof. Statements (i), (ii) and (iii) follow from the definition of the semigroup $\overline{\mathscr{B}}(G)$ and Proposition 1.6.

The proofs of statements (iv), (v) and (vi) follow from Proposition 1.6 and Theorem 1.17 of [8] and are similar to statements (ii), (iv) and (v) of Proposition 2.1.

(vii) We shall show that $\overline{\mathscr{B}}(G)\cdot\alpha\cdot\overline{\mathscr{B}}(G)=\overline{\mathscr{B}}(G)$ for every $\alpha\in\overline{\mathscr{B}}(G)$. We fix arbitrary $\alpha,\beta\in\overline{\mathscr{B}}(G)$ and show that there exist $\gamma,\delta\in\overline{\mathscr{B}}(G)$ such that $\gamma\cdot\alpha\cdot\delta=\beta$.

We consider the following cases:

(1) $\alpha=\alpha^g_h\in\mathscr{B}(G)$ and $\beta=\alpha^k_l\in\mathscr{B}(G)$;

(2) $\alpha=\alpha^g_h\in\mathscr{B}(G)$ and $\beta=\mathring{\alpha}^k_l\in\mathring{\mathscr{B}}(G)$;

(3) $\alpha=\mathring{\alpha}^g_h\in\mathring{\mathscr{B}}(G)$ and $\beta=\alpha^k_l\in\mathscr{B}(G)$;

(4) $\alpha=\mathring{\alpha}^g_h\in\mathring{\mathscr{B}}(G)$ and $\beta=\mathring{\alpha}^k_l\in\mathring{\mathscr{B}}(G)$,

where $g,h,k,l\in G$.

We put

$\gamma = \alpha_g^k$ and $\delta = \alpha_l^h$ in case (1);
$\gamma = \mathring{\alpha}_g^k$ and $\delta = \mathring{\alpha}_l^h$ in case (2);
$\gamma = \alpha_a^k$ and $\delta = \alpha_l^{a \cdot g^{-1} \cdot h}$, where $a \in G^+(g) \setminus \{g\}$, in case (3);
$\gamma = \mathring{\alpha}_g^k$ and $\delta = \mathring{\alpha}_l^h$ in case (4).

Elementary verifications show that $\gamma \cdot \alpha \cdot \delta = \beta$, and this completes the proof of assertion (vii).

Statement $(viii)$ follows from statements (iv) and (v).

The proof of the statements of the proposition for the semigroup $\overline{\mathscr{B}}^+(G)$ is similar. □

Proposition 2.3. *Let G be a linearly ordered group. Then for any distinct elements g and h in G such that $g \leqslant h$ in G (resp., in G^+) the subsemigroup $\mathscr{C}\left(\overline{g,h}\right)$ of $\mathscr{B}(G)$ (resp., $\mathscr{B}^+(G)$), which is generated by elements α_h^g and α_g^h, is isomorphic to the bicyclic semigroup, and hence for every idempotent α_g^g in $\mathscr{B}(G)$ (resp., in $\mathscr{B}^+(G)$) there exists a subsemigroup $\mathscr{C}$ in $\mathscr{B}(G)$ (resp., in $\mathscr{B}^+(G)$) such that α_g^g is a unit of $\mathscr{C}$ and $\mathscr{C}$ is isomorphic to the bicyclic semigroup.*

Proof. Since the semigroup $\mathscr{C}$ which is generated by elements α_h^g and α_g^h is isomorphic to the semigroup $\mathscr{C}_{\mathbb{N}}(\alpha,\beta)$ (this isomorphism $\mathfrak{i}\colon \mathscr{C} \to \mathscr{C}_{\mathbb{N}}(\alpha,\beta)$ can be determined on generating elements of $\mathscr{C}$ by the formulae $(\alpha_h^g)\mathfrak{i} = \alpha$ and $(\alpha_g^h)\mathfrak{i} = \beta$), we conclude that the first part of the proposition follows from Remark 1.1. Obviously, the element α_g^g is a unity of the semigroup $\mathscr{C}$. □

3. Congruences on the semigroups $\mathscr{B}(G)$ and $\mathscr{B}^+(G)$

The following lemma follows from the definition of a congruence on a semilattice.

Lemma 3.1. *Let $\mathfrak{C}$ be an arbitrary congruence on a semilattice S and let $\preccurlyeq$ be the natural partial order on S. Let a and b be idempotents of the semigroup S such that $a\mathfrak{C}b$. Then the relation $a \preccurlyeq b$ implies that $a\mathfrak{C}c$ for all idempotents $c \in S$ such that $a \preccurlyeq c \preccurlyeq b$.*

A linearly ordered group G is called *archimedean* if for each $a, b \in G^+ \setminus \{e\}$ there exist positive integers m and n such that $b \leqslant a^m$ and $a \leqslant b^n$ [7]. Linearly ordered archimedean groups may be described as follows (**Hölder's theorem**): *a linearly ordered group is archimedean if and only if it is isomorphic to some subgroup of the additive group of real numbers with the natural order* [13].

Theorem 3.2. *Let G be an archimedean linearly ordered group. Then every non-trivial congruence on $\mathscr{B}^+(G)$ is a group congruence.*

Proof. Suppose that $\mathfrak{C}$ is a non-trivial congruence on the semigroup $\mathscr{B}^+(G)$. Then there exist distinct elements α_b^a and α_d^c of the semigroup $\mathscr{B}^+(G)$ such that $\alpha_b^a\mathfrak{C}\alpha_d^c$. Since by Proposition 2.1(v) every $\mathscr{H}$-class of the semigroup $\mathscr{B}^+(G)$ is a singleton set, we conclude that either $\alpha_b^a\cdot(\alpha_b^a)^{-1}\neq\alpha_d^c\cdot(\alpha_d^c)^{-1}$ or $(\alpha_b^a)^{-1}\cdot\alpha_b^a\neq(\alpha_d^c)^{-1}\cdot\alpha_d^c$. We shall consider the case $\alpha_a^a=\alpha_b^a\cdot(\alpha_b^a)^{-1}\neq\alpha_d^c\cdot(\alpha_d^c)^{-1}=\alpha_c^c$. In the other case the proof is similar. Since by Proposition 2.1(ii) the band $E(\mathscr{B}^+(G))$ is a linearly ordered semilattice, without loss of generality we can assume that $\alpha_c^c\preccurlyeq\alpha_a^a$. Then by Proposition 2.1(i) we have that $a\leqslant c$ in G. Since $\alpha_b^a\mathfrak{C}\alpha_d^c$ and $\mathscr{B}^+(G)$ is an inverse semigroup, Lemma III.1.1 from [20] implies that $\left(\alpha_b^a\cdot(\alpha_b^a)^{-1}\right)\mathfrak{C}\left(\alpha_d^c\cdot(\alpha_d^c)^{-1}\right)$, i.e., $\alpha_a^a\mathfrak{C}\alpha_c^c$. Then we have

$$\begin{aligned}
&\alpha_a^c\cdot\alpha_a^a\cdot\alpha_c^a=\alpha_c^c;\\
&\alpha_a^c\cdot\alpha_c^c\cdot\alpha_c^a=\alpha_{c\cdot a^{-1}\cdot c}^{c\cdot a^{-1}\cdot c};\\
&\alpha_a^c\cdot\alpha_{c\cdot a^{-1}\cdot c}^{c\cdot a^{-1}\cdot c}\cdot\alpha_c^a=\alpha_{c\cdot(a^{-1}\cdot c)^2}^{c\cdot(a^{-1}\cdot c)^2};\\
&\ldots\quad\ldots\\
&\alpha_a^c\cdot\alpha_{c\cdot(a^{-1}\cdot c)^{n-1}}^{c\cdot(a^{-1}\cdot c)^{n-1}}\cdot\alpha_c^a=\alpha_{c\cdot(a^{-1}\cdot c)^{n}}^{c\cdot(a^{-1}\cdot c)^{n}},
\end{aligned}$$

and hence $\alpha_a^a\mathfrak{C}\alpha_{c\cdot(a^{-1}\cdot c)^{n}}^{c\cdot(a^{-1}\cdot c)^{n}}$ for every non-negative integer n. Since $a<c$ in G we get that $a^{-1}\cdot c$ is a positive element of the linearly ordered group G. Since the linearly ordered group G is archimedean we conclude that for every $g\in G$ with $g>a$ there exists a positive integer n such that $a^{-1}\cdot g<\left(a^{-1}\cdot c\right)^n$ and hence $g<c\cdot\left(a^{-1}\cdot c\right)^{n-1}$. Therefore Lemma 3.1 and Proposition 2.1(i) imply that $\alpha_a^a\mathfrak{C}\alpha_g^g$ for every $g\in G$ such that $a\leqslant g$.

If $a=e$ then all idempotents of the semigroup $\mathscr{B}^+(G)$ are $\mathfrak{C}$-equivalent. Since the semigroup $\mathscr{B}^+(G)$ is inverse we conclude that the quotient semigroup $\mathscr{B}^+(G)/\mathfrak{C}$ contains only one idempotent and by Lemma II.1.10 from [20] the semigroup $\mathscr{B}^+(G)/\mathfrak{C}$ is a group.

Suppose that $e<a$. Then by Proposition 2.3 we have that the semigroup $\mathscr{C}^*$ which is generated by elements α_g^e and α_e^g is isomorphic to the bicyclic semigroup for every element g in G^+ such that $e<a\leqslant g$. Hence the following conditions hold:

$$\ldots\preccurlyeq\alpha_{g^i}^{g^i}\preccurlyeq\alpha_{g^{i-1}}^{g^{i-1}}\preccurlyeq\ldots\preccurlyeq\alpha_g^g\preccurlyeq\alpha_a^a\quad\text{and}$$

$$\alpha_{g^i}^{g^i}\neq\alpha_{g^j}^{g^j},\qquad\text{for distinct positive integers } i \text{ and } j,$$

in $E(\mathscr{B}^+(G))$. Since the linearly ordered group G is archimedean we conclude that $\alpha_a^a\mathfrak{C}\alpha_{g^i}^{g^i}$ for every positive integer i. Since the semigroup $\mathscr{C}^*$ is isomorphic to the bicyclic semigroup, Corollary 1.32 of [8] and Lemma 3.1 imply that all idempotents of the semigroup $\mathscr{B}^+(G)$ are $\mathfrak{C}$-equivalent. Since the semigroup $\mathscr{B}^+(G)$ is inverse we conclude that the quotient semigroup

$\mathscr{B}^+(G)/\mathfrak{C}$ contains only one idempotent and by Lemma II.1.10 from [20] the semigroup $\mathscr{B}^+(G)/\mathfrak{C}$ is a group. $\square$

Theorem 3.3. *Let G be an archimedean linearly ordered group. Then every non-trivial congruence on $\mathscr{B}(G)$ is a group congruence.*

Proof. Suppose that $\mathfrak{C}$ is a non-trivial congruence on the semigroup $\mathscr{B}(G)$. Similar arguments as in the proof of Theorem 3.2 imply that there exist distinct idempotents α_a^a and α_b^b in the semigroup $\mathscr{B}(G)$ such that $\alpha_a^a\mathfrak{C}\alpha_b^b$ and $\alpha_b^b \preccurlyeq \alpha_a^a$, for $a, b \in G$ with $a \leqslant b$ in G. Then we have

$$\alpha_a^e \cdot \alpha_a^a \cdot \alpha_e^a = \alpha_e^e \quad \text{and} \quad \alpha_a^e \cdot \alpha_b^b \cdot \alpha_e^a = \alpha_b^{b\cdot a^{-1}} \cdot \alpha_e^a = \alpha_{b\cdot a^{-1}}^{b\cdot a^{-1}},$$

and hence $\alpha_e^e\mathfrak{C}\alpha_{b\cdot a^{-1}}^{b\cdot a^{-1}}$. Since $a \leqslant b$ in G we conclude that $e \leqslant b \cdot a^{-1}$ in G and hence Theorem 3.2 implies that $\alpha_c^c\mathfrak{C}\alpha_d^d$ for all $c, d \in G^+$.

We fix an arbitrary element $g \in G\setminus G^+$. Then we have that $g^{-1} \in G^+\setminus\{e\}$ and hence $\alpha_e^e\mathfrak{C}\alpha_{g^{-1}}^{g^{-1}}$. Since

$$\alpha_e^g \cdot \alpha_e^e \cdot \alpha_g^e = \alpha_g^g \quad \text{and} \quad \alpha_e^g \cdot \alpha_{g^{-1}}^{g^{-1}} \cdot \alpha_g^e = \alpha_{g^{-1}}^{g^{-1}\cdot e\cdot g} \cdot \alpha_g^e = \alpha_{g^{-1}}^e \cdot \alpha_g^e = \alpha_{g^{-1}\cdot e\cdot g}^e = \alpha_e^e$$

we conclude that $\alpha_e^e\mathfrak{C}\alpha_g^g$. Therefore all idempotents of the semigroup $\mathscr{B}(G)$ are $\mathfrak{C}$-equivalent. Since the semigroup $\mathscr{B}(G)$ is inverse we conclude that the quotient semigroup $\mathscr{B}(G)/\mathfrak{C}$ contains only one idempotent and by Lemma II.1.10 from [20] the semigroup $\mathscr{B}(G)/\mathfrak{C}$ is a group. $\square$

Remark 3.4. We observe that Proposition 1.5 implies that if G is a linearly ordered d-group then the statements similar to Propositions 2.1 and 2.3 and Theorems 3.2 and 3.3 hold for the semigroups $\mathring{\mathscr{B}}(G)$ and $\mathring{\mathscr{B}}^+(G)$.

Theorem 3.5. *If G is the lexicographic product $A\times_{\boldsymbol{lex}}H$ of non-singleton linearly ordered groups A and H then the semigroups $\mathscr{B}(G)$ and $\mathscr{B}^+(G)$ have non-trivial non-group congruences.*

Proof. We define a relation $\sim_{\mathfrak{c}}$ on the semigroup $\mathscr{B}(G)$ as follows:

$$\alpha_{(c_1,d_1)}^{(a_1,b_1)} \sim_{\mathfrak{c}} \alpha_{(c_2,d_2)}^{(a_2,b_2)} \quad \text{if and only if} \quad a_1 = a_2,\, c_1 = c_2 \quad \text{and} \quad d_1^{-1}b_1 = d_2^{-1}b_2.$$

Simple verifications show that $\sim_{\mathfrak{c}}$ is an equivalence relation on the semigroup $\mathscr{B}(G)$.

Next we shall prove that $\sim_{\mathfrak{c}}$ is a congruence on $\mathscr{B}(G)$. Suppose that $\alpha_{(c_1,d_1)}^{(a_1,b_1)} \sim_{\mathfrak{c}} \alpha_{(c_2,d_2)}^{(a_2,b_2)}$ for some $\alpha_{(c_1,d_1)}^{(a_1,b_1)}, \alpha_{(c_2,d_2)}^{(a_2,b_2)} \in \mathscr{B}(G)$. Let $\alpha_{(x,y)}^{(u,v)}$ be an arbitrary element of $\mathscr{B}(G)$. Then we have

$$\alpha_{(m_1,n_1)}^{(k_1,l_1)} = \alpha_{(c_1,d_1)}^{(a_1,b_1)} \cdot \alpha_{(x,y)}^{(u,v)} = \begin{cases} \alpha_{(x,y)}^{(u,v)\cdot(c_1,d_1)^{-1}\cdot(a_1,b_1)}, & \text{if } (c_1,d_1)\preccurlyeq(u,v); \\ \alpha^{(a_1,b_1)}_{(c_1,d_1)\cdot(u,v)^{-1}\cdot(x,y)}, & \text{if } (u,v)\preccurlyeq(c_1,d_1) \end{cases} =$$

$$= \begin{cases} \alpha_{(x,y)}^{(uc_1^{-1}a_1, vd_1^{-1}b_1)}, & \text{if } (c_1,d_1)\preccurlyeq(u,v); \\ \alpha^{(a_1,b_1)}_{(c_1u^{-1}x, d_1v^{-1}y)}, & \text{if } (u,v)\preccurlyeq(c_1,d_1) \end{cases} =$$

$$= \begin{cases} \alpha_{(x,y)}^{(uc_1^{-1}a_1, vd_1^{-1}b_1)}, & \text{if } c_1 < u; \\ \alpha_{(x,y)}^{(a_1, vd_1^{-1}b_1)}, & \text{if } c_1 = u \text{ and } d_1 \leqslant v; \\ \alpha^{(a_1,b_1)}_{(c_1u^{-1}x, d_1v^{-1}y)}, & \text{if } u < c_1; \\ \alpha^{(a_1,b_1)}_{(x, d_1v^{-1}y)}, & \text{if } u = c_1 \text{ and } v \leqslant d_1; \end{cases}$$

and

$$\alpha_{(m_2,n_2)}^{(k_2,l_2)} = \alpha_{(c_2,d_2)}^{(a_2,b_2)} \cdot \alpha_{(x,y)}^{(u,v)} = \begin{cases} \alpha_{(x,y)}^{(u,v)\cdot(c_2,d_2)^{-1}\cdot(a_2,b_2)}, & \text{if } (c_2,d_2)\preccurlyeq(u,v); \\ \alpha^{(a_2,b_2)}_{(c_2,d_2)\cdot(u,v)^{-1}\cdot(x,y)}, & \text{if } (u,v)\preccurlyeq(c_2,d_2) \end{cases} =$$

$$= \begin{cases} \alpha_{(x,y)}^{(uc_2^{-1}a_2, vd_2^{-1}b_2)}, & \text{if } (c_2,d_2)\preccurlyeq(u,v); \\ \alpha^{(a_2,b_2)}_{(c_2u^{-1}x, d_2v^{-1}y)}, & \text{if } (u,v)\preccurlyeq(c_2,d_2), \end{cases} =$$

$$= \begin{cases} \alpha_{(x,y)}^{(uc_2^{-1}a_2, vd_2^{-1}b_2)}, & \text{if } c_2 < u; \\ \alpha_{(x,y)}^{(a_2, vd_2^{-1}b_2)}, & \text{if } c_2 = u \text{ and } d_2 \leqslant v; \\ \alpha^{(a_2,b_2)}_{(c_2u^{-1}x, d_2v^{-1}y)}, & \text{if } u < c_2; \\ \alpha^{(a_2,b_2)}_{(x, d_2v^{-1}y)}, & \text{if } u = c_2 \text{ and } v \leqslant d_2. \end{cases}$$

Since $a_1 = a_2$, $c_1 = c_2$ and $d_1^{-1}b_1 = d_2^{-1}b_2$ we conclude that the following conditions hold:

(1) if $c_1 = c_2 < u$ then $k_1 = uc_1^{-1}a_1 = uc_2^{-1}a_2 = k_2$, $m_1 = x = m_2$ and
$$n_1^{-1}l_1 = y^{-1}vd_1^{-1}b_1 = y^{-1}vd_2^{-1}b_2 = n_2^{-1}l_2;$$

(2) if $c_1 = c_2 = u$ and $d_1 \leqslant v$ then $k_1 = a_1 = a_2 = k_2$, $m_1 = x = m_2$ and
$$n_1^{-1}l_1 = y^{-1}vd_1^{-1}b_1 = y^{-1}vd_2^{-1}b_2 = n_2^{-1}l_2;$$

(3) if $u < c_1 = c_2$ then $k_1 = a_1 = a_2 = k_2$, $m_1 = c_1u^{-1}x = c_2u^{-1}x = m_2$ and
$$n_1^{-1}l_1 = y^{-1}vd_1^{-1}b_1 = y^{-1}vd_2^{-1}b_2 = n_2^{-1}l_2;$$

(4) if $u=c_1=c_2$ and $v\leqslant d_1$ then $k_1=a_1=a_2=k_2$, $m_1=x=m_2$ and
$$n_1^{-1}l_1=y^{-1}vd_1^{-1}b_1=y^{-1}vd_2^{-1}b_2=n_2^{-1}l_2.$$

Hence we get that $\alpha_{(m_1,n_1)}^{(k_1,l_1)}\sim_{\mathfrak{c}}\alpha_{(m_2,n_2)}^{(k_2,l_2)}$. Similarly we have

$$\alpha_{(r_1,s_1)}^{(p_1,q_1)}=\alpha_{(x,y)}^{(u,v)}\cdot\alpha_{(c_1,d_1)}^{(a_1,b_1)}=\begin{cases}\alpha_{(c_1,d_1)}^{(a_1,b_1)\cdot(x,y)^{-1}\cdot(u,v)}, & \text{if } (x,y)\leqslant(a_1,b_1);\\ \alpha_{(x,y)\cdot(a_1,b_1)^{-1}\cdot(c_1,d_1)}^{(u,v)}, & \text{if } (a_1,b_1)\leqslant(x,y)\end{cases}=$$
$$=\begin{cases}\alpha_{(c_1,d_1)}^{(a_1x^{-1}u,b_1y^{-1}v)}, & \text{if } (x,y)\leqslant(a_1,b_1);\\ \alpha_{(xa_1^{-1}c_1,yb_1^{-1}d_1)}^{(u,v)}, & \text{if } (a_1,b_1)\leqslant(x,y),\end{cases}=$$
$$=\begin{cases}\alpha_{(c_1,d_1)}^{(a_1x^{-1}u,b_1y^{-1}v)}, & \text{if } x<a_1;\\ \alpha_{(c_1,d_1)}^{(u,b_1y^{-1}v)}, & \text{if } x=a_1 \text{ and } y\leqslant b_1;\\ \alpha_{(xa_1^{-1}c_1,yb_1^{-1}d_1)}^{(u,v)}, & \text{if } a_1<x;\\ \alpha_{(c_1,yb_1^{-1}d_1)}^{(u,v)}, & \text{if } a_1=x \text{ and } b_1\leqslant y,\end{cases}$$

and

$$\alpha_{(r_2,s_2)}^{(p_2,q_2)}=\alpha_{(x,y)}^{(u,v)}\cdot\alpha_{(c_2,d_2)}^{(a_2,b_2)}=\begin{cases}\alpha_{(c_2,d_2)}^{(a_2,b_2)\cdot(x,y)^{-1}\cdot(u,v)}, & \text{if } (x,y)\leqslant(a_2,b_2);\\ \alpha_{(x,y)\cdot(a_2,b_2)^{-1}\cdot(c_2,d_2)}^{(u,v)}, & \text{if } (a_2,b_2)\leqslant(x,y)\end{cases}=$$
$$=\begin{cases}\alpha_{(c_2,d_2)}^{(a_2x^{-1}u,b_2y^{-1}v)}, & \text{if } (x,y)\leqslant(a_2,b_2);\\ \alpha_{(xa_2^{-1}c_2,yb_2^{-1}d_2)}^{(u,v)}, & \text{if } (a_2,b_2)\leqslant(x,y)\end{cases}=$$
$$=\begin{cases}\alpha_{(c_2,d_2)}^{(a_2x^{-1}u,b_2y^{-1}v)}, & \text{if } x<a_2;\\ \alpha_{(c_2,d_2)}^{(u,b_2y^{-1}v)}, & \text{if } x=a_2 \text{ and } y\leqslant b_2;\\ \alpha_{(xa_2^{-1}c_2,yb_2^{-1}d_2)}^{(u,v)}, & \text{if } a_2<x;\\ \alpha_{(c_2,yb_2^{-1}d_2)}^{(u,v)}, & \text{if } a_2=x \text{ and } b_2\leqslant y.\end{cases}$$

Since $a_1=a_2$, $c_1=c_2$ and $d_1^{-1}b_1=d_2^{-1}b_2$ we conclude that the following conditions hold:

(1) if $x<a_1=a_2$ then $p_1=a_1x^{-1}u=a_2x^{-1}u=p_2$, $r_1=c_1=c_2=r_2$ and
$$s_1^{-1}q_1=d_1^{-1}b_1y^{-1}v=d_2^{-1}b_2y^{-1}v=s_2^{-1}q_2;$$

(2) if $x=a_1=a_2$ and $y\leqslant b_1$ then $p_1=u=p_2$, $r_1=c_1=c_2=r_2$ and
$$s_1^{-1}q_1=d_1^{-1}b_1y^{-1}v=d_2^{-1}b_2y^{-1}v=s_2^{-1}q_2;$$

(3) if $a_1=a_2<x$ then $p_1=u=p_2$, $r_1=xa_1^{-1}c_1=xa_2^{-1}c_2=r_2$ and
$$s_1^{-1}q_1=d_1^{-1}b_1y^{-1}v=d_2^{-1}b_2y^{-1}v=s_2^{-1}q_2;$$

(4) if $a_1=a_2=x$ and $b_1\leqslant y$ then $p_1=u=p_2$, $r_1=c_1=c_2=r_2$ and
$$s_1^{-1}q_1=d_1^{-1}b_1y^{-1}v=d_2^{-1}b_2y^{-1}v=s_2^{-1}q_2.$$

Hence we get that $\alpha^{(p_1,q_1)}_{(r_1,s_1)}\sim_{\mathfrak{c}}\alpha^{(p_2,q_2)}_{(r_2,s_2)}$.

We fix any $a_1,a_2,b_1,b_2\in G$. If $a_1\neq a_2$ then the elements $\alpha^{(a_1,b_1)}_{(a_1,b_1)}$ and $\alpha^{(a_2,b_2)}_{(a_2,b_2)}$ are idempotents of the semigroup $\mathscr{B}(G)$, and moreover, the elements $\alpha^{(a_1,b_1)}_{(a_1,b_1)}$ and $\alpha^{(a_2,b_2)}_{(a_2,b_2)}$ are not $\sim_c$-equivalent. Since a homomorphic image of an idempotent is an idempotent too, we conclude that $\left(\alpha^{(a_1,b_1)}_{(a_1,b_1)}\right)\pi_c\neq\left(\alpha^{(a_2,b_2)}_{(a_2,b_2)}\right)\pi_c$, where $\pi_c\colon\mathscr{B}(G)\to\mathscr{B}(G)/\sim_c$ is the natural homomorphism which is generated by the congruence $\sim_c$ on the semigroup $\mathscr{B}(G)$. This implies that the quotient semigroup $\mathscr{B}(G)/\sim_c$ is not a group, and hence $\sim_c$ is not a group congruence on the semigroup $\mathscr{B}(G)$.

The proof of the statement that the semigroup $\mathscr{B}^+(G)$ has a non-trivial non-group congruence is similar. □

Theorem 3.6. *Let G be a commutative linearly ordered group. Then the following conditions are equivalent:*

(*i*) *G is archimedean;*
(*ii*) *every non-trivial congruence on $\mathscr{B}(G)$ is a group congruence;*
(*iii*) *every non-trivial congruence on $\mathscr{B}^+(G)$ is a group congruence.*

Proof. Implications $(i)\Rightarrow(ii)$ and $(i)\Rightarrow(iii)$ follow from Theorems 3.3 and 3.2, respectively.

$(ii)\Rightarrow(i)$ Suppose the contrary that there exists a non-archimedean commutative linearly ordered group G such that every non-trivial congruence on $\mathscr{B}(G)$ is a group congruence. Then by Hahn's theorem (see [12] or [16, Section VII.3, Theorem 1]) G is isomorphic to a lexicographic product $\prod\limits_{\alpha\in\mathscr{J}}{}_{\mathbf{lex}}H_\alpha$ of some family of non-singleton subgroups $\{H_\alpha\mid\alpha\in\mathscr{J}\}$ of the additive group of real numbers with a non-singleton linearly ordered index set $\mathscr{J}$. We fix a non-maximal element $\alpha_0\in\mathscr{J}$, and put
$$A=\prod{}_{\mathbf{lex}}\{H_\alpha\mid\alpha\leqslant\alpha_0\}\qquad\text{and}\qquad H=\prod{}_{\mathbf{lex}}\{H_\alpha\mid\alpha_0<\alpha\}.$$
Then G is isomorphic to a lexicographic product $A\times_{\mathbf{lex}}H$ of non-singleton linearly ordered groups A and H, and hence by Theorem 3.5 the semigroup $\mathscr{B}(G)$ has a non-trivial non-group congruence. The obtained contradiction implies that the group G is archimedean.

The proof of implication $(iii)\Rightarrow(i)$ is similar to $(ii)\Rightarrow(i)$. □

On the semigroup $\overline{\mathscr{B}}(G)$ (resp., $\overline{\mathscr{B}}^+(G)$) we determine a relation $\sim_{\mathfrak{id}}$ in the following way. We define a map $\mathfrak{id}\colon\overline{\mathscr{B}}(G)\to\overline{\mathscr{B}}(G)$ (resp., $\mathfrak{id}\colon\overline{\mathscr{B}}^+(G)\to$

$\overline{\mathscr{B}}^{+}(G)$) by the formulae $(\alpha_h^g)\mathfrak{id} = \mathring{\alpha}_h^g$ and $(\mathring{\alpha}_h^g)\mathfrak{id} = \alpha_h^g$ for $g, h \in G$ (resp., $g, h \in G^{+}$). We put

$$\alpha \sim_{\mathfrak{id}} \beta \quad \text{if and only if} \quad \alpha = \beta \ \text{ or } \ (\alpha)\mathfrak{id} = \beta \ \text{ or } \ (\beta)\mathfrak{id} = \alpha$$

for $\alpha, \beta \in \overline{\mathscr{B}}(G)$ (resp., $\alpha, \beta \in \overline{\mathscr{B}}^{+}(G)$). Simple verifications show that $\sim_{\mathfrak{id}}$ is an equivalence relation on the semigroup $\overline{\mathscr{B}}(G)$ (resp., $\overline{\mathscr{B}}^{+}(G)$).

Proposition 3.7. *If G is a linearly ordered d-group then $\sim_{\mathfrak{id}}$ is a congruence on semigroups $\overline{\mathscr{B}}(G)$ and $\overline{\mathscr{B}}^{+}(G)$. Moreover, quotient semigroups $\overline{\mathscr{B}}(G)/\sim_{\mathfrak{id}}$ and $\overline{\mathscr{B}}(G)^{+}/\sim_{\mathfrak{id}}$ are isomorphic to semigroups $\mathscr{B}(G)$ and $\mathscr{B}^{+}(G)$, respectively.*

Proof. It is sufficient to show that if $\alpha \sim_{\mathfrak{id}} \beta$ and $\gamma \sim_{\mathfrak{id}} \delta$ then $(\alpha \cdot \gamma) \sim_{\mathfrak{id}} (\beta \cdot \delta)$ for $\alpha, \beta, \gamma, \delta \in \overline{\mathscr{B}}(G)$ (resp., $\alpha, \beta, \gamma, \delta \in \overline{\mathscr{B}}^{+}(G)$). Since the case $\alpha = \beta$ and $\gamma = \delta$ is trivial we consider the following cases:

(*i*) $\alpha = \alpha_b^a$, $\beta = \mathring{\alpha}_b^a$ and $\gamma = \delta = \alpha_d^c$;
(*ii*) $\alpha = \alpha_b^a$, $\beta = \mathring{\alpha}_b^a$ and $\gamma = \delta = \mathring{\alpha}_d^c$;
(*iii*) $\alpha = \mathring{\alpha}_b^a$, $\beta = \alpha_b^a$ and $\gamma = \delta = \alpha_d^c$;
(*iv*) $\alpha = \mathring{\alpha}_b^a$, $\beta = \alpha_b^a$ and $\gamma = \delta = \mathring{\alpha}_d^c$;
(*v*) $\alpha = \alpha_b^a$, $\beta = \mathring{\alpha}_b^a$, $\gamma = \mathring{\alpha}_d^c$ and $\delta = \alpha_d^c$;
(*vi*) $\alpha = \alpha_b^a$, $\beta = \mathring{\alpha}_b^a$, $\gamma = \alpha_d^c$ and $\delta = \mathring{\alpha}_d^c$;
(*vii*) $\alpha = \mathring{\alpha}_b^a$, $\beta = \alpha_b^a$, $\gamma = \mathring{\alpha}_d^c$ and $\delta = \alpha_d^c$;
(*viii*) $\alpha = \mathring{\alpha}_b^a$, $\beta = \alpha_b^a$, $\gamma = \alpha_d^c$ and $\delta = \mathring{\alpha}_d^c$;
(*ix*) $\alpha = \beta = \alpha_b^a$, $\gamma = \mathring{\alpha}_d^c$ and $\delta = \alpha_d^c$;
(*x*) $\alpha = \beta = \mathring{\alpha}_b^a$, $\gamma = \mathring{\alpha}_d^c$ and $\delta = \alpha_d^c$;
(*xi*) $\alpha = \beta = \alpha_b^a$, $\gamma = \alpha_d^c$ and $\delta = \mathring{\alpha}_d^c$; and
(*xii*) $\alpha = \beta = \mathring{\alpha}_b^a$, $\gamma = \alpha_d^c$ and $\delta = \mathring{\alpha}_d^c$,

where $a, b, c, d \in G$ (resp., $a, b, c, d \in G^{+}$).

In case (*i*) we have that

$$\alpha\cdot\gamma = \alpha_b^a\cdot\alpha_d^c = \begin{cases} \alpha_d^{c\cdot b^{-1}\cdot a}, & \text{if } b < c; \\ \alpha_d^a, & \text{if } b = c; \\ \alpha_{b\cdot c^{-1}\cdot d}^a, & \text{if } b > c, \end{cases} \quad \text{and} \quad \beta\cdot\delta = \mathring{\alpha}_b^a\cdot\alpha_d^c = \begin{cases} \mathring{\alpha}_d^{c\cdot b^{-1}\cdot a}, & \text{if } b < c; \\ \mathring{\alpha}_d^a, & \text{if } b = c; \\ \alpha_{b\cdot c^{-1}\cdot d}^a, & \text{if } b > c, \end{cases}$$

and hence $(\alpha \cdot \gamma) \sim_{\mathfrak{id}} (\beta \cdot \delta)$ in $\overline{\mathscr{B}}(G)$ (resp., in $\overline{\mathscr{B}}^{+}(G)$). In other cases verifications are similar.

Since the restriction $\Phi_{\sim_{\mathfrak{id}}}|_{\mathscr{B}(G)} \colon \mathscr{B}(G) \to \mathscr{B}(G)$ of the natural homomorphism $\Phi_{\sim_{\mathfrak{id}}} \colon \overline{\mathscr{B}}(G) \to \mathscr{B}(G)$ is a bijective map we conclude that the semigroup $(\overline{\mathscr{B}}(G))\Phi_{\sim_{\mathfrak{id}}}$ is isomorphic to the semigroup $\mathscr{B}(G)$. Similar arguments show that the semigroup $\overline{\mathscr{B}}^{+}(G)/\sim_{\mathfrak{id}}$ is isomorphic to $\mathscr{B}^{+}(G)$. □

Theorem 3.8. *Let G be an archimedean linearly ordered d-group. If $\mathfrak{C}$ is a non-trivial congruence on $\overline{\mathscr{B}}(G)$ (resp., on $\overline{\mathscr{B}}^{+}(G)$) then the quotient* *semigroup $\overline{\mathscr{B}}(G)/\mathfrak{C}$ (resp., $\overline{\mathscr{B}}^+(G)/\mathfrak{C}$) is either a group or $\overline{\mathscr{B}}(G)/\mathfrak{C}$ (resp., $\overline{\mathscr{B}}^+(G)/\mathfrak{C}$) is isomorphic to the semigroup $\mathscr{B}(G)$ (resp., $\mathscr{B}^+(G)$).*

Proof. Since the subsemigroup of idempotents of the semigroup $\overline{\mathscr{B}}(G)$ is linearly ordered, similar arguments as in the proof of Theorem 3.2 imply that there exist distinct idempotents ε and ι of $\overline{\mathscr{B}}(G)$ such that $\varepsilon\mathfrak{C}\iota$ and $\varepsilon \preccurlyeq \iota$. If the set $(\varepsilon,\iota)=\{\upsilon\in E(\overline{\mathscr{B}}(G)) \mid \varepsilon\prec\upsilon\prec\iota\}$ is non-empty then Lemma 3.1 and Theorem 3.2 imply that the quotient semigroup $\overline{\mathscr{B}}(G)/\mathfrak{C}$ is inverse and has only one idempotent, and hence by Lemma II.1.10 from [20] it is a group. Otherwise there exists $g\in G$ such that $\iota=\alpha_g^g$ and $\varepsilon=\mathring{\alpha}_g^g$. Since $\alpha_l^k=\alpha_g^k\cdot\alpha_g^g\cdot\alpha_l^g$ and $\mathring{\alpha}_l^k=\alpha_g^k\cdot\mathring{\alpha}_g^g\cdot\alpha_l^g$ for every $k,l\in G$ we conclude that the congruence $\mathfrak{C}$ coincides with the congruence $\sim_{\mathfrak{id}}$ on $\overline{\mathscr{B}}(G)$, and hence by Proposition 3.7 the quotient semigroup $\overline{\mathscr{B}}(G)/\mathfrak{C}$ is isomorphic to the semigroup $\mathscr{B}(G)$.

In the case of the semigroup $\overline{\mathscr{B}}^+(G)$ the proof is similar. □

Theorem 3.9. *Let G be a commutative linearly ordered d-group. Then the following conditions are equivalent:*

- (i) *G is archimedean;*
- (ii) *every non-trivial congruence on $\mathring{\mathscr{B}}(G)$ is a group congruence;*
- (iii) *every non-trivial congruence on $\mathring{\mathscr{B}}^+(G)$ is a group congruence;*
- (iv) *the semigroup $\overline{\mathscr{B}}(G)$ has a unique non-trivial non-group congruence;*
- (v) *the semigroup $\overline{\mathscr{B}}^+(G)$ has a unique non-trivial non-group congruence.*

Proof. The equivalence of statements (i), (ii) and (iii) follows from Proposition 1.5 and Theorem 3.6. Also Theorem 3.8 implies that implications (i) ⇒ (iv) and (i) ⇒ (v) hold.

Next we shall show that implication (iv) ⇒ (i) holds. Suppose the contrary: there exists a commutative linearly ordered non-archimedean d-group G such that the semigroup $\overline{\mathscr{B}}(G)$ has a unique non-trivial non-group congruence. Then by Proposition 3.7 we have that $\sim_{\mathfrak{id}}$ is a unique non-trivial non-group congruence on the semigroup $\overline{\mathscr{B}}(G)$. Therefore, similarly as in the proof of Theorem 3.6 we get that G is isomorphic to the lexicographic product $A\times_{\mathbf{lex}}H$ of non-singleton linearly ordered groups A and H, and hence by Theorem 3.5 the semigroup $\mathscr{B}(G)$ has a non-trivial non-group congruence $\sim$. We define a relation $\overline{\sim}$ on the semigroup $\overline{\mathscr{B}}(G)$ as follows:

(i) $\left(\alpha_{(c,d)}^{(a,b)},\alpha_{(r,s)}^{(p,q)}\right)\in\overline{\sim}$ if and only if $\left(\alpha_{(c,d)}^{(a,b)},\alpha_{(r,s)}^{(p,q)}\right)\in\sim$, for $\alpha_{(c,d)}^{(a,b)},\alpha_{(r,s)}^{(p,q)}\in\mathscr{B}(G)\subset\overline{\mathscr{B}}(G)$;

(ii) $\left(\alpha_{(r,s)}^{(p,q)},\mathring{\alpha}_{(r,s)}^{(p,q)}\right)$, $\left(\mathring{\alpha}_{(r,s)}^{(p,q)},\alpha_{(r,s)}^{(p,q)}\right)$, $\left(\mathring{\alpha}_{(r,s)}^{(p,q)},\mathring{\alpha}_{(r,s)}^{(p,q)}\right)\in\overline{\sim}$, for all $p,r\in A$ and $q,s\in H$;

(iii) $\left(\mathring{\alpha}_{(c,d)}^{(a,b)}, \mathring{\alpha}_{(r,s)}^{(p,q)}\right) \in \eqsim$ if and only if $\left(\alpha_{(c,d)}^{(a,b)}, \alpha_{(r,s)}^{(p,q)}\right) \in \sim$, for $\alpha_{(c,d)}^{(a,b)}, \alpha_{(r,s)}^{(p,q)} \in \mathscr{B}(G) \subset \overline{\mathscr{B}}(G)$ and $\mathring{\alpha}_{(c,d)}^{(a,b)}, \mathring{\alpha}_{(r,s)}^{(p,q)} \in \mathring{\mathscr{B}}(G) \subset \overline{\mathscr{B}}(G)$;

(iv) $\left(\mathring{\alpha}_{(c,d)}^{(a,b)}, \alpha_{(r,s)}^{(p,q)}\right) \in \eqsim$ if and only if $\left(\alpha_{(c,d)}^{(a,b)}, \alpha_{(r,s)}^{(p,q)}\right) \in \sim$, for $\alpha_{(c,d)}^{(a,b)}, \alpha_{(r,s)}^{(p,q)} \in \mathscr{B}(G) \subset \overline{\mathscr{B}}(G)$ and $\mathring{\alpha}_{(c,d)}^{(a,b)} \in \mathring{\mathscr{B}}(G) \subset \overline{\mathscr{B}}(G)$;

(v) $\left(\alpha_{(c,d)}^{(a,b)}, \mathring{\alpha}_{(r,s)}^{(p,q)}\right) \in \eqsim$ if and only if $\left(\alpha_{(c,d)}^{(a,b)}, \alpha_{(r,s)}^{(p,q)}\right) \in \sim$, for $\alpha_{(c,d)}^{(a,b)}, \alpha_{(r,s)}^{(p,q)} \in \mathscr{B}(G) \subset \overline{\mathscr{B}}(G)$ and $\mathring{\alpha}_{(r,s)}^{(p,q)} \in \mathring{\mathscr{B}}(G) \subset \overline{\mathscr{B}}(G)$.

Then simple verifications show that $\eqsim$ is a congruence on the semigroup $\overline{\mathscr{B}}(G)$, and moreover, the quotient semigroup $\overline{\mathscr{B}}(G)/\eqsim$ is isomorphic to the quotient semigroup $\mathscr{B}(G)/\sim$. This implies that the congruence $\eqsim$ is different from $\sim_{\mathfrak{i}\mathfrak{d}}$. This contradicts that $\sim_{\mathfrak{i}\mathfrak{d}}$ is a unique non-trivial non-group congruence on the semigroup $\overline{\mathscr{B}}(G)$. The obtained contradiction implies implication $(iv) \Rightarrow (i)$.

The proof of implication $(v) \Rightarrow (i)$ is similar to the proof of implication $(iv) \Rightarrow (i)$. □

Theorem 3.10. *Let G be a linearly ordered group and let $\mathfrak{C}_{mg}$ be the least group congruence on the semigroup $\mathscr{B}(G)$ (resp., $\mathscr{B}^+(G)$). Then the quotient semigroup $\mathscr{B}(G)/\mathfrak{C}_{mg}$ (resp., $\mathscr{B}^+(G)/\mathfrak{C}_{mg}$) is antiisomorphic to the group G.*

Proof. By Proposition 1.2(ii) and Lemma III.5.2 from [20] we have that elements α_b^a and α_d^c are $\mathfrak{C}_{mg}$-equivalent in $\mathscr{B}(G)$ (resp., in $\mathscr{B}^+(G)$) if and only if there exists $x \in G$ such that $\alpha_b^a \cdot \alpha_x^x = \alpha_d^c \cdot \alpha_x^x$. Then Proposition 2.1$(i)$ implies that $\alpha_b^a \cdot \alpha_g^g = \alpha_d^c \cdot \alpha_g^g$ for all $g \in G$ such that $g \geqslant x$ in G. If $g \geqslant b$ and $g \geqslant d$ then the definition of the semigroup operation in $\mathscr{B}(G)$ (resp., in $\mathscr{B}^+(G)$) implies that $\alpha_b^a \cdot \alpha_g^g = \alpha_g^{g\cdot b^{-1}\cdot a}$ and $\alpha_d^c \cdot \alpha_g^g = \alpha_g^{g\cdot d^{-1}\cdot c}$, and since G is a group we get that $b^{-1} \cdot a = d^{-1} \cdot c$.

Conversely, suppose that α_b^a and α_d^c are elements of the semigroup $\mathscr{B}(G)$ (resp., $\mathscr{B}^+(G)$) such that $b^{-1} \cdot a = d^{-1} \cdot c$. Then for any element $g \in G$ such that $g \geqslant b$ and $g \geqslant d$ in G we have $\alpha_b^a \cdot \alpha_g^g = \alpha_g^{g\cdot b^{-1}\cdot a}$ and $\alpha_d^c \cdot \alpha_g^g = \alpha_g^{g\cdot d^{-1}\cdot c}$, and hence, since $b^{-1} \cdot a = d^{-1} \cdot c$, we get that $\alpha_b^a \mathfrak{C}_{mg} \alpha_d^c$. Therefore, $\alpha_b^a \mathfrak{C}_{mg} \alpha_d^c$ in $\mathscr{B}(G)$ (resp., in $\mathscr{B}^+(G)$) if and only if $b^{-1} \cdot a = d^{-1} \cdot c$.

We determine a map $\mathfrak{f} \colon \mathscr{B}(G) \to G$ (resp., $\mathfrak{f} \colon \mathscr{B}^+(G) \to G$) by the formula $(\alpha_b^a)\mathfrak{f} = b^{-1} \cdot a$, for $a, b \in G$. Then we have

$$(\alpha_b^a \cdot \alpha_d^c)\mathfrak{f} = \begin{cases} (\alpha_d^{c\cdot b^{-1}\cdot a})\mathfrak{f}, & \text{if } b < c; \\ (\alpha_d^a)\mathfrak{f}, & \text{if } b = c; \\ (\alpha_{b\cdot c^{-1}\cdot d}^a)\mathfrak{f}, & \text{if } b > c, \end{cases} = \begin{cases} d^{-1} \cdot c \cdot b^{-1} \cdot a, & \text{if } b < c; \\ d^{-1} \cdot a, & \text{if } b = c; \\ (b \cdot c^{-1} \cdot d)^{-1} \cdot a, & \text{if } b > c, \end{cases} =$$

$$= \begin{cases} d^{-1} \cdot c \cdot b^{-1} \cdot a, & \text{if } b < c; \\ d^{-1} \cdot c \cdot b^{-1} \cdot a, & \text{if } b = c; \\ d^{-1} \cdot c \cdot b^{-1} \cdot a, & \text{if } b > c, \end{cases} = d^{-1} \cdot c \cdot b^{-1} \cdot a = (\alpha_d^c)\mathfrak{f} \cdot (\alpha_b^a)\mathfrak{f},$$

for $a,b,c,d\in G$. This completes the proof of the theorem. □

Hölder's theorem and Theorem 3.10 imply the following.

Theorem 3.11. *Let G be an archimedean linearly ordered group and let $\mathfrak{C}_{mg}$ be the least group congruence on the semigroup $\mathscr{B}(G)$ (resp., $\mathscr{B}^+(G)$). Then the quotient semigroup $\mathscr{B}(G)/\mathfrak{C}_{mg}$ (resp., $\mathscr{B}^+(G)/\mathfrak{C}_{mg}$) is isomorphic to the group G.*

Theorems 3.2, 3.3 and 3.11 imply the following.

Corollary 3.12. *Let G be an archimedean linearly ordered group and let $\mathfrak{C}_{mg}$ be the least group congruence on the semigroup $\mathscr{B}(G)$ (resp., $\mathscr{B}^+(G)$). Then every non-isomorphic image of the semigroup $\mathscr{B}(G)$ (resp., $\mathscr{B}^+(G)$) is isomorphic to some homomorphic image of the group G.*

Theorem 3.13. *Let G be a linearly ordered d-group and let $\mathfrak{C}_{mg}$ be the least group congruence on the semigroup $\overline{\mathscr{B}}(G)$ (resp., $\overline{\mathscr{B}}^+(G)$). Then the quotient semigroup $\overline{\mathscr{B}}(G)/\mathfrak{C}_{mg}$ (resp., $\overline{\mathscr{B}}^+(G)/\mathfrak{C}_{mg}$) is antiisomorphic to the group G.*

Proof. Similar arguments as in the proofs of Theorem 3.10 and Proposition 3.7 show that the following assertions are equivalent:

(*i*) $\alpha_b^a\mathfrak{C}_{mg}\alpha_d^c$ in $\overline{\mathscr{B}}(G)$ (resp., in $\overline{\mathscr{B}}^+(G)$);
(*ii*) $\alpha_b^a\mathfrak{C}_{mg}\mathring{\alpha}_d^c$ in $\overline{\mathscr{B}}(G)$ (resp., in $\overline{\mathscr{B}}^+(G)$);
(*iii*) $\mathring{\alpha}_b^a\mathfrak{C}_{mg}\mathring{\alpha}_d^c$ in $\overline{\mathscr{B}}(G)$ (resp., in $\overline{\mathscr{B}}^+(G)$);
(*iv*) $b^{-1}\cdot a=d^{-1}\cdot c$.

We determine a map $\mathfrak{f}\colon \mathscr{B}(G)\to G$ (resp., $\mathfrak{f}\colon \mathscr{B}^+(G)\to G$) by the formulae $(\alpha_b^a)\mathfrak{f}=b^{-1}\cdot a$ and $(\mathring{\alpha}_b^a)\mathfrak{f}=b^{-1}\cdot a$, for $a,b\in G$. Then we have

$$(\alpha_b^a\cdot\alpha_d^c)\mathfrak{f}=(\alpha_d^c)\mathfrak{f}\cdot(\alpha_b^a)\mathfrak{f},$$

$$(\mathring{\alpha}_b^a\cdot\mathring{\alpha}_d^c)\mathfrak{f}=\begin{cases}(\mathring{\alpha}_d^{c\cdot b^{-1}\cdot a})\mathfrak{f}, & \text{if } b<c;\\ (\mathring{\alpha}_d^{a})\mathfrak{f}, & \text{if } b=c;\\ (\mathring{\alpha}_{b\cdot c^{-1}\cdot d}^{a})\mathfrak{f}, & \text{if } b>c,\end{cases}=\begin{cases}d^{-1}\cdot c\cdot b^{-1}\cdot a, & \text{if } b<c;\\ d^{-1}\cdot a, & \text{if } b=c;\\ (b\cdot c^{-1}\cdot d)^{-1}\cdot a, & \text{if } b>c,\end{cases}=$$
$$=\begin{cases}d^{-1}\cdot c\cdot b^{-1}\cdot a, & \text{if } b<c;\\ d^{-1}\cdot c\cdot b^{-1}\cdot a, & \text{if } b=c;\\ d^{-1}\cdot c\cdot b^{-1}\cdot a, & \text{if } b>c,\end{cases}=d^{-1}\cdot c\cdot b^{-1}\cdot a=(\mathring{\alpha}_d^c)\mathfrak{f}\cdot(\mathring{\alpha}_b^a)\mathfrak{f},$$

$$(\alpha_b^a\cdot\mathring{\alpha}_d^c)\mathfrak{f}=\begin{cases}(\mathring{\alpha}_d^{c\cdot b^{-1}\cdot a})\mathfrak{f}, & \text{if } b<c;\\ (\mathring{\alpha}_d^{a})\mathfrak{f}, & \text{if } b=c;\\ (\alpha_{b\cdot c^{-1}\cdot d}^{a})\mathfrak{f}, & \text{if } b>c,\end{cases}=\begin{cases}d^{-1}\cdot c\cdot b^{-1}\cdot a, & \text{if } b<c;\\ d^{-1}\cdot a, & \text{if } b=c;\\ (b\cdot c^{-1}\cdot d)^{-1}\cdot a, & \text{if } b>c,\end{cases}=$$
$$=\begin{cases}d^{-1}\cdot c\cdot b^{-1}\cdot a, & \text{if } b<c;\\ d^{-1}\cdot c\cdot b^{-1}\cdot a, & \text{if } b=c;\\ d^{-1}\cdot c\cdot b^{-1}\cdot a, & \text{if } b>c,\end{cases}=d^{-1}\cdot c\cdot b^{-1}\cdot a=(\alpha_d^c)\mathfrak{f}\cdot(\mathring{\alpha}_b^a)\mathfrak{f},$$

$$(\mathring{\alpha}_b^a \cdot \alpha_d^c)\mathfrak{f} = \begin{cases} (\alpha_d^{c\cdot b^{-1}\cdot a})\mathfrak{f}, & \text{if } b < c; \\ (\mathring{\alpha}_d^a)\mathfrak{f}, & \text{if } b = c; \\ (\mathring{\alpha}_{b\cdot c^{-1}\cdot d}^a)\mathfrak{f}, & \text{if } b > c, \end{cases} = \begin{cases} d^{-1}\cdot c\cdot b^{-1}\cdot a, & \text{if } b < c; \\ d^{-1}\cdot a, & \text{if } b = c; \\ (b\cdot c^{-1}\cdot d)^{-1}\cdot a, & \text{if } b > c, \end{cases} =$$

$$= \begin{cases} d^{-1}\cdot c\cdot b^{-1}\cdot a, & \text{if } b < c; \\ d^{-1}\cdot c\cdot b^{-1}\cdot a, & \text{if } b = c; \\ d^{-1}\cdot c\cdot b^{-1}\cdot a, & \text{if } b > c, \end{cases} = d^{-1}\cdot c\cdot b^{-1}\cdot a = (\mathring{\alpha}_d^c)\mathfrak{f}\cdot(\alpha_b^a)\mathfrak{f},$$

for $a, b, c, d \in G$. This completes the proof of the theorem. □

Hölder's theorem and Theorem 3.13 imply the following.

Theorem 3.14. *Let G be an archimedean linearly ordered d-group and let $\mathfrak{C}_{mg}$ be the least group congruence on the semigroup $\overline{\mathscr{B}}(G)$ (resp., $\overline{\mathscr{B}}^+(G)$). Then the quotient semigroup $\overline{\mathscr{B}}(G)/\mathfrak{C}_{mg}$ (resp., $\overline{\mathscr{B}}^+(G))/\mathfrak{C}_{mg}$) is isomorphic to the group G.*

Theorems 3.8 and 3.14 imply the following.

Corollary 3.15. *Let G be an archimedean linearly ordered d-group, T be a semigroup and $h\colon \overline{\mathscr{B}}(G) \to T$ $\big($resp., $h\colon \overline{\mathscr{B}}^+(G) \to T\big)$ be a homomorphism. Then only one of the following conditions holds:*

(i) h is a monomorphism;

(ii) the image $\big(\overline{\mathscr{B}}(G)\big)h$ $\big($resp., $\big(\overline{\mathscr{B}}^+(G)\big)h\big)$ is isomorphic to some homomorphic image of the group G;

(iii) the image $\big(\overline{\mathscr{B}}(G)\big)h$ $\big($resp., $\big(\overline{\mathscr{B}}^+(G)\big)h\big)$ is isomorphic to the semigroup $\mathscr{B}(G)$ (resp., $\mathscr{B}^+(G)$).

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