

## On estimation in multilevel models with block circular symmetric covariance structure

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ABSTRACT. In this article we consider a multilevel model with block circular symmetric covariance structure. Maximum likelihood estimation of the parameters of this model is discussed. We show that explicit maximum likelihood estimators of variance components exist under certain restrictions on the parameter space.

### 1. Introduction

Very often data arise in natural hierarchies, for example, children are nested within families, students are grouped within classrooms and employees are clustered within workplaces. The existence of such data hierarchies is not accidental and should be accounted for when conducting a statistical analysis. Multilevel models (Goldstein, 2003) refer to a class of multivariate statistical models developed for the analysis of hierarchically structured data. One can note the existence of many other names for these models, including, hierarchical linear model, random coefficients model, and hierarchical mixed linear model. To a certain extent, the emergence of names is due to the statistical properties of different modeling strategies used to analyze multilevel data.

The distinguishing feature of hierarchical data is that observations within a corresponding group (hierarchy) are usually more similar than observations from different groups (hierarchies). Moreover, since hierarchical structures violate the independence assumption, techniques for dealing with this problem have to be developed.

In this article we consider the problem of estimation in balanced multilevel models with a block circular symmetric covariance structure. In the framework of multilevel models, this structure has been utilized in many

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applications, to describe the situations with a spatial circular layout on one factor and an exchangeable feature on another factor. Estimation in linear models with patterned covariance matrices has got a lot of attention. Olkin and Press (1969) and Olkin (1972) provided maximum likelihood estimators (MLEs) for the parameters in a circular symmetric model, but without patterned blocks. Szatrowski (1980) and Szatrowski and Miller (1980) discussed the multivariate normal model with a linear covariance structure and gave a necessary and sufficient condition of explicit MLEs for both the mean and covariance matrices. Marin and Dhorne (2002, 2003) gave a necessary and sufficient condition of an optimal unbiased estimator for a statistical model with linear Toeplitz covariance structure.

The aim of this article is to extend models that are block circular symmetric (Olkin, 1972) to patterned blocks, when both circular symmetry and exchangeability are present in data. The ultimate goal is to derive explicit MLEs of all parameters. This can be achieved by first estimating the eigenvalues of the covariance matrix and thereafter imposing conditions on the parameter space. In this article explicit estimators of the eigenvalues are presented as well as the number of constraints needed to obtain explicit estimators of the original parameters. However, the derivations of these estimators will be postponed to another publication, since there are many scenarios which should be investigated.

The organization of the article is as follows. Section 2 introduces the basic model and notation, as well as some results concerning commutativity and spectral decomposition of the building blocks of the covariance matrix. In Section 3 the main results concerning the spectral properties of the covariance matrix are presented. In Section 4 explicit MLEs of eigenvalues of the patterned covariance matrices are derived and the estimability of (co)variance components is discussed in terms of model reparameterization (restrictions). An illustrative example is given in Section 5.

## 2. Preliminaries

In this section a model with block circular covariance structure is introduced and spectral properties of the matrices corresponding to such a dependence structure are given. Let us consider the following mixed linear model

$$\mathbf{y} = \mu \mathbf{1}_p + \mathbf{Z}_1 \boldsymbol{\gamma}_1 + \mathbf{Z}_2 \boldsymbol{\gamma}_2 + \mathbf{I}_p \boldsymbol{\epsilon}, \quad (2.1)$$

where  $\mu$  is an unknown constant parameter,  $\boldsymbol{\gamma}_1$ ,  $\boldsymbol{\gamma}_2$  and  $\boldsymbol{\epsilon}$  are independently normally distributed random variables with zero means and variance-covariance matrices  $\boldsymbol{\Sigma}_1$ ,  $\boldsymbol{\Sigma}_2$ , and  $\sigma^2 \mathbf{I}_p$ , respectively. Here  $\mathbf{Z}_1 = \mathbf{I}_{n_2} \otimes \mathbf{1}_{n_1}$ ,  $\mathbf{Z}_2 = \mathbf{I}_{n_2} \otimes \mathbf{I}_{n_1}$ ,  $\mathbf{1}_s$  is a column vector of size  $s$  with all elements equal to one and  $\mathbf{I}_s$  is the identity matrix of order  $s$ ,  $p = n_1 n_2$ . The symbol  $\otimes$  denotes

the Kronecker product. Thus,

$$\begin{aligned}\mathbf{y} &\sim N_p(\mu\mathbf{1}_p, \boldsymbol{\Sigma}), \\ \boldsymbol{\Sigma} &= \mathbf{Z}_1\boldsymbol{\Sigma}_1\mathbf{Z}'_1 + \boldsymbol{\Sigma}_2 + \sigma^2\mathbf{I}_p.\end{aligned}\quad (2.2)$$

In our model we suppose that the covariance matrix  $\boldsymbol{\Sigma}_1 : n_2 \times n_2$  has the following structure (compound symmetry):

$$\boldsymbol{\Sigma}_1 = a\mathbf{I}_{n_2} + b(\mathbf{J}_{n_2} - \mathbf{I}_{n_2}), \quad (2.3)$$

where  $a$  and  $b$  are unknown parameters, and  $\mathbf{J}_{n_2}$  is a matrix of size  $n_2 \times n_2$  with all elements equal to one. The covariance matrix  $\boldsymbol{\Sigma}_2 : p \times p$  has a block compound symmetric pattern with a symmetric circular Toeplitz (SC-Toeplitz) matrix in each block, i.e.

$$\boldsymbol{\Sigma}_2 = \mathbf{I}_{n_2} \otimes \boldsymbol{\Sigma}^{(1)} + (\mathbf{J}_{n_2} - \mathbf{I}_{n_2}) \otimes \boldsymbol{\Sigma}^{(2)}, \quad (2.4)$$

where the SC-Toeplitz matrix  $\boldsymbol{\Sigma}^{(h)} = (\sigma_{ij}^{(h)})$  depends on  $[n_1/2] + 1$  parameters, the symbol  $[\cdot]$  stands for the integer part, and for  $i, j = 1, \dots, n_1$ ,  $h = 1, 2$ ,

$$\sigma_{ij}^{(h)} = \begin{cases} \tau_{|j-i|+(h-1)([n_1/2]+1)}, & \text{if } |j-i| \leq [\frac{n_1}{2}], \\ \tau_{n_1-|j-i|+(h-1)([n_1/2]+1)}, & \text{otherwise,} \end{cases} \quad (2.5)$$

and  $\tau'_q$ s are unknown parameters,  $q = 0, \dots, 2[n_1/2] + 1$ . For example, when  $n_1 = 4$ ,

$$\boldsymbol{\Sigma}^{(1)} = \begin{pmatrix} \tau_0 & \tau_1 & \tau_2 & \tau_1 \\ \tau_1 & \tau_0 & \tau_1 & \tau_2 \\ \tau_2 & \tau_1 & \tau_0 & \tau_1 \\ \tau_1 & \tau_2 & \tau_1 & \tau_0 \end{pmatrix}, \quad \boldsymbol{\Sigma}^{(2)} = \begin{pmatrix} \tau_3 & \tau_4 & \tau_5 & \tau_4 \\ \tau_4 & \tau_3 & \tau_4 & \tau_5 \\ \tau_5 & \tau_4 & \tau_3 & \tau_4 \\ \tau_4 & \tau_5 & \tau_4 & \tau_3 \end{pmatrix}.$$

The covariance matrix  $\boldsymbol{\Sigma}$  given in (2.2) is a sum of three symmetric matrices  $\mathbf{Z}_1\boldsymbol{\Sigma}_1\mathbf{Z}'_1$ ,  $\boldsymbol{\Sigma}_2$  and  $\sigma^2\mathbf{I}_p$ , which commute (see Lemma 2.1), and hence can be simultaneously diagonalized. This property will be utilized to obtain the eigenvalues of  $\boldsymbol{\Sigma}$ , which in turn can be used to derive explicit maximum likelihood estimators of the unknown parameters.

**Lemma 2.1.** *The matrices  $\mathbf{Z}_1\boldsymbol{\Sigma}_1\mathbf{Z}'_1$  and  $\boldsymbol{\Sigma}_2$  are commuting normal matrices.*

*Proof.* Since  $\mathbf{Z}_1\boldsymbol{\Sigma}_1\mathbf{Z}'_1$  and  $\boldsymbol{\Sigma}_2$  both are symmetric they are also normal matrices. Due to the specific structure of  $\boldsymbol{\Sigma}_1$  given in (2.3), we first observe that

$$\mathbf{Z}_1\boldsymbol{\Sigma}_1\mathbf{Z}'_1 = a\mathbf{I}_{n_2} \otimes \mathbf{J}_{n_1} + b(\mathbf{J}_{n_2} - \mathbf{I}_{n_2}) \otimes \mathbf{J}_{n_1}. \quad (2.6)$$

Since both  $\boldsymbol{\Sigma}^{(1)}$  and  $\boldsymbol{\Sigma}^{(2)}$  which define  $\boldsymbol{\Sigma}_2$  in (2.4) commute with  $\mathbf{J}_{n_1}$ , it is straightforward to verify that  $\mathbf{Z}_1\boldsymbol{\Sigma}_1\mathbf{Z}'_1\boldsymbol{\Sigma}_2 = \boldsymbol{\Sigma}_2\mathbf{Z}_1\boldsymbol{\Sigma}_1\mathbf{Z}'_1$ .  $\square$

In the next theorem we give the eigenvalues and eigenvectors of an SC-Toeplitz matrix, since they are important for the subsequent inference.

**Theorem 2.2.** *Let  $\mathbf{T} = \{t_{ij}\} : n \times n$  be an SC-Toeplitz matrix, i.e.*

$$t_{ij} = \begin{cases} t_{|j-i|}, & \text{if } |j-i| \leq \lfloor \frac{n}{2} \rfloor, \\ t_{n-|j-i|}, & \text{otherwise.} \end{cases} \quad (2.7)$$

The eigenvalues of  $\mathbf{T}$  are given by

$$\lambda_k = \sum_{j=0}^{n-1} t_j \cos\left(\frac{2\pi}{n}(k-1)(n-j)\right), \quad k = 1, \dots, n.$$

The corresponding eigenvectors  $\mathbf{w}^1, \dots, \mathbf{w}^n$  are defined through

$$\begin{aligned} w_j^k = & \frac{1}{\sqrt{n}} \left( \cos\left(\frac{2\pi}{n}(j-1)(k-1)\right) \right. \\ & \left. + \sin\left(\frac{2\pi}{n}(j-1)(k-1)\right) \right), \quad j, k = 1, \dots, n. \end{aligned} \quad (2.8)$$

*Proof.* For derivation of the eigenvalues and eigenvectors we refer readers to Basilevsky (1983) and Olkin and Press (1969).  $\square$

**Corollary 2.3.** *The matrix  $\mathbf{T}$  defined in (2.7) has the following properties.*

(i)  $t_{ij} = t_{ij'}$ ,  $j' = n - j + 2$ ,  $j = 2, \dots, n$ .

(ii)  $\lambda_i = \lambda_{n-i+2}$ ,  $i = 2, \dots, n$ .

(iii) The eigenvectors of  $\mathbf{T}$  defined in (2.8) are independent of the elements of  $\mathbf{T}$ .

(iv) Let  $\mathbf{W} = (\mathbf{w}^1, \dots, \mathbf{w}^n)$ , with  $\mathbf{w}^k = (w_1^k, \dots, w_n^k)'$ ,  $k = 1, \dots, n$ . Then  $\mathbf{W}\mathbf{W}' = \mathbf{I}_n$  and  $\mathbf{1}'_n \mathbf{W} = (\sqrt{n}, 0, \dots, 0)$ .

Eigenvalues of the matrices  $\mathbf{Z}_1 \boldsymbol{\Sigma}_1 \mathbf{Z}'_1$  and  $\boldsymbol{\Sigma}_2$  together with the corresponding eigenvectors will be presented in the following theorems.

**Theorem 2.4.** *A symmetric matrix  $\mathbf{Z}_1 \boldsymbol{\Sigma}_1 \mathbf{Z}'_1 : n_1 n_2 \times n_1 n_2$  of the form given in (2.6) has three distinct eigenvalues:*

$$\begin{aligned} \lambda_1 &= n_1(a-b) + n_2 n_1 b && \text{with multiplicity } 1, \\ \lambda_2 &= n_1(a-b) && \text{with multiplicity } (n_2 - 1), \\ \lambda_3 &= 0 && \text{with multiplicity } n_2(n_1 - 1). \end{aligned}$$

The set  $(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{n_2}, \mathbf{v}_{n_2+1}, \dots, \mathbf{v}_{n_1 n_2})$  comprises the eigenvectors of  $\mathbf{Z}_1 \boldsymbol{\Sigma}_1 \mathbf{Z}'_1$  which are of the form

$$\mathbf{v}_h = \mathbf{w}_2^{i_2} \otimes \mathbf{w}_1^{i_1}, \quad (2.9)$$

where elements of the vectors  $\mathbf{w}_k^{i_k}$  are defined by (2.8),  $i_k = 1, \dots, n_k$ ,  $h = 1, \dots, n_1 n_2$ , and  $k = 1, 2$ . Moreover, the eigenvector corresponding to  $\lambda_1$  is

$\mathbf{v}_1 = \mathbf{w}_2^1 \otimes \mathbf{w}_1^1 = (n_1 n_2)^{-1/2} \mathbf{1}_{n_2} \otimes \mathbf{1}_{n_1}$ ; the eigenvectors corresponding to  $\lambda_2$  are  $\mathbf{v}_h = \mathbf{w}_2^{h_2} \otimes \mathbf{w}_1^1 = \mathbf{w}_2^{h_2} \otimes n_1^{-1/2} \mathbf{1}_{n_1}$ ,  $h_2 = 2, \dots, n_2$ ; and the eigenvectors corresponding to  $\lambda_3$  are  $\mathbf{v}_h = \mathbf{w}_2^{h_2} \otimes \mathbf{w}_1^{h_1}$ , where  $h_2 = 1, \dots, n_2$ ,  $h_1 = 2, \dots, n_1$ .

*Proof.* Let us define the following orthogonal matrix

$$\mathbf{\Gamma} = \mathbf{\Gamma}_2 \otimes \mathbf{\Gamma}_1,$$

where the matrix  $\mathbf{\Gamma}_k$  comprises eigenvectors of an SC-Toeplitz matrix of order  $n_k$  which are specified by (2.8), i.e.  $\mathbf{\Gamma}_k = (\mathbf{w}_k^1, \dots, \mathbf{w}_k^{n_k})$ ,  $k = 1, 2$ . Observing that the first column in  $\mathbf{\Gamma}_k$  is  $\mathbf{w}_k^1 = n_k^{-1/2} \mathbf{1}_{n_k}$ , it follows that

$$\mathbf{\Gamma}'_k \mathbf{J}_{n_k} \mathbf{\Gamma}_k = \begin{pmatrix} n_k & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{n_k-1} \end{pmatrix},$$

where  $\mathbf{0}_{n_k-1} : (n_k - 1) \times (n_k - 1)$  is a matrix with all elements equal to zero. Using the expression for  $\mathbf{Z}_1 \mathbf{\Sigma}_1 \mathbf{Z}'_1$  given in (2.6) we obtain

$$\begin{aligned} \mathbf{\Gamma}' \mathbf{Z}_1 \mathbf{\Sigma}_1 \mathbf{Z}'_1 \mathbf{\Gamma} &= (a - b) \mathbf{I}_{n_2} \otimes \begin{pmatrix} n_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{n_1-1} \end{pmatrix} \\ &+ b \begin{pmatrix} n_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{n_2-1} \end{pmatrix} \otimes \begin{pmatrix} n_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{n_1-1} \end{pmatrix}. \end{aligned}$$

This is a diagonal matrix and therefore the eigenvalues follow immediately. Due to the structure of the matrix  $\mathbf{Z}_1 \mathbf{\Sigma}_1 \mathbf{Z}'_1$  the eigenvectors defined in (2.9) can be easily verified.  $\square$

In the next theorem the eigenvalues and the eigenvectors of the matrix  $\mathbf{\Sigma}_2$  are presented, using the block structure of  $\mathbf{\Sigma}_2$ .

**Theorem 2.5.** *Let  $\mathbf{\Sigma}_2$  follow the structure specified in (2.4), and let  $\lambda_1^{(i)}, \dots, \lambda_{n_1}^{(i)}$  be eigenvalues given in Theorem 2.2 of the SC-Toeplitz matrix  $\mathbf{\Sigma}^{(i)}$  in (2.5),  $i = 1, 2$ . Then,  $\mathbf{\Sigma}_2$  has eigenvalues*

$$\lambda_{1h} = \lambda_h^{(1)} + (n_2 - 1) \lambda_h^{(2)}, \quad (2.10)$$

$$\lambda_{2h} = \lambda_h^{(1)} - \lambda_h^{(2)}, \quad (2.11)$$

where  $h = 1, \dots, n_1$ .

Furthermore, if  $n_1$  is odd, the multiplicity of  $\lambda_{i1}$  is  $(n_2 - 1)^{i-1}$ , and the eigenvalues  $\lambda_{i2}, \dots, \lambda_{in_1}$  are of multiplicity  $2(n_2 - 1)^{i-1}$ ,  $i = 1, 2$ . If  $n_1$  is even, the multiplicities of both  $\lambda_{i1}$  and  $\lambda_{i \frac{n_1}{2}}$  are  $(n_2 - 1)^{i-1}$ , and other eigenvalues  $\lambda_{i2}, \dots, \lambda_{in_1}$  are of multiplicity  $2(n_2 - 1)^{i-1}$ ,  $i = 1, 2$ . Thus, the number of distinct eigenvalues for  $\mathbf{\Sigma}_2$  is  $2(\lfloor n_1/2 \rfloor + 1)$ .

The set  $(\mathbf{v}_1^1, \dots, \mathbf{v}_1^{n_1}, \mathbf{v}_2^1, \dots, \mathbf{v}_2^{n_1(n_2-1)})$  comprises the eigenvectors of  $\Sigma_2$  which are of the following form:

$$\begin{aligned}\mathbf{v}_1^i &= \mathbf{w}_2^1 \otimes \mathbf{w}_1^{h_1} = n_2^{-1/2} \mathbf{1}_{n_2} \otimes \mathbf{w}_1^{h_1}, \quad i, h_1 = 1, \dots, n_1, \\ \mathbf{v}_2^j &= \mathbf{w}_2^{h_2} \otimes \mathbf{w}_1^{h_1}, \quad j=1, \dots, n_1(n_2-1), h_2=2, \dots, n_2,\end{aligned}$$

where elements of the vectors  $\mathbf{w}_k^{i_k}$  are defined in (2.8),  $i_k = 1, \dots, n_k$  and  $k = 1, 2$ .

*Proof.* Let us define the following orthogonal matrix

$$\mathbf{\Gamma} = \mathbf{\Gamma}_2 \otimes \mathbf{\Gamma}_1 = (\mathbf{v}_1^1, \dots, \mathbf{v}_1^{n_1}, \mathbf{v}_2^1, \dots, \mathbf{v}_2^{n_1(n_2-1)}),$$

where the matrix  $\mathbf{\Gamma}_k$  consists of eigenvectors of an SC-Toeplitz matrix of order  $n_k$  which are specified in (2.8), i.e.  $\mathbf{\Gamma}_k = (\mathbf{w}_k^1, \dots, \mathbf{w}_k^{n_k})$ ,  $k = 1, 2$ , and

$$\mathbf{\Gamma}'_1 \Sigma^{(i)} \mathbf{\Gamma}_1 = \mathbf{\Lambda}^{(i)} = \text{diag}(\lambda_1^{(i)}, \lambda_2^{(i)}, \dots, \lambda_{n_1}^{(i)}), \quad i = 1, 2. \quad (2.12)$$

Observing that the first column in  $\mathbf{\Gamma}_k$  is  $\mathbf{w}_k^1 = n_k^{-1/2} \mathbf{1}_{n_k}$ ,  $k = 1, 2$ , it follows that

$$\mathbf{\Gamma}'_k \mathbf{J}_{n_k} \mathbf{\Gamma}_k = \begin{pmatrix} n_k & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{n_k-1} \end{pmatrix}.$$

Then

$$\begin{aligned}\mathbf{\Gamma}' \Sigma_2 \mathbf{\Gamma} &= (\mathbf{\Gamma}'_2 \otimes \mathbf{\Gamma}'_1) \left[ \mathbf{I}_{n_2} \otimes (\Sigma^{(1)} - \Sigma^{(2)}) + \mathbf{J}_{n_2} \otimes \Sigma^{(2)} \right] (\mathbf{\Gamma}_2 \otimes \mathbf{\Gamma}_1) \\ &= \mathbf{I}_{n_2} \otimes (\mathbf{\Lambda}^{(1)} - \mathbf{\Lambda}^{(2)}) + \begin{pmatrix} n_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{n_2-1} \end{pmatrix} \otimes \mathbf{\Lambda}^{(2)}.\end{aligned} \quad (2.13)$$

From the expression in (2.13) the eigenvalues with corresponding multiplicities can be obtained.

To verify that  $\mathbf{v}_1^h$  is an eigenvector of  $\Sigma_2$  corresponding to the eigenvalue  $\lambda_{1h}$ ,  $i = 1, 2$  one should check that  $\Sigma_2 \mathbf{v}_1^h = \lambda_{1h} \mathbf{v}_1^h$ ,  $h = 1, \dots, n_1$ . Indeed, for  $\mathbf{v}_1^h = \mathbf{w}_2^1 \otimes \mathbf{w}_1^{h_1} = n_2^{-1/2} \mathbf{1}_{n_2} \otimes \mathbf{w}_1^{h_1}$ , we have

$$\begin{aligned}\Sigma_2 \mathbf{v}_1^h &= \left( \mathbf{I}_{n_2} \otimes \Sigma^{(1)} + (\mathbf{J}_{n_2} - \mathbf{I}_{n_2}) \otimes \Sigma^{(2)} \right) (n_2^{-1/2} \mathbf{1}_{n_2} \otimes \mathbf{w}_1^{h_1}) \\ &= n_2^{-1/2} \mathbf{1}_{n_2} \otimes \Sigma^{(1)} \mathbf{w}_1^{h_1} + n_2^{-1/2} (n_2 - 1) \mathbf{1}_{n_2} \otimes \Sigma^{(2)} \mathbf{w}_1^{h_1} \\ &= n_2^{-1/2} \mathbf{1}_{n_2} \otimes (\lambda_h^{(1)} \mathbf{w}_1^{h_1}) + n_2^{-1/2} (n_2 - 1) \mathbf{1}_{n_2} \otimes (\lambda_h^{(2)} \mathbf{w}_1^{h_1}) \\ &= (\lambda_h^{(1)} + (n_2 - 1) \lambda_h^{(2)}) (n_2^{-1/2} \mathbf{1}_{n_2} \otimes \mathbf{w}_1^{h_1}) \\ &= \lambda_{1h} \mathbf{v}_1^h,\end{aligned}$$

where  $h_1 = 1, \dots, n_1$ . Similarly, one can verify that  $\Sigma_2 \mathbf{v}_2^h = \lambda_{2h} \mathbf{v}_2^h$ ,  $h = 1, \dots, n_1(n_2 - 1)$ .  $\square$

An alternative formulation of the spectrum of  $\Sigma_2$  in Theorem 2.5 is given in the following corollary.

**Corollary 2.6.** *Let  $\boldsymbol{\tau}$  and  $\boldsymbol{\lambda}$  be the vectors representing distinct elements and distinct eigenvalues of  $\boldsymbol{\Sigma}_2$ , given in (2.5) and Theorem 2.5, respectively. Then*

$$\boldsymbol{\lambda} = \mathbf{B}_2 \boldsymbol{\tau},$$

where the nonsingular coefficient matrix  $\mathbf{B}_2$  has the following form:

$$\mathbf{B}_2 = \begin{pmatrix} \mathbf{A} & (n_2 - 1)\mathbf{A} \\ \mathbf{A} & -\mathbf{A} \end{pmatrix},$$

where  $\mathbf{A} = \{a_{ij}\}$  is a  $(\lfloor \frac{n_1}{2} \rfloor + 1) \times (\lfloor \frac{n_1}{2} \rfloor + 1)$  matrix and

$$a_{ij} = 2^{\mathbf{1}_{\{1 < j < \lfloor n_1/2 \rfloor + 1\}}} \cos(2\pi(i-1)(n_1-j+1)/n_1), \quad i, j = 1, \dots, n_1,$$

and  $\mathbf{1}_{\{\cdot\}}$  is the indicator function.

Moreover, by inverting  $\mathbf{B}_2$ ,  $\boldsymbol{\tau}$  can be expressed as follows:

$$\boldsymbol{\tau} = \mathbf{B}_2^{-1} \boldsymbol{\lambda},$$

where, since  $\mathbf{A}^2 = n_1 \mathbf{I}_{n_1}$ ,

$$\mathbf{B}_2^{-1} = \frac{1}{n_1 n_2} \begin{pmatrix} \mathbf{A} & (n_2 - 1)\mathbf{A} \\ \mathbf{A} & -\mathbf{A} \end{pmatrix}.$$

### 3. Spectral properties of $\boldsymbol{\Sigma}$

In this section we will present the spectral properties of the covariance matrix  $\boldsymbol{\Sigma}$  given in (2.2) which will be used when deriving MLEs for the variance-covariance parameters.

**Theorem 3.1.** *Let the matrix  $\boldsymbol{\Sigma}$  be defined as in (2.2). There exists an orthogonal matrix  $\mathbf{Q}$  such that  $\mathbf{Q}' \boldsymbol{\Sigma} \mathbf{Q} = \mathbf{D}$ , where  $\mathbf{D}$  is a diagonal matrix containing the eigenvalues of  $\boldsymbol{\Sigma}$ . Moreover,*

$$\mathbf{D} = \text{Diag}\{\mathbf{D}_1, \mathbf{I}_{n_2-1} \otimes \mathbf{D}_2\},$$

where

$$\mathbf{D}_1 = \text{diag}(\sigma^2 + n_1 a + n_1(n_2 - 1)b + \lambda_{11}, \sigma^2 + \lambda_{12}, \dots, \sigma^2 + \lambda_{1n_1}),$$

$$\mathbf{D}_2 = \text{diag}(\sigma^2 + n_1(a - b) + \lambda_{21}, \sigma^2 + \lambda_{22}, \dots, \sigma^2 + \lambda_{2n_1}),$$

and  $\lambda_{ih}$  are given by (2.10)-(2.11) in Theorem 2.5,  $i = 1, 2$ ,  $h = 1, \dots, n_1$ .

The matrix  $\mathbf{Q}$  whose columns are the orthonormal eigenvectors of  $\boldsymbol{\Sigma}$  equals

$$\mathbf{Q} = \mathbf{V}_{\mathbf{D}_1} \otimes \mathbf{V}_{\mathbf{D}_2}, \quad (3.1)$$

where

$$\mathbf{V}_{\mathbf{D}_1} = (\mathbf{w}_2^1, \dots, \mathbf{w}_2^{n_2}), \quad \mathbf{V}_{\mathbf{D}_2} = (\mathbf{w}_1^1, \dots, \mathbf{w}_1^{n_1}),$$

and the vectors  $\mathbf{w}_k^i$  are given by (2.8),  $k = 1, 2$ ,  $i = 1, \dots, n_k$ .

*Proof.* Recall that  $\Sigma$  is a sum of three symmetric commuting matrices  $\sigma^2 \mathbf{I}_p$ ,  $\mathbf{Z}_1 \Sigma_1 \mathbf{Z}_1'$  and  $\mathbf{Z}_2$ , and hence they can be simultaneously diagonalized. Define  $\mathbf{Q}$  as in (3.1) and then we get

$$\begin{aligned} \mathbf{Q}' \Sigma \mathbf{Q} &= \mathbf{Q}' (\sigma^2 \mathbf{I}_p + \mathbf{Z}_1 \Sigma_1 \mathbf{Z}_1' + \Sigma_2) \mathbf{Q} \\ &= \sigma^2 \mathbf{I}_p + \left[ \mathbf{I}_{n_2} \otimes \begin{pmatrix} (a-b)n_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{n_1-1} \end{pmatrix} \right. \\ &\quad \left. + \begin{pmatrix} n_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{n_2-1} \end{pmatrix} \otimes \begin{pmatrix} bn_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{n_1-1} \end{pmatrix} \right] \\ &\quad + \left[ \mathbf{I}_{n_2} \otimes (\Lambda^{(1)} - \Lambda^{(2)}) + \begin{pmatrix} n_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{n_2-1} \end{pmatrix} \otimes \Lambda^{(2)} \right], \end{aligned}$$

where  $\Lambda^{(i)}$ ,  $i = 1, 2$  are defined in (2.12). From the last expression, the distinct eigenvalues  $\eta_i$  of  $\Sigma$  with the corresponding multiplicities  $m_i$ ,  $i = 1, \dots, 2([n_1/2] + 1)$ , can be obtained directly.

Both of the two following tables present the spectrum of  $\Sigma$ . It is seen from Table 1 that there are four types of eigenvalues of  $\Sigma$ . There is a clear picture of how the results of Theorem 2.4 and Theorem 2.5 are connected and build up the eigenstructure of  $\Sigma$ .

TABLE 1. Let  $d_i$  be the eigenvalues of  $\Sigma$  given in (2.2) with the corresponding eigenvectors  $\mathbf{u}_i$  and multiplicities  $m_i$ ,  $i = 1, \dots, n_1 n_2$ . Here  $\mathbf{w}_k^1 = n_k^{-1/2} \mathbf{1}_{n_k}$ , and vectors  $\mathbf{w}_k^{h_k}$  are defined in (2.8),  $h_k = 2, \dots, n_k$ ,  $k = 1, 2$ . The eigenvalues  $\lambda_{kh_1}$  are defined in Theorem 2.5.

$d_i$	$m_i$	$\mathbf{u}_i$
$\sigma^2 + n_1(a-b) + n_2 n_1 b + \lambda_{11}$	1	$\mathbf{w}_2^1 \otimes \mathbf{w}_1^1$
$\sigma^2 + \lambda_{1h_1}$	1	$\mathbf{w}_2^1 \otimes \mathbf{w}_1^{h_1}$
$\sigma^2 + n_1(a-b) + \lambda_{21}$	$n_2 - 1$	$\mathbf{w}_2^{h_2} \otimes \mathbf{w}_1^1$
$\sigma^2 + \lambda_{2h_1}$	$n_2 - 1$	$\mathbf{w}_2^{h_2} \otimes \mathbf{w}_1^{h_1}$

However, taking into account that  $\lambda_{ks} = \lambda_{kr}$ , where  $r = n_1 - s + 2$ ,  $k = 1, 2$ ,  $s = 2, \dots, n_1$ , in Table 2 the distinct eigenvalues of  $\Sigma$  are presented.

The eigenvectors for  $\Sigma$  corresponding to the distinct eigenvalues provided in Table 2 can be easily verified. Thus, the proof of Theorem 3.1 is completed.  $\square$

Now let  $\boldsymbol{\theta}$  be a vector of the unknown (co)variance parameters in  $\Sigma$ , i.e.

$$\boldsymbol{\theta} = (\sigma^2, a, b, \tau_0, \dots, \tau_{2[n_1/2]+1})'. \quad (3.2)$$

TABLE 2. Distinct eigenvalues  $\eta_i$  of  $\Sigma$  given in (2.2) with corresponding multiplicities  $m_i$ .

$\eta_i$	$m_i$	
	odd $n_1$	even $n_1$
$\eta_1$	1	1
$\eta_2, \dots, \eta_{\lfloor \frac{n_1}{2} \rfloor + 1}$	2	2, $\eta_{\frac{n_1}{2}}$ has multiplicity 1.
$\eta_{\lfloor \frac{n_1}{2} \rfloor + 2}$	$n_2 - 1$	$n_2 - 1$
$\eta_{\lfloor \frac{n_1}{2} \rfloor + 3}, \dots, \eta_{2(\lfloor \frac{n_1}{2} \rfloor + 1)}$	$2(n_2 - 1)$	$2(n_2 - 1)$ , $\eta_{n_1+1}$ has multiplicity $n_2 - 1$ .

One important observation is that all the information about the unknown parameters in  $\Sigma$ , equivalently in  $\theta$ , is contained in the eigenvalues of  $\Sigma$  obtained in Theorem 3.1. The following theorem demonstrates the relationship between distinct eigenvalues of the covariance matrix  $\Sigma$  and unknown parameters in  $\theta$ .

**Theorem 3.2.** *Let  $\eta$  be the vector representing the distinct eigenvalues of  $\Sigma$  given in (2.2), and let the vector of unknown parameters  $\theta$  be specified by (3.2). Then*

$$\eta = L\theta, \quad (3.3)$$

where  $L = (B_1 : B_2)$ , the matrix  $B_2$  is given in Corollary 2.6,

$$B_1 = \begin{pmatrix} 1 & n_1 & n_1(n_2 - 1) \\ \mathbf{1}_{\lfloor n_1/2 \rfloor} & \mathbf{0}_{\lfloor n_1/2 \rfloor} & \mathbf{0}_{\lfloor n_1/2 \rfloor} \\ 1 & n_1 & -n_1 \\ \mathbf{1}_{\lfloor n_1/2 \rfloor} & \mathbf{0}_{\lfloor n_1/2 \rfloor} & \mathbf{0}_{\lfloor n_1/2 \rfloor} \end{pmatrix},$$

and  $\mathbf{1}_{\lfloor n_1/2 \rfloor}$  and  $\mathbf{0}_{\lfloor n_1/2 \rfloor}$  are vectors of length  $\lfloor \frac{n_1}{2} \rfloor$ .

#### 4. Maximum likelihood estimation

In this section MLEs for the eigenvalues of  $\Sigma$  given in (2.2) will be derived. Let  $\mathbf{y}_1, \dots, \mathbf{y}_n$  be a random sample from  $N_p(\mu\mathbf{1}_p, \Sigma)$ , and

$$\mathbf{Y} = (\mathbf{y}_1, \dots, \mathbf{y}_n) \sim N_{p,n}(\mu\mathbf{1}_p\mathbf{1}'_n, \Sigma, \mathbf{I}_n),$$

i.e.,  $\mathbf{Y}$  is matrix normally distributed which means that the columns of  $\mathbf{Y}$  are independent normally distributed  $p$ -vectors with an unknown covariance matrix  $\Sigma$  and expectation of  $\mathbf{Y}$  equals  $\mu\mathbf{1}_p\mathbf{1}'_n$ . It is equivalent to

$$\text{vec}\mathbf{Y} \sim N_{pn}(\mu\mathbf{1}_{pn}, \mathbf{I}_n \otimes \Sigma),$$

where  $\text{vec}(\mathbf{Y})$  denotes the vectorization of the matrix  $\mathbf{Y}$ .

The log-likelihood function is given by

$$\ln L(\mu, \Sigma) = c - \frac{1}{2} |\mathbf{I}_n \otimes \Sigma| - \frac{1}{2} [(\text{vec} \mathbf{Y} - \mu \mathbf{1}_{pn})' (\mathbf{I}_n \otimes \Sigma)^{-1} (\text{vec} \mathbf{Y} - \mu \mathbf{1}_{pn})],$$

where  $c = -\frac{1}{2} pn \ln(2\pi)$ .

First we consider the MLE of  $\mu$ . The partial derivative,

$$\frac{\partial \ln L}{\partial \mu} = \mathbf{1}'_{pn} (\mathbf{I}_n \otimes \Sigma)^{-1} \text{vec} \mathbf{Y} - \mathbf{1}'_{pn} (\mathbf{I}_n \otimes \Sigma)^{-1} \mathbf{1}_{pn} \mu,$$

yields the normal equation

$$\mathbf{1}'_{pn} (\mathbf{I}_n \otimes \Sigma)^{-1} \text{vec} \mathbf{Y} = \mathbf{1}'_{pn} (\mathbf{I}_n \otimes \Sigma)^{-1} \mathbf{1}_{pn} \mu,$$

and then the MLE of  $\mu$  is given by

$$\hat{\mu} = [\mathbf{1}'_{pn} (\mathbf{I}_n \otimes \Sigma^{-1}) \mathbf{1}_{pn}]^{-1} \mathbf{1}'_{pn} (\mathbf{I}_n \otimes \Sigma^{-1}) \text{vec} \mathbf{Y}, \quad (4.1)$$

if  $\Sigma$  is known. Since  $\mathbf{1}_p$  is an eigenvector of  $\Sigma$ , (4.1) becomes the least squares estimator (Szatrowski, 1980). Thus, the MLE for  $\mu$  equals

$$\hat{\mu} = (\mathbf{1}'_{pn} \mathbf{1}_{pn})^{-1} \mathbf{1}'_{pn} \text{vec} \mathbf{Y}. \quad (4.2)$$

Next, the eigenvalues of the covariance matrix  $\Sigma$  will be estimated. Since  $\Sigma$  is a symmetric matrix, it can be decomposed as  $\Sigma = \mathbf{Q} \mathbf{D}(\boldsymbol{\eta}) \mathbf{Q}'$ , where  $\mathbf{Q}$  is an orthogonal matrix whose  $p$  columns are orthonormal eigenvectors of  $\Sigma$ ,  $\mathbf{D}(\boldsymbol{\eta})$  is a  $p \times p$  diagonal matrix with the eigenvalues of  $\Sigma$  on the main diagonal. Recall, that  $\boldsymbol{\eta} = (\eta_1, \dots, \eta_{2(\lfloor \frac{n-1}{2} \rfloor + 1)})$  represent  $2(\lfloor \frac{n-1}{2} \rfloor + 1)$  distinct eigenvalues of  $\Sigma$  with the corresponding multiplicities  $m_i$  given in Table 2. Moreover,  $\mathbf{Q}$  given in (3.1) is independent of  $\mathbf{D}(\boldsymbol{\eta})$ .

When  $\mu$  is replaced by its MLE,  $\hat{\mu}$ , we have

$$\begin{aligned} L(\mu, \boldsymbol{\eta}) &\leq L(\hat{\mu}, \boldsymbol{\eta}) \\ &= (2\pi)^{-\frac{1}{2}pn} |\mathbf{D}(\boldsymbol{\eta})|^{-\frac{n}{2}} e^{-\frac{1}{2} \text{tr} \{ [\mathbf{D}(\boldsymbol{\eta})]^{-1} [\mathbf{Q}'(\mathbf{Y} - \hat{\mu} \mathbf{1}_p \mathbf{1}'_n)(\mathbf{Y} - \hat{\mu} \mathbf{1}_p \mathbf{1}'_n)' \mathbf{Q}] \}}, \end{aligned}$$

where  $\text{tr}$  denotes the trace. Now,

$$\begin{aligned} L(\hat{\mu}, \boldsymbol{\eta}) &\leq (2\pi)^{-\frac{1}{2}pn} |\mathbf{D}(\boldsymbol{\eta})|^{-\frac{n}{2}} e^{-\frac{1}{2} \text{tr} \{ [\mathbf{D}(\boldsymbol{\eta})]^{-1} \mathbf{H} \}} \\ &= (2\pi)^{-\frac{1}{2}pn} |\mathbf{D}(\boldsymbol{\eta})|^{-\frac{n}{2}} e^{-\frac{1}{2} \text{tr} \{ [\mathbf{D}(\boldsymbol{\eta})]^{-1} \mathbf{H}_d \}}, \end{aligned}$$

where

$$\mathbf{H} = \mathbf{Q}'(\mathbf{Y} - \hat{\mu} \mathbf{1}_p \mathbf{1}'_n)(\mathbf{Y} - \hat{\mu} \mathbf{1}_p \mathbf{1}'_n)' \mathbf{Q} \text{ and } \mathbf{H}_d = (h_j) = \text{diag}(\mathbf{H}). \quad (4.3)$$

Thus,

$$L(\hat{\mu}, \boldsymbol{\eta}) \leq (2\pi)^{-\frac{1}{2}pn} \prod_{i=1}^{2(\lfloor \frac{n-1}{2} \rfloor + 1)} \eta_i^{-(nm_i/2)} \exp \left\{ -\frac{1}{2} \sum_{i=1}^{2(\lfloor \frac{n-1}{2} \rfloor + 1)} \eta_i^{-1} \sum_{j=1}^{m_i} h_j \right\}.$$

By taking the derivative with respect to  $\eta_i$ ,  $i = 1, \dots, 2(\lfloor \frac{n_1}{2} \rfloor + 1)$ , the MLE of  $\eta_i$  is obtained by solving the following normal equation

$$-\frac{nm_i}{2\eta_i} + \frac{\sum_{j=1}^{m_i} h_j}{2\eta_i^2} = 0,$$

since then the upper bound of  $L(\hat{\mu}, \boldsymbol{\eta})$  is obtained.

**Theorem 4.1.** *The MLEs of the distinct eigenvalues  $\eta_i$  of  $\boldsymbol{\Sigma}$  are*

$$\hat{\eta}_i = \frac{\sum_{j=1}^{m_i} h_j}{nm_i}, \quad i = 1, \dots, 2(\lfloor n_1/2 \rfloor + 1), \quad (4.4)$$

where  $h_j$  is  $j$ -th diagonal element of the matrix  $\mathbf{H}_d$  in (4.3) and  $m_i$  is the multiplicity of  $\eta_i$  given in Table 2.

Based on  $\hat{\boldsymbol{\eta}} = (\hat{\eta}_1, \dots, \hat{\eta}_{2(\lfloor n_1/2 \rfloor + 1)})$ , an estimator of  $\boldsymbol{\Sigma}$  is given by  $\hat{\boldsymbol{\Sigma}} = \mathbf{Q}\mathbf{D}(\hat{\boldsymbol{\eta}})\mathbf{Q}'$ , where  $\mathbf{D}(\hat{\boldsymbol{\eta}})$  is a  $p \times p$  diagonal matrix.

The elements of the covariance matrix  $\boldsymbol{\Sigma}$  are functions of unknown parameters in  $\boldsymbol{\theta} = (\sigma^2, a, b, \tau_0, \dots, \tau_{2\lfloor n_1/2 \rfloor + 1})'$ , i.e.  $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}(\boldsymbol{\theta})$ . It can be shown that the system of linear equations given in (3.3) is consistent. If the number of distinct eigenvalues of  $\boldsymbol{\Sigma}$ , i.e. the number of elements in  $\boldsymbol{\eta}$ , equals the number of elements in  $\boldsymbol{\theta}$ , the MLE for  $\boldsymbol{\theta}$  has an explicit expression (Szatrowski and Miller, 1980), which is obtained by solving the linear system in (3.3). If the number of elements in  $\boldsymbol{\eta}$  is less than the number elements in  $\boldsymbol{\theta}$ ,  $\boldsymbol{\theta}$  is estimable only under some constraints on  $\boldsymbol{\theta}$ .

In the last theorem we prove that in the circular symmetric model with patterned blocks given in (2.1),  $\boldsymbol{\theta}$  is non-estimable unless some constraints will be imposed.

**Theorem 4.2.** *Let  $s_1$  be the number of the distinct eigenvalues of  $\boldsymbol{\Sigma}$  defined in (2.2), and  $s_2$  be the number of unknown parameters in  $\boldsymbol{\Sigma}$ , then  $s_2 - s_1 = 3$ .*

*Proof.* According to the definition of  $\boldsymbol{\Sigma}$  in (2.2), the number of unknown parameters is  $3 + 2(\lfloor n_1/2 \rfloor + 1)$ , i.e. it is  $n_1 + 4$  for odd  $n_1$  and  $n_1 + 5$  for even  $n_1$ . Moreover, recall that  $\boldsymbol{\Sigma}$  given in (2.2) is the sum of three matrices:

$$\boldsymbol{\Sigma} = \underbrace{\sigma^2 \mathbf{I}}_{1 \text{ parameter}} + \underbrace{\mathbf{Z}_1 \boldsymbol{\Sigma}_1 \mathbf{Z}_1'}_{2 \text{ parameters}} + \underbrace{\boldsymbol{\Sigma}_2}_{2(\lfloor \frac{n_1}{2} \rfloor + 1) \text{ parameters}}$$

From Table 2, it follows that  $\boldsymbol{\Sigma}$  has  $2(\lfloor \frac{n_1}{2} \rfloor + 1)$  distinct eigenvalues. Thus,  $s_2 - s_1 = 3$ .  $\square$

We may note that the model given in (2.1) belongs to the class of invariant normally distributed models, see Andersson (1975). Jensen (1988) remarked that the invariant normal models belonged to normally distributed models where the covariance matrix was parametrized according to a Jordan algebra

structure. Indeed, Jensen referred to the seminal paper by Seely (1971) on quadratic subspaces characterization which is a special Jordan algebra. One of the interesting results in Seely's work is that he connects optimal results of the obtained estimators with the Jordan algebra, especially commutative Jordan algebra. The use of Jordan algebra in statistical inference and applications has been further discussed by many authors, see for instance Malley (1986). However, in this work, we are more focused on explicit estimators, because the given models are overparametrized, and postpone optimality consideration for a following publication.

## 5. Example

In the section, an example will illustrate the obtained results. Let us consider model (2.1) when  $n_2=2$  and  $n_1=4$ , i.e.

$$\mathbf{y}_j = \mathbf{1}_8\mu + (\mathbf{I}_2 \otimes \mathbf{1}_4)\gamma_1 + \mathbf{I}_8\gamma_2 + \epsilon.$$

In this case  $\mathbf{y}_j \sim N_8(\mathbf{1}_8\mu, \Sigma)$ ,  $j = 1, \dots, n$ , and

$$\Sigma = \sigma^2\mathbf{I}_8 + \Sigma_1 \otimes \mathbf{J}_4 + \Sigma_2,$$

where

$$\Sigma_1 \otimes \mathbf{J}_4 = \left( \begin{array}{cccc|cccc} a & a & a & a & b & b & b & b \\ a & a & a & a & b & b & b & b \\ a & a & a & a & b & b & b & b \\ a & a & a & a & b & b & b & b \\ \hline b & b & b & b & a & a & a & a \\ b & b & b & b & a & a & a & a \\ b & b & b & b & a & a & a & a \\ b & b & b & b & a & a & a & a \end{array} \right),$$

$$\Sigma_2 = \left( \begin{array}{cccc|cccc} \tau_0 & \tau_1 & \tau_2 & \tau_1 & \tau_3 & \tau_4 & \tau_5 & \tau_4 \\ \tau_1 & \tau_0 & \tau_1 & \tau_2 & \tau_4 & \tau_3 & \tau_4 & \tau_5 \\ \tau_2 & \tau_1 & \tau_0 & \tau_1 & \tau_5 & \tau_4 & \tau_3 & \tau_4 \\ \tau_1 & \tau_2 & \tau_1 & \tau_0 & \tau_4 & \tau_5 & \tau_4 & \tau_3 \\ \hline \tau_3 & \tau_4 & \tau_5 & \tau_4 & \tau_0 & \tau_1 & \tau_2 & \tau_1 \\ \tau_4 & \tau_3 & \tau_4 & \tau_5 & \tau_1 & \tau_0 & \tau_1 & \tau_2 \\ \tau_5 & \tau_4 & \tau_3 & \tau_4 & \tau_2 & \tau_1 & \tau_0 & \tau_1 \\ \tau_4 & \tau_5 & \tau_4 & \tau_3 & \tau_1 & \tau_2 & \tau_1 & \tau_0 \end{array} \right).$$

The vector of the unknown (co)variance components is the following

$$\boldsymbol{\theta} = (\sigma^2, a, b, \tau_0, \tau_1, \tau_2, \tau_3, \tau_4, \tau_5)'$$

and the eigenvalues of  $\Sigma$  are given by

$$\begin{aligned}\eta_1 &= \sigma^2 + 4(a+b) + \tau_0 + 2\tau_1 + \tau_2 + \tau_3 + 2\tau_4 + \tau_5, \\ \eta_2 &= \sigma^2 + \tau_0 - 2\tau_1 + \tau_2 + \tau_3 - 2\tau_4 + \tau_5, \\ \eta_3 &= \sigma^2 + \tau_0 - \tau_2 + \tau_3 - \tau_5, \\ \eta_4 &= \sigma^2 + 4(a-b) + \tau_0 + 2\tau_1 + \tau_2 - \tau_3 - 2\tau_4 - \tau_5, \\ \eta_5 &= \sigma^2 + \tau_0 - 2\tau_1 + \tau_2 - \tau_3 + 2\tau_4 - \tau_5, \\ \eta_6 &= \sigma^2 + \tau_0 - \tau_2 - \tau_3 + \tau_5.\end{aligned}$$

The multiplicities of the eigenvalues are 1, 1, 2, 1, 1 and 2, respectively.

Using (4.2), the MLE for  $\mu$  is

$$\hat{\mu} = \frac{\sum_{j=1}^n \sum_{i=1}^8 y_{ij}}{8n},$$

and the MLEs for  $\eta_k$ ,  $k = 1, \dots, 6$ , given in (4.4) are

$$\begin{aligned}\hat{\eta}_1 &= \frac{1}{n} \sum_{j=1}^n (\mathbf{v}'_1 \mathbf{y}_j)^2 - 8\hat{\mu}^2, & \hat{\eta}_4 &= \frac{1}{n} \sum_{j=1}^n (\mathbf{v}'_5 \mathbf{y}_j)^2, \\ \hat{\eta}_2 &= \frac{1}{n} \sum_{j=1}^n (\mathbf{v}'_2 \mathbf{y}_j)^2, & \hat{\eta}_5 &= \frac{1}{n} \sum_{j=1}^n (\mathbf{v}'_6 \mathbf{y}_j)^2, \\ \hat{\eta}_3 &= \frac{1}{2n} \sum_{j=1}^n (\mathbf{v}'_3 \mathbf{y}_j)^2 + \frac{1}{2n} \sum_{j=1}^n (\mathbf{v}'_4 \mathbf{y}_j)^2, & \hat{\eta}_6 &= \frac{1}{2n} \sum_{j=1}^n (\mathbf{v}'_7 \mathbf{y}_j)^2 + \frac{1}{2n} \sum_{j=1}^n (\mathbf{v}'_8 \mathbf{y}_j)^2,\end{aligned}$$

where  $\mathbf{v}_p$ ,  $p = 1, \dots, 8$ , are the orthonormal eigenvectors of  $\Sigma$ .

An estimator of  $\Sigma$  can be calculated as follows

$$\hat{\Sigma} = \sum_{k=1}^6 \hat{\eta}_k \mathbf{E}_k,$$

where

$$\begin{aligned}\mathbf{E}_1 &= \frac{1}{8} \mathbf{J}_8, & \mathbf{E}_2 &= \frac{1}{8} \mathbf{J}_4 \otimes \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \\ \mathbf{E}_3 &= \frac{1}{4} \mathbf{J}_2 \otimes \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \otimes \mathbf{I}_2, & \mathbf{E}_4 &= \frac{1}{4} \mathbf{I}_2 \otimes \mathbf{J}_4 - \frac{1}{8} \mathbf{J}_8, \\ \mathbf{E}_5 &= \frac{1}{8} \mathbf{I}_2 \otimes \mathbf{J}_2 \otimes \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \frac{1}{8} (\mathbf{J}_2 - \mathbf{I}_2) \otimes \mathbf{J}_2 \otimes \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \\ \mathbf{E}_6 &= \frac{1}{4} \mathbf{I}_2 \otimes \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \otimes \mathbf{I}_2 + \frac{1}{4} (\mathbf{J}_2 - \mathbf{I}_2) \otimes \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \mathbf{I}_2.\end{aligned}$$

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## References

- Andersson, S. (1975), *Invariant normal models*, Ann. Statist. **3**, 132–154.
- Basilevsky, A. (1983), *Applied Matrix Algebra in the Statistical Sciences*, North-Holland, New York.
- Goldstein, H. (2003), *Multilevel Statistical Models*, Wiley, New York.
- Jensen, S. T. (1988) *Covariance hypotheses which are linear in both the covariance and the inverse covariance*, Ann. Statist. **16**, 302–322.
- Malley, J. D. (1986), *Optimal Unbiased Estimation of Variance Components*, Lecture Notes in Statistics **39**, Springer-Verlag, Berlin.
- Marin, J.-M., and Dhorne, T. (2002), *Linear Toeplitz covariance structure models with optimal estimators of variance components*, Linear Algebra Appl. **354**, 195–212.
- Marin, J.-M., and Dhorne, T. (2003), *Optimal quadratic unbiased estimation for models with linear Toeplitz covariance structure*, Statistics **37**, 85–99.
- Olkin, I. (1972), *Testing and estimation for structures which are circularly symmetric in blocks*; In: Proceedings of the Symposium on Multivariate Analysis, Dalhousie, Nova Scotia, pp. 183–195.
- Olkin, I., and Press, S. J. (1969), *Testing and estimation for a circular stationary model*, Ann. Math. Statist. **40**, 1358–1373.
- Seely, J. (1971), *Quadratic subspaces and completeness*, Ann. Math. Statist. **42**, 710–721.
- Szatrowski, T. H. (1980), *Necessary and sufficient conditions for explicit solutions in the multivariate normal estimation problem for patterned means and covariances*, Ann. Statist. **8**, 802–810.
- Szatrowski, T. H., and Miller, J. J. (1980), *Explicit maximum likelihood estimates from balanced data in the mixed model of the analysis of variance*, Ann. Statist. **8**, 811–819.

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