ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 16, Number 1, 2012
Available online at www.math.ut.ee/acta/

An illustrated introduction to Caïssan squares: the magic of chess

George P. H. Styan

Abstract. We study various properties of $n \times n$ Caïssan magic squares. Following the seminal 1881 article [22] by "Ursus" [Henry James Kesson (b. c. 1844)] in *The Queen,* we define a magic square to be Caïssan whenever it is pandiagonal and knight-Nasik so that all paths of length n by a chess bishop are magic (pandiagonal, Nasik, CSP1-magic) and by a (regular) chess knight are magic (CSP2-magic). We also study Caïssan beauties, which are pandiagonal and both CSP2- and CSP3-magic; a CSP3-path is by a special knight that leaps over 3 instead of 2 squares. Our paper ends with a bibliography of over 100 items (many with hyperlinks) listed chronologically from the 14th century onwards. We give special attention to items by (or connected with) "Ursus": Henry James Kesson (b. c. 1844), Andrew Hollingworth Frost (1819–1907), Charles Planck (1856–1935), and Pavle Bidev (1912–1988). We have tried to illustrate our findings as much as possible, and whenever feasible, with images of postage stamps or other philatelic items.

1. Caïssan magic squares

1.1. Classic magic squares. An $n \times n$ "magic square" or "fully-magic square" is an array of numbers, usually integers, such that the numbers in all the rows, columns and two main diagonals add up to the same number, the magic sum $m \neq 0$. When only the numbers in all the rows and columns

Received February 3, 2012.

2010 *Mathematics Subject Classification.* 00A08, 01A55, 15A24.

Key words and phrases. Alternate-couplets property, Pavle Bidev (1912–1988), Caïssan beauties, Caïssan magic squares, "Cavendish", CSP2-magic, CSP3-magic, EP magic matrices, Equal Projectors, 4-pac magic squares, 4-ply magic squares, Andrew Hollingworth Frost (1819–1907), involution-associated, involutory matrix, Henry Jones (1831–1899), Sir William Jones (1746–1794), Henry James Kesson (b. c. 1844), keyed magic matrices, knight-Nasik, magic key, most-perfect magic squares, Nasik squares, pandiagonal magic square, philatelic items, Charles Planck (1856–1935), postage stamps, "Ursus".

Based, in part, on the talk [113] given on 30 June 2011 at The 9th Tartu Conference on Multivariate Statistics & The 20th International Workshop on Matrices and Statistics, Tartu, Estonia, 26 June–1 July 2011.

add up to the magic sum $m \neq 0$, then we have a "semi-magic square". The associated matrix $\mathbf{M}$, say, will be called a "magic matrix" [1] and we will assume in this paper that $\operatorname{rank}(\mathbf{M}) \geq 2$. An $n \times n$ magic square is said to be "classic" whenever it comprises n^2 consecutive integers, usually $1, 2, \ldots, n^2$ (or $0, 1, \ldots, n^2 - 1$), each precisely once; the magic sum is then $m = n(n^2 + 1)/2$ (or $n(n^2 - 1)/2$), and so when $n = 8$, the magic sum $m = 260$ (or 252). Other names for a classic magic square include natural, normal or pure.

1.2. Pandiagonal magic squares and magic paths. We define an $n \times n$ magic square to be "pandiagonal" whenever all $2n$ diagonal *paths* with wrap-around parallel to (and including) the two main diagonals are magic. A *path* here is continuous of length n (with wrap-around) and with consecutive entries a chess-bishop's (chess-piece's) move apart. Our paths all proceed in the same direction. A path is said to be magic whenever its n elements add up to the magic sum m. In our $n \times n$ pandiagonal magic square, therefore, all its rook's and bishop's (and queen's, king's and pawn's) paths are magic and so it has (at least) $4n$ magic paths. In a semi-magic square there are $2n$ magic paths for the rooks (and kings).

A pandiagonal magic square has also been called continuous, diabolic, Indian, Jaina, or Nasik. The usage here of the word *Nasik* stems from a series of papers by Andrew H. Frost ([14, (1865)], [16, 17, 18, (1877)], [30, (1896)]) in which a magic square is defined to be "Nasik"[2] whenever it is pandiagonal[3] and satisfies "several other conditions"[4]. More recent usage indicates that a magic square is Nasik whenever it is (just) pandiagonal. The term *Nasik square* was apparently first defined by Andrew's older brother, the mathematician Percival Frost (1817–1898)[5] in his *Introduction* to the paper by A. H. Frost [14, (1865)]. In that paper [14, p. 94] an $n \times n$ Nasik

[1] The term magic matrix seems to have been introduced in 1956 by Charles Fox [58, 59, 60].

[2] Andrew Hollingworth Frost (1819–1907) was a British missionary with the Church Missionary Society living in Nasik, India, in 1853–1869 [99]. Nasik is in the northwest of Maharashtra state (180 km from Mumbai and 220 km from Pune), India and India Security Press in Nasik is where a wide variety of items like postage stamps, passports, visas, and non-postal adhesives are printed.

[3] Percival Frost in his introduction to [14, (1865)] by his younger brother Andrew H. Frost, says that "Mr. A. Frost has investigated a very elegant method of constructing squares, in which not only do the rows and columns form a constant sum, but also the same constant sum is obtained by the *same number of summations in the directions of the diagonals*—I shall call them Nasik Squares".

[4] Andrew H. Frost [30, (1896)] defines a Nasik square to be "A square containing n cells in each side, in which are placed the natural numbers from 1 to n^2 in such an order that a constant sum $\frac{1}{2}n(n^2 + 1)$ (here called W) is obtained by adding the numbers on n of the cells, these cells *lying in a variety of directions defined by certain laws.*"

[5] For an obituary of Percival Frost (1817–1898) see [32].

square is said to satisfy $4n$ (not completely specified) conditions, as does a pandiagonal magic square.

Andrew H. Frost [30, (1896)] gives a method by which Nasik squares of the nth order can be formed for all values of n; a Nasik square is defined as "A square containing n cells in each side, in which are placed the natural numbers from 1 to n in such an order that a constant sum $\frac{1}{2}n(n^2+1)$ is obtained by adding the numbers on n of the cells, these cells lying in a variety of directions denned by certain laws."

1.3. Caïssa: the *patron goddess* of chess players. The *patron goddess* of chess players was named Caïssa by Sir William Jones (1746–1794), the English philologist and scholar of ancient India, in a poem [7] entitled "Caïssa" published in 1763.

The first so-called "Caïssan magic square" that we have found seems to be that presented in 1881 by "Ursus"[6] (1881), who introduces his seminal article [22] in this way:

> ONCE UPON A TIME, when Orpheus was a little boy, long before the world was blessed with the "Eastern Question", there dwelt in the Balkan forests a charming nymph by name Caïssa. The sweet Caïssa roamed from tree to tree in Dryad meditation fancy free. As for trees, she was, no doubt, most partial to the box[7] and the ebony[8] See Figure 1.3.2.
>
> Mars read of her charms in a "society" paper, saw her photograph, and started by the first express for Thracia. He came, he saw, but he conquered not – in fact, was ingloriously repulsed. As he was wandering by the Danube blurting out, to give vent to his chagrin, many unparliamentary expressions, a Naiad advised him to go to Euphron, who ruled over a sort of Lowtherian Arcadia. Euphron good-naturedly produced from his stores a mimic war game, warranted to combine amusement with instruction.

[6] According to Irving Finkel [25], the adjective Caïssan was suggested to "Ursus" (Henry James Kesson) by "Cavendish" (Henry Jones), who originated it. Moreover, Iltyd Nicholl suggests "Kesson" itself was a nom-de plume, deriving from the site called "Nassek" (Nasik, Nashik) where a contention-producing magic-square had been earlier discovered over a gateway.

[7] *Buxus* is a genus of about 70 species in the family *Buxaceae.* Common names include box (majority of English-speaking countries) or boxwood (North America). See also Figure 1.3.2.

[8] Ebony is a general name for very dense black wood. In the strictest sense it is yielded by several species in the genus *Diospyros,* but other heavy, black (or dark coloured) woods (from completely unrelated trees) are sometimes also called ebony. Some well-known species of ebony include *Diospyros ebenum* (Ceylon ebony), native to southern India and Sri Lanka, *Diospyros crassiflora* (Gaboon ebony), native to western Africa, and *Diospyros celebica* (Makassar ebony), native to Indonesia and prized for its luxuriant, multi-coloured wood grain.

The warrior – in mufti – took the game to Caïssa, taught her how to play, called it after her name, and, throwing off his disguise, proposed. The delighted Caïssa consented to become Mrs Mars. He spent the honeymoon in felling box trees and ebony ditto, she in fashioning the wood to make tesselated boards and quaint chessmen to send all over the uncivilised world.

Such is our profane version of the story, told so prettily in Ovidian verse by Sir William Jones, the Orientalist. Such is Caïssa, as devoutly believed in by Phillidorians as though she were in Lemprière's *Biographical Dictionary.* Our Caïssa, however, shall herself be a chessman – we beg her pardon – chesswoman. Like the "queen" in the Russian game, she shall have the power of every man, moving as a king, rook, bishop, or pawn (like the queen in our game), and also as a knight. Caïssa then is complete mistress of the martial board.

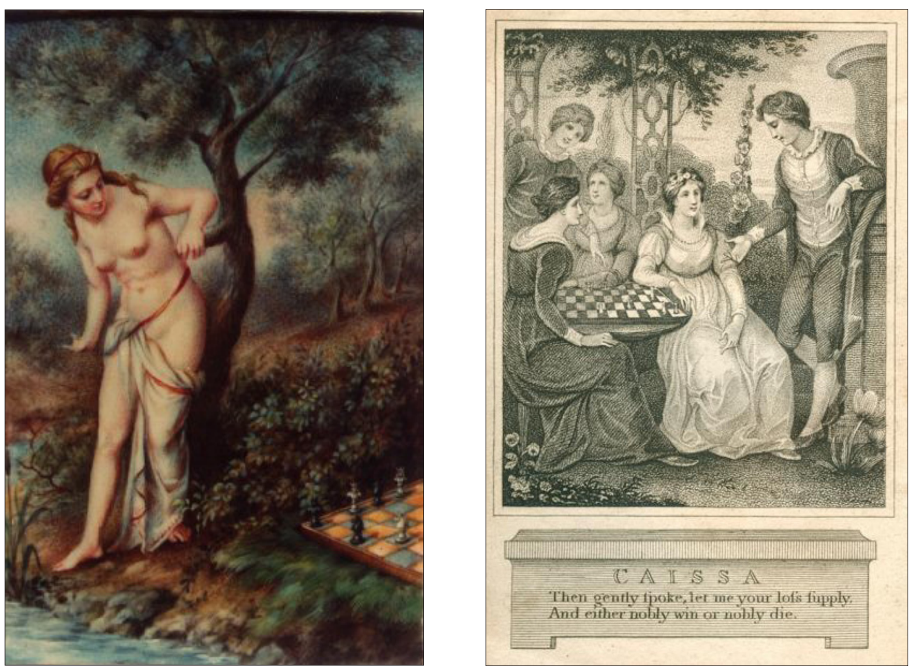


FIGURE 1.3.1: (left panel) Illustration of Caïssa [*Wikipedia*], apparently by Domenico Maria Fratto (1669–1763) [9]; (right panel) Frontispiece from *The Poetical Works of Sir William Jones* [9, vol. 1 (1810)].

[9]It seems the artist died in 1763, the year the poem was published.

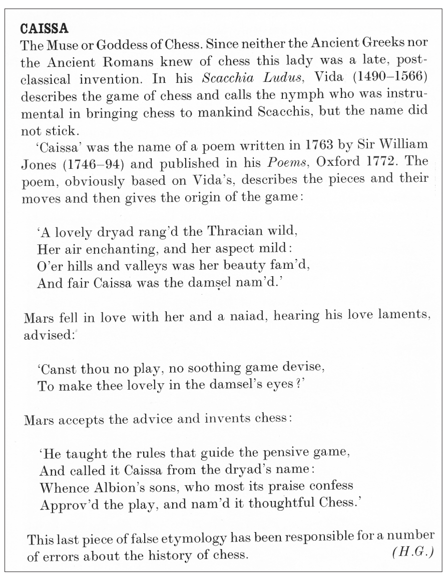
FIGURE 1.3.2: (left panel) White pieces made of boxwood, black piece is ebonized, not ebony; (right) Elephant carvings from Ceylon (Sri Lanka), from ebony, likely Gaboon ebony (*Diospyros dendro*).

And Ursus [22] ends his article in this way:

> The idea of the foregoing Caïssan magic squares was suggested partly by the Brahminical squares, that from time immemorial have been used by the Hindoos as talismans; partly by the article [16] by the Rev. A. H. Frost, M.A., of St. John's College, Camb., on "Nasik Squares," in No 57 of *The Quarterly Journal of Pure and Applied Mathematics*" (1877). Our squares are, however, mostly original, as are the methods of construction, though one or two, notably that for the fifteen-square, may by Girtonians and Newnhamites readily be translated into the (mathematically) purer language adopted in Mr. Frost's able paper.
>
> The epithet "Caïssan" described the distinguishing characteristics of the squares, as the paths include all possible continuous chess moves. We commenced with Caïssa's mythical history; let us conclude with a few facts less apocryphal. We have alluded to the Brahminical squares. Of these the favorites are the four-square and the eight-square. We have shown that the latter is, from our point of view, the first perfect one; in fact as the Policeman of Penzance [19, (1879)] would say, "taking one consideration with another", it may be pronounced the best of the lot. Now *chaturanga*, the great-grandfather of chess, is of unknown antiquity and Duncan Forbes has given translations of venerable Sanskrit manuscripts, which describe the pieces, moves, and mode of playing the game. *Chaturanga* means four arms, i.e., the four parts of an army game–elephants, camels, boats (for fighting on the rivers, Sanskrit *roka*, a boat, hence rook), and foot-soldiers, the whole commanded by the kaisar or king.
>
> At first there were four armies, two as allies against two others; afterwards the allies united, one of the kings becoming prime minister with almost unlimited powers, and in the chivalrous west being denominated Queen. We have already noticed that the Russians, who are inveterate chess players, and who probably got their national game directly from the fountainhead, endow their queen with the powers of all the other pieces. So then in the west we have king, queen, bishop (ex-elephant),

knight (ex-camel), castle or rook, and pawn, the line soldier on whom, after all, military success depends. Sir William Jones, who wrote "Caïssa" when he was a boy, thinking in Greek, but then little versed in Sanskrit, turned the corrupt Saxon "chess" into the pseudo-classic Caïssa. Strange coincidence, for much later, his admirer, Duncan Forbes, traces "chess", by such links as the French *échecs*, English *check* and *check mate*, German *Schachmatt*, to the Persian *Shah Met*, Shah being their rendering of the Sanskrit or Aryan *Kaisar*, still retained by Germans in *Kaiser*, and by Russians in *Czar*.

Sir William's Caïssa would pass very well for the feminine of Kaisar; but this is not the coincidence we wish to accentuate. Both "chess" and "Caïssan" squares, under whatever name the reader pleases, have been known in India – the nursery of civilisation – from the remotest antiquity; hence there would seem to be a close connection between them. India is now the greatest possession of the British crown, and our Queen, as everyone knows, bears the masculine title of Kaisar-in-Hind[10].



CAISSA

The Muse or Goddess of Chess. Since neither the Ancient Greeks nor the Ancient Romans knew of chess this lady was a late, post-classical invention. In his *Scacchia Ludus*, Vida (1490–1566) describes the game of chess and calls the nymph who was instrumental in bringing chess to mankind Scacchis, but the name did not stick.

'Caissa' was the name of a poem written in 1763 by Sir William Jones (1746–94) and published in his *Poems*, Oxford 1772. The poem, obviously based on Vida's, describes the pieces and their moves and then gives the origin of the game:

'A lovely dryad rang'd the Thracian wild,
Her air enchanting, and her aspect mild:
O'er hills and valleys was her beauty fam'd,
And fair Caissa was the damsel nam'd.'

Mars fell in love with her and a naiad, hearing his love laments, advised:

'Canst thou no play, no soothing game devise,
To make thee lovely in the damsel's eyes?'

Mars accepts the advice and invents chess:

'He taught the rules that guide the pensive game,
And called it Caissa from the dryad's name:
Whence Albion's sons, who most its praise confess
Approv'd the play, and nam'd it thoughtful Chess.'

This last piece of false etymology has been responsible for a number of errors about the history of chess. *(H.G.)*

FIGURE 1.3.3: (left panel) "Caissa: The Muse or Goddess of Chess" from [65, p. 54]; (right panel) Women chess players and magic chessboard: Rwanda 2005.

[10]The Kaisar-i-Hind (sometimes misspelt as Kaiser-i-Hind or Kaisar-in-Hind) was a medal awarded by the British monarch between 1900 and 1947 to civilians of any nationality who rendered distinguished service in the advancement of the interests of the British Raj. The medal name literally means "Emperor of India" in the vernacular of the Hindi and Urdu languages. The world kaisar, meaning "emperor" is derived from the Latin "Caesar", derived from the name of the Roman general and statesman, Julius Caesar. It is a cognate with the German title Kaiser, which was borrowed from the Latin at an earlier date.

The poem "Caïssa" (Fig. 1.3.3, left panel) by Sir William Jones[11] was based on the poem "Scacchi, Ludus" published in 1527 by the Italian humanist, bishop and poet Marco Girolamo [Marcus Hieronymus] Vida (c. 1485–1566), giving a mythical origin of chess that has become well known in the chess world. As observed by Murray [41, p. 793]:

> [26] Vida's description of the moves and rules, and the game (a Queen's Gambit), contain nothing of material importance. The name *Scacchis*, which Vida bestowed upon the nymph who was the means of teaching chess to mankind, has not commended itself to players, and *Caïssa*, the creation of Sir William Jones (1763), has supplanted it entirely.

1.4. "Ursus": Henry James Kesson (b. c. 1844). It seems that the first person to explicitly connect Caïssa with magic squares was "Ursus" in a three-part article entitled "Caïssan magic squares" published in *The Queen: The Lady's Newspaper & Court Chronicle*[12] 1881. We believe that this "Ursus" was the *nom de plume* of Henry James Kesson (b. c. 1844), about whom we know very little. We do know that "Ursus" was a regular contributor to *The Queen* with a five-part article [20] on "Magic squares and circles" published two years earlier in 1879 and a six-part article [21] on "Trees in rows" in 1880, as well as many double acrostics (puzzles and solutions). Falkener (1892, p. 337) cites work on magic squares by "H. J. Kesson (Ursus)" in *The Queen,* 1879–1881. Whyld (1978), quoting Bidev (1977), says that the "pioneering work [on Caïssan magic squares] was done about a century ago by a London mathematician named Kesson, who under the pen-name Ursus, wrote a series of articles in *The Queen*".

Moreover, we believe that this Henry James Kesson is the one listed in the 1851 British Census as age 7, "born in St Pancras", with father John Kesson, age 40, "attendant at British Museum"[13]. The 1851 Census apparently also stated that the family then (1851) lived at 40 Chichester Place, Grays Inn

[11]The father of Sir William Jones (1746–1794) was William Jones, FRS (1675–1749), a Welsh mathematician, whose most noted contribution was his proposal for the use of the Greek letter π as the symbol to represent the ratio of the circumference of a circle to its diameter.

[12]Published from 1864–1922, *The Queen: The Lady's Newspaper & Court Chronicle* was started by Samuel Orchart Beeton (1831–1877) and Isabella Mary Beeton *née* Mayson (1836–1865), and Frederick Greenwood (1830–1909). Isabella Beeton is universally known as "Mrs Beeton", the principal author of *Mrs Beeton's Book of Household Management,* first published in 1861 and still in print.

[13]The book, *The Cross and the Dragon* [13] is by John Kesson "of the British Museum". This is probably the John Kesson (d. 1876) who translated *Travels in Scotland* by J. G. Kohl from German into English [10] and *The Childhood of King Erik Menved* by B. S. Ingemann from Danish into English [12]. It is just possible that the Scottish novelist, playwright and radio producer Jessie Kesson (1916–1994), born as Jessie Grant McDonald,

Lane, St Pancras. As well as Henry James Kesson and his father John, the family then comprised Maria Kesson [John's wife], then also age 40, born in Finsbury, and children Maria Jane, then age 13, born Islington, Lucy Emma, age 9 and Arnold age 3, both born in St Pancras.

Horn [63, Document F] mentions a "[School]master: Henry James Kesson, Trained Two Years; Certificated First Class. Emily Kesson, Sewing Mistress and General Assistant. ..." in 1890 in Austrey, a village at the northern extremity of the county of Warwickshire, near Newton Regis and No Man's Heath, and close to the Leicestershire villages of Appleby Magna, Norton-juxta-Twycross and Orton on the Hill.

FIGURE 1.4.1: (left panel) Marco Girolamo [Marcus Hieronymus] Vida (c. 1485–1566); (right panel) Sir William Jones (1746–1794), 250th birth anniversary: India 1997, *Scott*[14] 1626.

may be a descendant (in law) since in 1934 she married Johnnie Kesson, a cattleman, living in Abriachan (near Inverness) and then Rothienorman (near Aberdeen).

[14] *Scott* numbers are as published in the *Scott Standard Postage Stamp Catalogue* [114].

FIGURE 1.4.2: (left panel) The Lincoln Imp [38]; (right panel) Lincoln Cathedral: Montserrat 1978, *Scott* 387.

In addition, the 1894 book on magic squares by "Cavendish"[15] is dedicated to Henry James Kesson "by his sincere friend, the Author" and it seems that our "Ursus" is the H. J. Kesson, who wrote the words for four "operetta-cantatas for young people" (with music by Benjamin John W. Hancock [27, 28, 29, 34, 35] and a booklet on the legend of the Lincoln Imp[16] (1904).

Following [22], we define a magic square to be "Caïssan" whenever it is pandiagonal and all the "knight's paths" are magic. And so in an $n \times n$ Caïssan magic square, all the $8n$ paths by a chess piece (rook, bishop, knight, queen, king) are magic. The knights we consider here are the regular chess-knights, and we may call such Caïssan squares "regular Caïssan squares".

"Caïssa's special path" is defined by [22, p. 142] as a special knight's path where the "special knight" (Caïssa) can move 3 steps instead of 2 (e.g., down 1 and over 3 or up 3 and over 1). We will call such a path Caïssa's

[15]"Cavendish" was the *nom de plume* of Henry Jones (1831–1899), an English author well-known as a writer and authority on card games and who in 1877 founded the "The Championships, Wimbledon, or simply Wimbledon, the oldest tennis tournament in the world and considered the most prestigious".

[16]According to a 14th-century legend two mischievous creatures called imps were sent by Satan to do evil work on Earth. After causing mayhem in Northern England, the two imps headed to Lincoln Cathedral where they smashed tables and chairs and tripped up the Bishop. When an angel came out of a book of hymns and told them to stop, one of the imps was brave and started throwing rocks at the angel but the other imp cowered under the broken tables and chairs. The angel turned the first imp to stone giving the second imp a chance to escape.

special path of type 3 (CSP3). A "special knight" with paths of type CSP3 is called a "jumping rukh" by Calvo [75, (1994)]: "Most important of all is the fact that the 'jumping rukh' which accesses a third square vertically or horizontally, as depicted in the theory of Kohtz [49, (1918)] appears here in an astonishing manner despite starting from any square;" see also Kohtz & Brunner [47, p. 5 (1917)]. The regular knight's path, therefore, is of type 2 (CSP2).

In his study of a 15×15 Caïssan magic square [22, p. 391 (Fig. R)], which we examine in Section 7 below, Caïssan special paths of types 4, 5, 6, and 7 (CSP4, CSP5, CSP6, CSP7) are (also) considered.

1.5. "Caïssan beauties" and "knight-Nasik" magic squares. The term "knight-Nasik" has been used by (at least) Planck [39, 43, 44, 105], Woodruff [42, 46], Andrews [45], Foster [48], and Marder [52] to mean either a pandiagonal magic square with all $4n$ regular-knight's paths of type CSP2 magic, or just a magic square, not necessarily pandiagonal, with all $4n$ regular-knight's paths of type CSP2 magic. Ursus (1881) defined a Caïssan magic square to be pandiagonal with all $4n$ regular-knight's paths of type CSP2 magic.

The first use of the term "knight-Nasik" *per se* that we have found is by Planck [39, p. 17, footnote] who finds that "the well-known [8×8 magic square]

$$\begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix}_8 \tag{1.5.1}$$

is said to be *knight-Nasik*". It is not clear to us what the notation used in (1.5.1) means, and hence we have the following open question:

Open question 1.5.1. We have not yet been able to identify the magic matrix defined by (1.5.1) but we expect it to be pandiagonal, with all $4n$ regular-knight's paths of type CSP2 magic.

We will use the following terminology:

Definition 1.5.1. We define an $n \times n$ fully-magic matrix, often classic and 8×8, to be

(1) "CSP2-magic" when all $4n$ regular-knight's paths of type CSP2 and each of length n are magic,

(2) "CSP3-magic" when all $2n$ special-knight's paths of type CSP3 and each of length n are magic.

Quotes & Queries

by **K.Whyld**, Moorland House, Kelsey Road, Caistor, Lincs. LN7 6SF.

No.3942 - Arising from Professor Bidev's article in July *BCM* a request has been made for an example of the magic squares to which he refers, and more background. In a paper on the subject, published in *Bonus Socius*, The Hague 1977, Bidev says that the pioneering work was done about a century ago by a London mathematician named Kesson who, under the pen-name Ursus, wrote a series of articles 'Magic Squares and Caissan Magic Squares' in *The Queen*, and he adds that Kesson did not reach the same conclusion because he did not know the original moves of the pieces. The German problemist and chess historian, Johannes Kohtz, investigated around 1918 the connection between chess and magic squares at the dawn of the game, but his work remains unpublished.

1	58	3	60	8	63	6	61
16	55	14	53	9	50	11	52
17	42	19	44	24	47	22	45
32	39	30	37	25	34	27	36
57	2	59	4	64	7	62	5
56	15	54	13	49	10	51	12
41	18	43	20	48	23	46	21
40	31	38	29	33	26	35	28

The book *From Magic Squares to Chess*, by N.M.Rudin, was published in 1969, and Pavle Bidev's important *Sah Simbol Kosmosa* appeared in 1972. Unfortunately these are both difficult for us because the first is in Russian and the second in Serbo-Croatian, but Bidev provides a good summary in German and a rather shorter one in English. The specimen square comes from Bidev's book.

The usual properties of a magic square are that each column and file and the two long diagonals total the same number, which on a 8×8 board is 260. This example has many other features. The two squares occupied by the white rooks plus the two occupied by their pawns total the half-constant of 130. The same applies for Black, and again for the corresponding knight, bishop and king plus queen sets of squares. The eight squares available to knights before pawns are moved total 260. The total of all squares covered or occupied by knights in the initial position is 520. The bishop's ancestor, the alfil, could move only to the next but one square diagonally, and thus could reach only eight squares on the board, and no two alfils could meet. Each of these four sets of eight squares totals 260. The four sets of four squares occupied by rooks, knights, bishops, and king plus queen, equal the half-constant. If the king moves through an octagon: Ke1-f2-f3-e4-d4-c3-c2-d1, or, Ke1-f1-g2-g3-f4-e4-d3-d2, the squares total 260, and the same for Black. This short outline barely scratches the surface. Professor Bidev concludes that the moves of the pieces were determined by the properties of the Nasik magic squares.

FIGURE 1.5.1: Quotes & Queries No. 3942 by K. Whyld [66, (1978)].

The bishop's predecessor in "shatranj" (medieval chess) was the "alfil" (meaning "elephant" in Arabic, "éléphant" in French, "olifant" in Dutch), which could leap two squares along any diagonal, and could jump over an intervening piece. As a consequence, each alfil was restricted to eight squares, and no alfil could attack another. Even today, the word for the bishop in chess is "alfil" in Spanish and "alfiere" in Italian. Whyld [66] mentions the original alfil, and discusses several properties of the 8×8 Ursus magic matrix $\mathbf{U}$ (Fig. 1.5.1).

Definition 1.5.2 We define an $n \times n$ fully-magic matrix, often classic and 8×8, to be a

(1) "Caïssan magic matrix" when it is pandiagonal ($2n$ magic paths) and CSP2-magic ($4n$ magic paths) and so there are (at least) $8n$ magic paths each of length n in all,

(2) "special-Caïssan magic matrix" when it is pandiagonal ($2n$ magic paths) and CSP3-magic ($2n$ magic paths) and so there are (at least) $6n$ magic paths each of length n in all,

(3) a "Caïssan beauty" (CB) when it is pandiagonal ($2n$ magic paths), and both CSP2- and CSP3-magic ($6n$ magic paths)and so there are (at least) $10n$ magic paths each of length n in all.

"Cavendish" [26, (1894)] defines a Caïssan magic square as one being (just) pandiagonal. Planck [33, (1900)], however, citing the "late Henry Jones (Cavendish)" and we assume referring to the book on magic squares by Cavendish (1894), says that this "nomenclature is of doubtful propriety" and he defines a Caïssan magic square as being magic in all regular chess-move paths with wrap-around. And so such Caïssan magic squares are both pandiagonal with all CSP2 regular-knight's paths being magic.

Possibly the first person, however, to consider Caïssan magic squares was Simon de la Loubère (1642–1729), a French diplomat, writer, mathematician and poet. From his Siamese travels, he brought to France a very simple method for creating n-odd magic squares, now known as the "Siamese method" or the "de la Loubère method". This method apparently was initially brought from Surat, India, by a *médecin provençal* by the name of M. Vincent [6]. According to Marder [52, p. 5 (1940)] (see also Andrews [45, p. 165 (1917)]), the following words[17] come from the pen of La Loubère:

[17] We have not found any mention of chess, however, in the discussion of magic squares by Simon de La Loubère [6, vol. 2, pp. 227–247 (1693)].

> In these $[8 \times 8]$ squares it is necessary not merely that the summation of the rows, columns and diagonals should be alike, but that the sum of any eight numbers in one direction as in the moves of a bishop or a knight should also be alike.

As "an example of one of these squares" Andrews [45, p. 165, Fig. 262 (1917)] presents the Ursus matrix **U** given by Ursus [22, p. 142, Fig. D (1881)], see our Figure 1.6.1 below. See also Falkener [24] and Whyld [66].

1.6. The 8×8 **Ursus Caïssan magic matrix U.** The first so-called "Caïssan magic square" (CMS) that we have found is that presented by Ursus [22, p. 142, Fig. D (1881)] and displayed in our Figure 1.6.1. A magic regular-knight's path (CSP2) is marked with red circles (left panel) and a magic special-knight's path of type CSP3 is marked with red boxes (right panel).

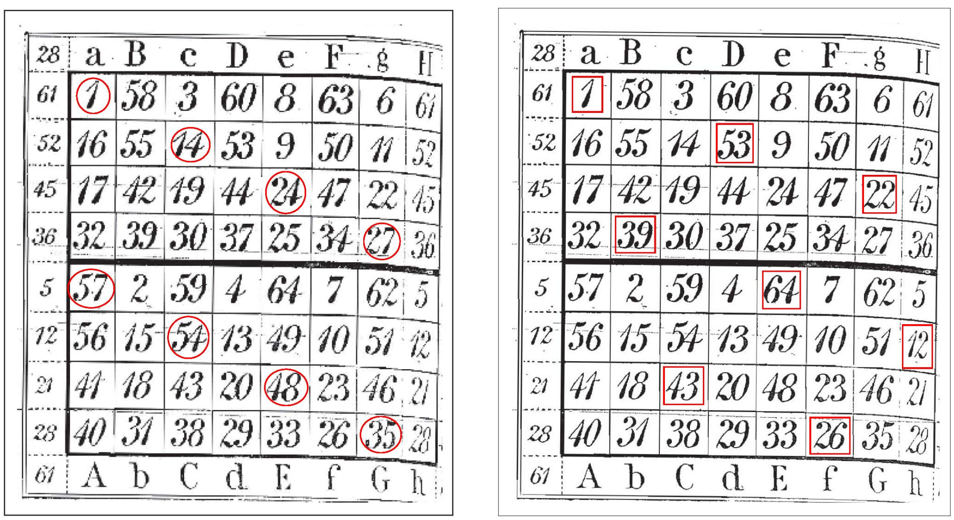


FIGURE 1.6.1: The 8×8 Caïssan magic square given by Ursus [22, p. 142, Fig. D (1881)] and defined by the Ursus matrix **U** (1.6.1), with a knight's path (CSP2) marked with red circles (left panel) and (right panel) with a special-knight's path (CSP3) marked with red boxes.

We will denote the *Ursus magic square* in Figure 1.6.1 by the *Ursus magic matrix*

$$\mathbf{U} = \begin{pmatrix} 1 & 58 & 3 & 60 & 8 & 63 & 6 & 61 \\ 16 & 55 & 14 & 53 & 9 & 50 & 11 & 52 \\ 17 & 42 & 19 & 44 & 24 & 47 & 22 & 45 \\ 32 & 39 & 30 & 37 & 25 & 34 & 27 & 36 \\ 57 & 2 & 59 & 4 & 64 & 7 & 62 & 5 \\ 56 & 15 & 54 & 13 & 49 & 10 & 51 & 12 \\ 41 & 18 & 43 & 20 & 48 & 23 & 46 & 21 \\ 40 & 31 & 38 & 29 & 33 & 26 & 35 & 28 \end{pmatrix}. \tag{1.6.1}$$

60	53	44	37	4	13	20	29
3	14	19	30	59	54	43	38
58	55	42	39	2	15	18	31
1	16	17	32	57	56	41	40
61	52	45	36	5	12	21	28
6	11	22	27	62	51	46	35
63	50	47	34	7	10	23	26
8	9	24	25	64	49	48	33

61	52	45	36	5	12	21	28
6	11	22	27	62	51	46	35
63	50	47	34	7	10	23	26
8	9	24	25	64	49	48	33
60	53	44	37	4	13	20	29
3	14	19	30	59	54	43	38
58	55	42	39	2	15	18	31
1	16	17	32	57	56	41	40

FIGURE 1.6.2: (left panel) The Nārāyaṇa Caïssan beauty $\mathbf{B}_{\mathrm{Nya}}$ [1, p. 396 (1356)], see also Fig. 1.6.3; (right panel) Frost's Nasik square $\mathbf{B}_{\mathrm{Fst}}$ [16, p. 48 (1877)].

We note that the classic Ursus magic matrix $\mathbf{U}$ is a *Caïssan beauty* (CB), i.e.,

(1) $\mathbf{U}$ is fully-magic ($2n+2$ magic paths),

(2) $\mathbf{U}$ is pandiagonal ($2n$ magic paths),

(3) $\mathbf{U}$ is CSP2-magic ($4n$ magic paths),

(4) $\mathbf{U}$ is CSP3-magic ($2n$ magic paths),

and so there are (at least) $10n$ magic paths of length n in all (2 magic paths are counted in both (1) and (2) above). Moreover, $\mathbf{U}$ has several other properties, see Table 1.6.1 below.

The oldest published magic square that we have found that is also a Caïssan beauty is given by Nārāyaṇa [1, p. 396, (1356)] over 650 years ago, see Fig. 1.6.2 (left panel) and Fig. 1.6.3, and also [51, 56, 70, 72, 73]. We will denote this Nārāyaṇa Caïssan beauty matrix by $\mathbf{B}_{\mathrm{Nya}}$ and we find that

$$\mathbf{U} = \mathbf{B}'_{\mathrm{Nya}}\mathbf{H}, \qquad \mathbf{B}_{\mathrm{Nya}} = \mathbf{FHU}', \tag{1.6.2}$$

where $\mathbf{F} = \mathbf{F}_n$ is the $n \times n$ flip matrix:

$$\mathbf{F} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{1.6.3}$$

and the 8×8 matrix

$$\mathbf{H} = \begin{pmatrix} \mathbf{0} & \mathbf{I}_4 \\ \mathbf{I}_4 & \mathbf{0} \end{pmatrix} = \mathbf{F}_2 \otimes \mathbf{I}_4 \tag{1.6.4}$$

where $\otimes$ denotes Kronecker product, and so the Ursus matrix $\mathbf{U}$ is formed by a particular rearrangement of the rows of $\mathbf{B}'_{\mathrm{Nya}}$, the transposed Nārāyaṇa matrix. Did Nārāyaṇa know that his magic matrix $\mathbf{B}_{\mathrm{Nya}}$ is a Caïssan beauty?

Five-hundred years after Nārāyaṇa [1, (1356)] and just four years before Ursus [22, (1881)], A. H. Frost [16, pp. 48–49 (1877)] observed, in his seminal paper on Nasik squares, that in the "Nasik square" $\mathbf{B}_{\mathrm{Fst}}$ (Fig. 1.6.2, right panel) there are "10 summations" through each element. Did Frost know that his magic matrix $\mathbf{B}_{\mathrm{Fst}}$ is a Caïssan beauty? We find that

$$\mathbf{U} = \mathbf{B}'_{\mathrm{Fst}}\mathbf{F}, \qquad \mathbf{B}_{\mathrm{Fst}} = \mathbf{FU}'; \tag{1.6.5}$$

with $\mathbf{F}$ the 8×8 flip matrix and $\mathbf{H}$ is as in (1.6.4). The Ursus matrix $\mathbf{U}$, therefore, may be formed by flipping the columns of $\mathbf{B}'_{\mathrm{Fst}}$, the transposed Frost matrix. Amela [116, 117, 118] has pointed out that both the Nārāyaṇa matrix $\mathbf{B}_{\mathrm{Nya}}$ and the Frost matrix $\mathbf{B}_{\mathrm{Fst}}$ may be constructed by "superposition", see (1.6.7) and Fig. 1.6.3. Following [72, p. 73], we construct two *samagarbha* (even-womb) magic squares, the first $\mathbf{C}_{aka}$ called *chādaka* (covering one) and the other $\mathbf{C}_{dya}$ called *chādya* (one to be covered), with the 8×8 magic matrices

$$\mathbf{C}_{aka}=\begin{pmatrix} 4&5&4&5&4&5&4&5\\ 3&6&3&6&3&6&3&6\\ 2&7&2&7&2&7&2&7\\ 1&8&1&8&1&8&1&8\\ 5&4&5&4&5&4&5&4\\ 6&3&6&3&6&3&6&3\\ 7&2&7&2&7&2&7&2\\ 8&1&8&1&8&1&8&1 \end{pmatrix}; \quad \mathbf{C}_{dya}=\begin{pmatrix} 24&16&8&0&32&40&48&56\\ 32&40&48&56&24&16&8&0\\ 24&16&8&0&32&40&48&56\\ 32&40&48&56&24&16&8&0\\ 24&16&8&0&32&40&48&56\\ 32&40&48&56&24&16&8&0\\ 24&16&8&0&32&40&48&56\\ 32&40&48&56&24&16&8&0 \end{pmatrix}. \tag{1.6.6}$$

See also Cavendish [26, pp. 29–32, (1894)]. We note that both $\mathbf{C}_{aka}$ and $\mathbf{C}_{dya}$ are Caïssan beauties (but clearly not classic), and hence the Nārāyaṇa and Frost Caïssan beauties are, respectively,

$$\mathbf{B}_{\mathrm{Nya}} = \mathbf{C}_{aka} + \mathbf{C}_{dya}\mathbf{F}, \qquad \mathbf{B}_{\mathrm{Fst}} = (\mathbf{C}_{aka} + \mathbf{C}_{dya})\mathbf{F}, \tag{1.6.7}$$

where $\mathbf{F} = \mathbf{F}_8$, the 8×8 flip matrix, and the Ursus magic matrix

$$\mathbf{U} = \mathbf{B}'_{\mathrm{Fst}}\mathbf{F} = \mathbf{F}(\mathbf{C}_{aka} + \mathbf{C}_{dya})'\mathbf{F} \tag{1.6.8}$$

$$= \mathbf{B}'_{\mathrm{Nya}}\mathbf{H} = (\mathbf{C}_{aka} + \mathbf{C}_{dya}\mathbf{F})'\mathbf{H}, \tag{1.6.9}$$

with $\mathbf{H} = \begin{pmatrix} \mathbf{0} & \mathbf{I}_4 \\ \mathbf{I}_4 & \mathbf{0} \end{pmatrix}$. We note that the classic Caïssan beauty

$$\mathbf{B}_{kdy} = \mathbf{C}_{aka} + \mathbf{C}_{dya} = \mathbf{F}\mathbf{U}'\mathbf{F} \tag{1.6.10}$$

is EP and satisfies all the properties as listed in Table 1.6.1, as does the Ursus magic matrix

$$\mathbf{U} = \mathbf{F}\mathbf{B}'_{kdy}\mathbf{F}. \tag{1.6.11}$$

Neither the Nārāyaṇa nor the Frost Caïssan beauty, however, is EP.

Falkener (1892), citing Frost [14, 15, 16, 23, (1865, 1866, 1877, 1882)] and Ursus [22, (1881)], finds the term "Indian magic square" to be more appropriate than "Caïssan magic square" (with Nasik being in India and Sir William Jones's poem Caïssa set in eastern Europe) and presents the Ursus magic square $\mathbf{U}$ and gives many properties in addition to its being pandiagonal and CSP2-magic. McClintock [31, pp. 111, 113 (1897)], who apparently does not mention Caïssa, presents two 8×8 magic squares, which we find to be pandiagonal and CSP2-magic.

Planck [33, Fig. II, p. 97 (1900)] presents the Caïssan magic square

$$\mathbf{B}_{\mathrm{Pnk}} = (1 \oplus \mathbf{F}_7)\mathbf{U}, \tag{1.6.12}$$

where the direct sum

$$1 \oplus \mathbf{F}_7 = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{F}_7 \end{pmatrix}. \tag{1.6.13}$$

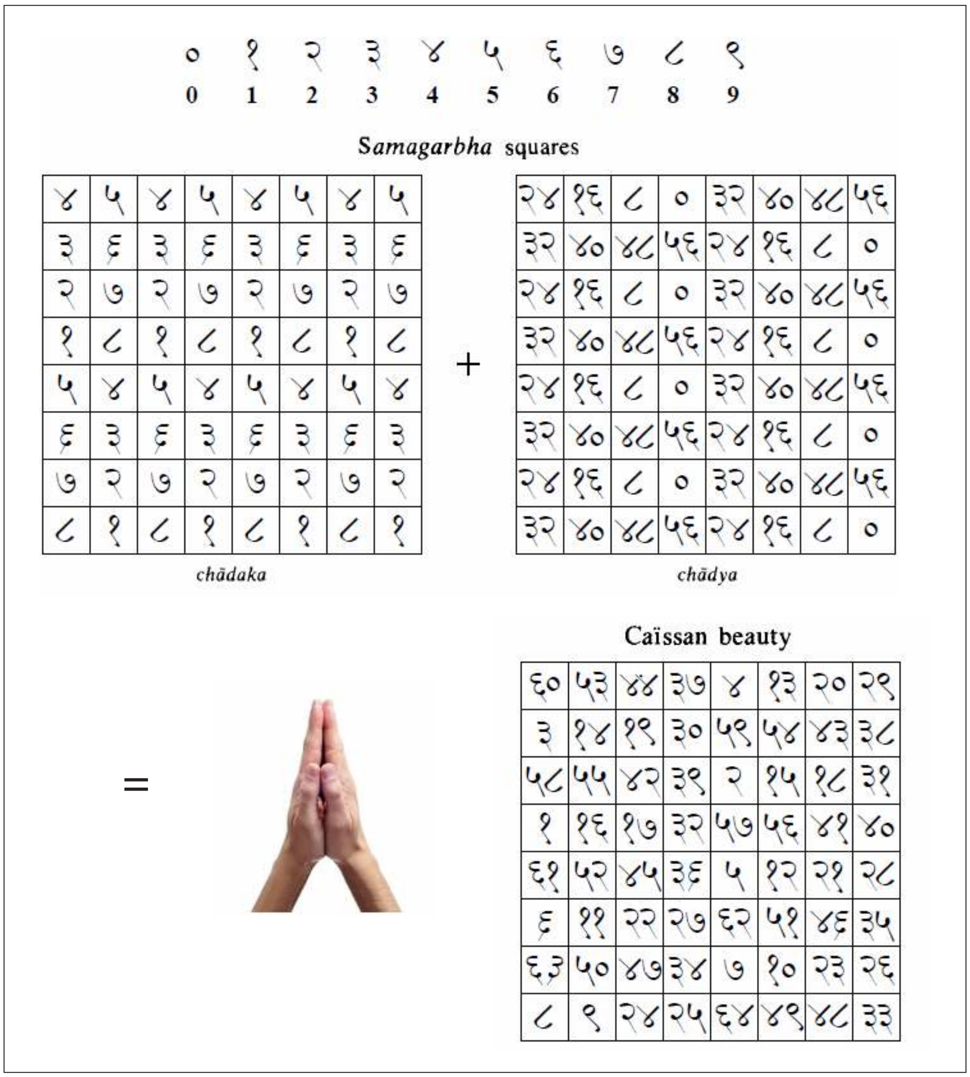


FIGURE 1.6.3: Construction by Amela [118] of the Nārāyaṇa Caïssan beauty $\mathbf{B}_{\mathrm{Nya}}$ [1, p. 396 (1356)], see also Fig. 1.6.2 (left panel), with "superposition made in the manner of the folding the palms of the hands (1.6.7)" [72, p. 73].

We find that $\mathbf{B}_{\text{Pnk}}$ (1.6.12) is a Caïssan beauty and which Planck [33] noted is 4-ply[18], i.e., the four numbers in each of the n^2 subsets of order 2×2 of 4 contiguous numbers (with wrap-around) add up to the same sum $4m/n = m/k$. Planck [33, p. 96] claimed that $\mathbf{B}_{\text{Pnk}}$ (1.6.12) was "the first complete Caïssan square of 64 cells that has been constructed".

We find that the Nārāyaṇa, Frost and Planck Caïssan beauties, respectively $\mathbf{B}_{\text{Nya}}, \mathbf{B}_{\text{Fst}}, \mathbf{B}_{\text{Pnk}}$, satisfy conditions (1)–(6) in Table 1.6.1 but are not EP (item (7) in Table 1.6.1) and do not satisfy conditions (8)–(10). Drury [106] identified 46080 classic Caïssan beauties (CBs) and we claim that only 192, or less than half-a-percent ($\frac{1}{2}$%), are EP; precisely 96 have magic key $\kappa = 2688$ and precisely 96 have magic key $\kappa = 8736$. In addition to being pandiagonal, CSP2-magic and CSP3-magic, we claim that all 46080 CBs satisfy properties (1)–(7) in Table 1.6.1.

TABLE 1.6.1: Some properties of the classic 8×8 Ursus Caïssan beauty matrix $\mathbf{U}$ (1.6.1).

(1) $\text{rank}(\mathbf{U}) = 3$,

(2) $\mathbf{U}$ has index 1, i.e., $\text{rank}(\mathbf{U}) = \text{rank}(\mathbf{U}^2)$,

(3) $\mathbf{U}$ is "keyed", i.e.,

$$\det(\lambda\mathbf{I} - \mathbf{U}) = \lambda^5(\lambda - m)(\lambda^2 - \kappa), \tag{1.6.14}$$

where the magic sum $m = 260$ and the "magic key" $\kappa = 2688$,

(4) all odd powers $\mathbf{U}^{2p+1}$ including the group inverse $\mathbf{U}^{\#}$ are linear in $\mathbf{U}$, and all even powers $\mathbf{U}^{2p}$ are linear in $\mathbf{U}^2$, $p = 0, 1, 2, \ldots$; setting $p = -1$ yields the group inverse $\mathbf{U}^{\#}$,

(5) $\mathbf{U}$ is 4-ply, i.e., the four numbers in each of the 64 subsets of order 2×2 of 4 contiguous numbers (with wrap-around) add up to the same sum $m/2$.

(6) $\mathbf{U}$ is $\mathbf{H}$-associated, i.e., $\mathbf{U} + \mathbf{HUH} = 2m\bar{\mathbf{E}}$, where $\mathbf{H} = \mathbf{F}_2 \otimes \mathbf{I}_4$ is involutory and every entry of $\bar{\mathbf{E}}$ is equal to $1/8$,

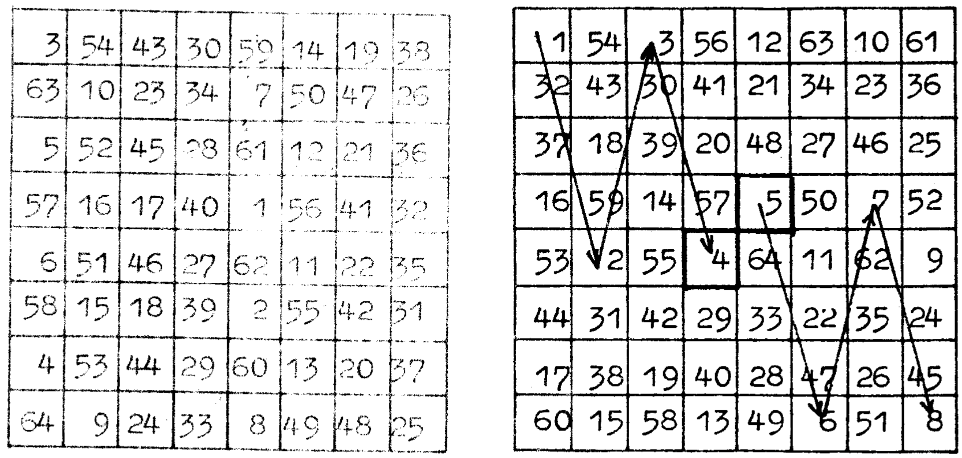
(7) $\mathbf{U}$ is EP, i.e., the Moore–Penrose inverse $\mathbf{U}^+$ commutes with $\mathbf{U}$,

(8) the Moore–Penrose inverse $\mathbf{U}^+$ is $\mathbf{H}$-associated and coincides with the group inverse $\mathbf{U}^{\#}$,

(9) $\mathbf{U}^2$ is symmetric.

---

[18] An anonymous referee notes that in his thesis, Christian Eggermont uses the term "2×2-ply for equiconstant 2-by-2's (4 elements)" for Planck's "4-ply"; see also Derksen, Eggermont & van den Essen [94].



3	54	43	30	59	14	19	38
63	10	23	34	7	50	47	26
5	52	45	28	61	12	21	36
57	16	17	40	1	56	41	32
6	51	46	27	62	11	22	35
58	15	18	39	2	55	42	31
4	53	44	29	60	13	20	37
64	9	24	33	8	49	48	25

1	54	3	56	12	63	10	61
32	43	30	41	21	34	23	36
37	18	39	20	48	27	46	25
16	59	14	57	5	50	7	52
53	2	55	4	64	11	62	9
44	31	42	29	33	22	35	24
17	38	19	40	28	47	26	45
60	15	58	13	49	6	51	8

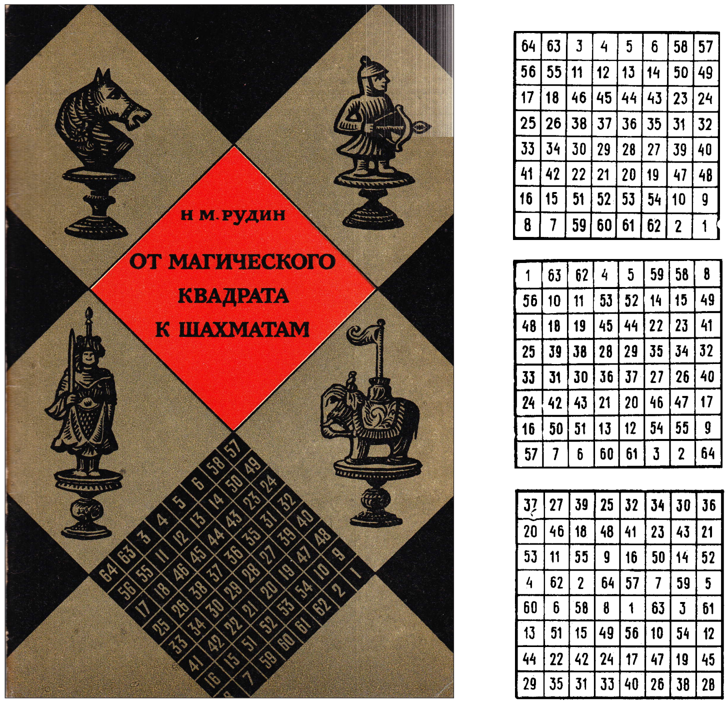
FIGURE 1.6.4: (left panel) $\mathbf{B}_{\mathrm{Csh}}$ from Cashmore [40, Fig. 8 (1907)]; (right panel) "Jaina–Nasik" 8×8 $\mathbf{B}_{\mathrm{Bdv}}$ from Bidev [68, Fig. 18 (1981)] with a broken magic CSP4-path marked.

Two other Caïssa beauties that we have found are given in Fig. 1.6.4. In the left panel $\mathbf{B}_{\mathrm{Csh}}$ comes from Cashmore's 1907 article [40, Fig. 8] on "Chess magic squares" and in the right panel $\mathbf{B}_{\mathrm{Bdv}}$ is from Bidev's "Kritik" [68, Fig. 18 (1981)] of [40] with a broken magic CSP4-path marked. Both $\mathbf{B}_{\mathrm{Csh}}$ and $\mathbf{B}_{\mathrm{Bdv}}$ satisfy conditions (1)–(7) in Table 1.6.1 and neither is EP and neither satisfies conditions (8)–(10). Indeed the Ursus Caïssan beauty matrix $\mathbf{U}$ is the only EP Caïssan beauty that we have found *per se* published in the literature though we recall that the classic Caïssan beauty $\mathbf{B}_{\mathrm{Chd}}$ (1.6.10) is EP.

In our report [113] we posed the following:

Open question 1.6.1. Does there exist an 8×8 classic magic matrix with all CSP2 and CSP3 knight's paths magic but which is *not* pandiagonal?

The answer is "yes" with three such "CSP23 Caïssan magic matrices" $\mathbf{C}_{\mathrm{R1}}$, $\mathbf{C}_{\mathrm{R2}}$, $\mathbf{C}_{\mathrm{R3}}$ given in Rudin [62] and in our Fig. 1.6.5. All three matrices are CSP2- and CSP3-magic and EP but none is pandiagonal! Moreover all three satisfy all 10 conditions in Table 1.6.1 except that condition (7) is satisfied with the involutory matrix $\mathbf{V} = \mathbf{F}$.



64	63	3	4	5	6	58	57
56	55	11	12	13	14	50	49
17	18	46	45	44	43	23	24
25	26	38	37	36	35	31	32
33	34	30	29	28	27	39	40
41	42	22	21	20	19	47	48
16	15	51	52	53	54	10	9
8	7	59	60	61	62	2	1

1	63	62	4	5	59	58	8
56	10	11	53	52	14	15	49
48	18	19	45	44	22	23	41
25	39	38	28	29	35	34	32
33	31	30	36	37	27	26	40
24	42	43	21	20	46	47	17
16	50	51	13	12	54	55	9
57	7	6	60	61	3	2	64

37	27	39	25	32	34	30	36
20	46	18	48	41	23	43	21
53	11	55	9	16	50	14	52
4	62	2	64	57	7	59	5
60	6	58	8	1	63	3	61
13	51	15	49	56	10	54	12
44	22	42	24	17	47	19	45
29	35	31	33	40	26	38	28

FIGURE 1.6.5: From *Ot magičeskogo kvadrata k šakhmatam* [*From Magic Squares to Chess*], by N. M. Rudin [62, (1969)]: (left panel) Cover 1 with the CSP23 magic matrix $\mathbf{C}_{\mathrm{R}2}$; (top right panel) $\mathbf{C}_{\mathrm{R}1}$ [62, No. 4, p. 11]; (centre right) $\mathbf{C}_{\mathrm{R}2}$ [62, No. 29, p. 18], (bottom right) $\mathbf{C}_{\mathrm{R}3}$ [62, No. 30, p. 18].

We note that rearranging the rows and columns of $\mathbf{C}_{\mathrm{R}3}$ yields the Ursus matrix $\mathbf{U}$,

$$\mathbf{C}_{\mathrm{R}3} = \mathbf{P}\mathbf{U}\mathbf{P}', \qquad \mathbf{P}'\mathbf{C}_{\mathrm{R}3}\mathbf{P} = \mathbf{U}, \tag{1.6.15}$$

with the CSP23 Caïssan magic matrix $\mathbf{C}_{\mathrm{R}3}$ and the permutation matrix $\mathbf{P}$,

$$\mathbf{C}_{\mathrm{R}3} = \begin{pmatrix} 37 & 27 & 39 & 25 & 32 & 34 & 30 & 36 \\ 20 & 46 & 18 & 48 & 41 & 23 & 43 & 21 \\ 53 & 11 & 55 & 9 & 16 & 50 & 14 & 52 \\ 4 & 62 & 2 & 64 & 57 & 7 & 59 & 5 \\ 60 & 6 & 58 & 8 & 1 & 63 & 3 & 61 \\ 13 & 51 & 15 & 49 & 56 & 10 & 54 & 12 \\ 44 & 22 & 42 & 24 & 17 & 47 & 19 & 45 \\ 29 & 35 & 31 & 33 & 40 & 26 & 38 & 28 \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.6.16}$$

And so $\mathbf{C}_{\mathrm{R}3}$ and $\mathbf{U}$ have the same characteristic polynomial though $\mathbf{C}_{\mathrm{R}3}$ is $\mathbf{F}$-associated while $\mathbf{U}$ is $\mathbf{H}$-associated.

2. Some magic matrix properties

A key purpose in this report is to identify various matrix-theoretic properties of Caïssan magic squares.

2.1. "V-associated" magic matrices: "H-associated", "F-associated". An important property of $n \times n$ magic matrices involves an $n \times n$ involutory matrix $\mathbf{V}$ that is symmetric, centrosymmetric, and has all row totals equal to 1; it defines an involution in that $\mathbf{V}^2 = \mathbf{I}_n$, the $n \times n$ identity matrix. The matrix $\mathbf{A}$ is centrosymmetric whenever $\mathbf{A} = \mathbf{FAF}$, where $\mathbf{F} = \mathbf{F}_n$ is the $n \times n$ flip matrix:

$$\mathbf{F} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{2.1.1}$$

Definition 2.1.1. We define an $n \times n$ magic matrix $\mathbf{M}$ with magic sum m to be $\mathbf{V}$-associated whenever

$$\mathbf{M} + \mathbf{VMV} = 2m\bar{\mathbf{E}}, \tag{2.1.2}$$

where all the elements of $\bar{\mathbf{E}}$ are equal to $1/n$. Here the involutory matrix $\mathbf{V}$ is symmetric, centrosymmetric, and has all row totals equal to 1 and defines an involution in that $\mathbf{V}^2 = \mathbf{I}_n$, the $n \times n$ identity matrix.

Theorem 2.1.1 [102, p. 21]**.** *The Moore–Penrose inverse* $\mathbf{M}^+$ *of the* $\mathbf{V}$*-associated magic matrix* $\mathbf{M}$ *is also* $\mathbf{V}$*-associated.*

The $n \times n$ Ursus magic matrix $\mathbf{U}$ (1.6.1) with $n = 8$

$$\mathbf{U} = \begin{pmatrix} 1 & 58 & 3 & 60 & 8 & 63 & 6 & 61 \\ 16 & 55 & 14 & 53 & 9 & 50 & 11 & 52 \\ 17 & 42 & 19 & 44 & 24 & 47 & 22 & 45 \\ 32 & 39 & 30 & 37 & 25 & 34 & 27 & 36 \\ 57 & 2 & 59 & 4 & 64 & 7 & 62 & 5 \\ 56 & 15 & 54 & 13 & 49 & 10 & 51 & 12 \\ 41 & 18 & 43 & 20 & 48 & 23 & 46 & 21 \\ 40 & 31 & 38 & 29 & 33 & 26 & 35 & 28 \end{pmatrix} \tag{2.1.3}$$

is $\mathbf{V}$-associated with $\mathbf{V}$ equal to the $n \times n$ centrosymmetric involutory matrix $\mathbf{H}$ (1.6.4) with $n = 2h$ even

$$\mathbf{H} = \begin{pmatrix} \mathbf{0} & \mathbf{I}_h \\ \mathbf{I}_h & \mathbf{0} \end{pmatrix} = \mathbf{F}_2 \otimes \mathbf{I}_h\,, \quad h = n/2\,, \tag{2.1.4}$$

where $\mathbf{F}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the 2×2 "flip matrix". The Ursus matrix $\mathbf{U}$ is, therefore, $\mathbf{H}$-associated.

In an $n \times n$ $\mathbf{H}$-associated magic matrix with magic sum m and $n = 2h$ even, the pairs of entries $h = n/2$ apart along the diagonals all add to $m/4$. And such a matrix is necessarily pandiagonal [31, 78, 96, 97, 104]. The converse holds for $n = 4$ but not for $n \geq 6$.

Theorem 2.1.2. *An* $\mathbf{H}$*-associated* $n \times n$ *magic matrix with* n *even is pandiagonal. When* $n = 4$ *then a pandiagonal magic matrix is* $\mathbf{H}$*-associated.*

Theorem 2.1.3. *An* $\mathbf{H}$*-associated* 8×8 *magic matrix is both special-knight* (CSP3) *magic and alfil-magic.*

The $n \times n$ magic matrix $\mathbf{M}$ with magic sum m is $\mathbf{F}$-associated, whenever

$$\mathbf{M} + \mathbf{FMF} = 2m\bar{\mathbf{E}}, \tag{2.1.5}$$

where $\mathbf{F} = \mathbf{F}_n$ is the $n \times n$ flip matrix. In an $n \times n$ $\mathbf{F}$-associated magic matrix with magic sum m the sums of pairs of entries diametrically equidistant from the centre are all equal to $2m/n$. In the literature an $\mathbf{F}$-associated magic square is often called (just) *associated* (with no qualifier to the word *associated*) or "regular" or "symmetrical" (e.g., Heinz & Hendricks [81, pp. 8, 166]).

Theorem 2.1.4 [84, 89]**.** *Suppose that the magic matrix* $\mathbf{M}$ *is* $\mathbf{F}$*-associated. Then* $\mathbf{M}^2$ *is centrosymmetric.*

Proof. By definition $\mathbf{M} + \mathbf{FMF} = 2m\bar{\mathbf{E}}$, which yields, by pre- and post-multiplication by $\mathbf{M}$ that $\mathbf{MFMF} = 2m^2\bar{\mathbf{E}} - \mathbf{M}^2 = \mathbf{FMFM}$. Hence $\mathbf{FM}^2\mathbf{F} = 2m^2\bar{\mathbf{E}} - \mathbf{FMFM} = 2m^2\bar{\mathbf{E}} - (2m^2\bar{\mathbf{E}} - \mathbf{M}^2) = \mathbf{M}^2$. $\square$

Theorem 2.1.5. *Suppose that the* $n \times n$ *magic matrix* $\mathbf{M}$ *with magic sum* m *is* $\mathbf{H}$*-associated with* $n = 2h$ *even. Then* $\mathbf{M}^2$ *and* $\mathbf{MHM}$ *are block-Latin, i.e.,*

$$\mathbf{M}^2 = \begin{pmatrix} \mathbf{K}_1 & \mathbf{L}_1 \\ \mathbf{L}_1 & \mathbf{K}_1 \end{pmatrix}, \quad \mathbf{MHM} = \begin{pmatrix} \mathbf{K}_2 & \mathbf{L}_2 \\ \mathbf{L}_2 & \mathbf{K}_2 \end{pmatrix} \tag{2.1.6}$$

for some $h \times h$ *matrices* $\mathbf{K}_1, \mathbf{K}_2, \mathbf{L}_1, \mathbf{L}_2$*. Moreover,*

$$\mathbf{K}_1 + \mathbf{L}_2 = \mathbf{K}_2 + \mathbf{L}_1 = m^2\bar{\mathbf{E}}_h, \tag{2.1.7}$$

where $\bar{\mathbf{E}}_h$ *is the* $n \times h$ *matrix with all entries equal to* $1/h$, *with* $h = n/2$.

Proof. By definition, $\mathbf{M}+\mathbf{HMH} = 2m\bar{\mathbf{E}}_n$, which pre- and post-multiplied by $\mathbf{M}$, respectively, yields

$$\mathbf{MHMH} = \mathbf{HMHM} = 2m^2\bar{\mathbf{E}}_n - \mathbf{M}^2. \tag{2.1.8}$$

Let the $n \times h$ matrices $\mathbf{J}_1 = (\mathbf{I}_h, \mathbf{0})'$ and $\mathbf{J}_2 = (\mathbf{0}, \mathbf{I}_h)'$. Then from (2.1.8) we obtain, since $\mathbf{J}_2 = \mathbf{HJ}_1$, that

$$\mathbf{J}_1'\mathbf{MHMJ}_2 = \mathbf{J}_2'\mathbf{MHMJ}_1 = m^2\bar{\mathbf{E}}_h - \mathbf{J}_1'\mathbf{M}^2\mathbf{J}_1 = m^2\bar{\mathbf{E}}_h - \mathbf{J}_2'\mathbf{M}^2\mathbf{J}_2, \tag{2.1.9}$$

$$\mathbf{J}_1'\mathbf{MHMJ}_1 = \mathbf{J}_2'\mathbf{MHMJ}_2 = m^2\bar{\mathbf{E}}_h - \mathbf{J}_1'\mathbf{M}^2\mathbf{J}_2 = m^2\bar{\mathbf{E}}_h - \mathbf{J}_2'\mathbf{M}^2\mathbf{J}_1, \tag{2.1.10}$$

and our proof is complete. □

Theorem 2.1.6 [83, 101]**.** *Suppose that the* $2p \times 2q$ *block-Latin matrix*

$$\mathbf{A} = \begin{pmatrix} \mathbf{K} & \mathbf{L} \\ \mathbf{L} & \mathbf{K} \end{pmatrix}, \tag{2.1.11}$$

where $\mathbf{K}$ *and* $\mathbf{L}$ *are both* $p \times q$. *Then*

$$\mathit{rank}(\mathbf{A}) = \mathit{rank}(\mathbf{K}+\mathbf{L}) + \mathit{rank}(\mathbf{K}-\mathbf{L}). \tag{2.1.12}$$

When the magic matrix $\mathbf{M}$ in Theorem 2.1.5 has rank 3 and index 1, then $\mathbf{M}^2$ has rank 3 and from (2.1.6) it follows that $\mathbf{K}_1 + \mathbf{L}_1$ and $\mathbf{K}_1 - \mathbf{L}_1$ each have rank at most 3, with rank$(\mathbf{K}_1 + \mathbf{L}_1) = 3$ if and only if $\mathbf{L}_1 = \mathbf{K}_1$, and then rank$(\mathbf{K}_1) = 3$. When, however, $\mathbf{M}$ defines a Caïssan beauty with rank 3 and index 1, our findings are that $\mathbf{K}_1 + \mathbf{L}_1$ has rank 2 and $\mathbf{K}_1 - \mathbf{L}_1$ has rank 1, and when the Caïssan beauty has rank 3 and index 3, we find that $\mathbf{M}^2$ has rank 2, and that both $\mathbf{K}_1 + \mathbf{L}_1$ and $\mathbf{K}_1 - \mathbf{L}_1$ have rank 1.

Motivated by an observation of A. C. Thompson [76, (1994)], see also the "A–D method" used by Planck [50, (1919)], we note that reversing the first $h = n/2$ rows and the first $h = n/2$ columns of an $n \times n$ $\mathbf{H}$-associated magic matrix with $n = 2h$ (even) makes it $\mathbf{F}$-associated and vice versa, since $\mathbf{HT} = \mathbf{TF}$ and $\mathbf{FT} = \mathbf{TH}$, where the $n \times n$ involutory *Thompson matrix*

$$\mathbf{T} = \begin{pmatrix} \mathbf{F}_h & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_h \end{pmatrix}, \; h = n/2. \tag{2.1.13}$$

We will refer to this as "Thompson's trick" and applying it to the Ursus matrix $\mathbf{U}$ yields the $\mathbf{F}$-associated "Ursus–Thompson matrix"

$$\mathbf{U}^* = \begin{pmatrix} 37 & 30 & 39 & 32 & 25 & 34 & 27 & 36 \\ 44 & 19 & 42 & 17 & 24 & 47 & 22 & 45 \\ 53 & 14 & 55 & 16 & 9 & 50 & 11 & 52 \\ 60 & 3 & 58 & 1 & 8 & 63 & 6 & 61 \\ 4 & 59 & 2 & 57 & 64 & 7 & 62 & 5 \\ 13 & 54 & 15 & 56 & 49 & 10 & 51 & 12 \\ 20 & 43 & 18 & 41 & 48 & 23 & 46 & 21 \\ 29 & 38 & 31 & 40 & 33 & 26 & 35 & 28 \end{pmatrix}, \tag{2.1.14}$$

which is $\mathbf{F}$-associated but neither pandiagonal nor CSP2-magic.

Open question 2.1.1. Does there exist an 8×8 magic square that is both CSP2-magic and $\mathbf{F}$-associated?

2.2. *4-ply* and the *alternate couplets* property: *4-pac* magic matrices.

McClintock (1897) considered $n \times n$ magic squares with $n = 4k$ doubly-even that have an *alternate couplets* property.

Definition 2.2.1 (McClintock [31, (1897)]). The $n \times n$ magic matrix $\mathbf{M}$, with $n = 4k$ doubly-even, and magic sum m, has the "alternate couplets" property whenever

$$\mathbf{RM} = \mathbf{M}_2\mathbf{J}_3' \tag{2.2.1}$$

for some $n \times 2$ matrix $\mathbf{M}_2$ with row totals $\frac{1}{2}m$. The $2 \times n$ $(= 4k)$ matrix

$$\mathbf{J}_3' = \mathbf{e}_{2k}' \otimes \mathbf{I}_2 = (\mathbf{I}_2,\ \mathbf{I}_2,\ \ldots,\ \mathbf{I}_2), \tag{2.2.2}$$

with $\mathbf{e}_{2k}'$ the $1 \times 2k$ $(= n/2)$ vector with each entry equal to 1, and the $n \times n$ "couplets-summing" matrix

$$\mathbf{R} = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}. \tag{2.2.3}$$

Theorem 2.2.1. *Let the $n \times n$ magic matrix $\mathbf{M}$, with $n = 4k$ doubly-even, and magic sum m, have the "alternate couplets" property* (2.2.1)*. Then the 2×2 matrix*

$$\mathbf{J}_3'\mathbf{M}_2 = m\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} m & m \\ m & m \end{pmatrix}. \tag{2.2.4}$$

Proof. We note first that

$$\mathbf{J}_3'\mathbf{R} = (\mathbf{e}_{2k}' \otimes \mathbf{I}_2)\mathbf{R} = \mathbf{E}_{2,n}, \tag{2.2.5}$$

where $\mathbf{E}_{2,n}$ is the $2 \times n$ matrix with every entry equal to 1. Hence

$$\mathbf{J}_3'\mathbf{RMJ}_3 = \mathbf{E}_{2,n}\mathbf{MJ}_3 = m\mathbf{E}_{2,n}(\mathbf{e}_{2k} \otimes \mathbf{I}_2) = 4m\mathbf{E}_{2,2}\,. \tag{2.2.6}$$

On the other hand we have

$$\mathbf{J}_3'\mathbf{RMJ}_3 = \mathbf{J}_3'\mathbf{M}_2\mathbf{J}_3'\mathbf{J}_3 = 4\mathbf{J}_3'\mathbf{M}_2. \tag{2.2.7}$$

Equating (2.2.6) and (2.2.7) yields (2.2.4) at once. □

Planck ([33, (1900)]) observed that the magic matrix formed from the Ursus matrix $\mathbf{U}$ by flipping (reversing) rows 2–8 is "4-ply".

Definition 2.2.2 (Planck [33, (1900)])**.** The $n \times n$ magic square with $n = 4k$ doubly-even and magic sum m is "4-ply" whenever the four numbers in each of the n^2 subsets of order 2×2 of 4 contiguous numbers (with wrap-around) add up to the same sum $4m/n = m/k$, i.e.,

$$\mathbf{RMR}' = 4m\bar{\mathbf{E}} = \frac{m}{k}\mathbf{E}, \tag{2.2.8}$$

where all the entries of the $n \times n$ matrix $\mathbf{E}$ are equal to 1 and $\bar{\mathbf{E}} = \frac{1}{n}\mathbf{E}$.

Theorem 2.2.2 (McClintock [31, (1897)])**.** *Let $\mathbf{M}$ denote an $n \times n$ magic matrix with $n = 4k$ doubly-even and magic sum m. Then $\mathbf{M}$ is 4-ply* (*Definition* 2.2.2) *if and only if it has the alternate couplets property* (*Definition* 2.2.1)*.*

Proof. If $\mathbf{M}$ has the alternate couplets property, then from (2.2.1)

$$\mathbf{RM} = \mathbf{M}_2\mathbf{J}_3' = \mathbf{M}_2(\mathbf{e}_{2k}' \otimes \mathbf{I}_2); \quad k = n/4\,. \tag{2.2.9}$$

Postmultiplying (2.2.9) by $\mathbf{R}'$ yields (2.2.8)

$$\mathbf{RMR}' = 4m\bar{\mathbf{E}} \tag{2.2.10}$$

at once since the numbers in the rows of $\mathbf{M}_2$ all add to $4m/n = m/k$, and so $\mathbf{M}$ is 4-pac. $\square$

To go the other way our proof is not quite so quick. With $n = 8$ we define

$$\mathbf{Q}_1 = \frac{1}{4}\begin{pmatrix} 3 & 1 & -1 & 1 & -1 & 1 & -1 & -3 \\ -3 & 3 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & -3 & 3 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -3 & 3 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & -3 & 3 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & -3 & 3 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & -3 & 3 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & -3 & 3 \end{pmatrix},$$
$$\mathbf{Q}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}. \tag{2.2.11}$$

We postmultiply (2.2.10) by $\mathbf{Q}_1$. Then (2.2.9) follows since $\mathbf{R}'\mathbf{Q}_1 = \mathbf{Q}_2$ and $\mathbf{EQ}_1 = \mathbf{0}$.

Definition 2.2.3. We will say that the $n \times n$ magic matrix $\mathbf{M}$ with $n = 4k$ doubly-even is *4-pac*[19] whenever (with wrap-around) it is 4-ply (Definition 2.2.2) or equivalently has the alternate couplets property (Definition 2.2.1).

From Theorem 2.2.2 it follows at once, therefore, that if a magic matrix $\mathbf{M}$ is 4-pac, then so its transpose $\mathbf{M}'$. An example is the Ursus matrix $\mathbf{U}$ with

[19]We choose the term *4-pac,* in part, since *ac* are the initial letters of the two words *alternate couplets*. According to *Wikipedia* "A *sixpack* is a set of six canned or bottled drinks, typically soft drink or beer, which are sold as a single unit." In Germany, "Yesterday I had dinner with seven courses!", "Wow, and what did you have?", "Oh, a sixpack and a hamburger!" – [109].

$$\mathbf{U} = \begin{pmatrix} 1 & 58 & 3 & 60 & 8 & 63 & 6 & 61 \\ 16 & 55 & 14 & 53 & 9 & 50 & 11 & 52 \\ 17 & 42 & 19 & 44 & 24 & 47 & 22 & 45 \\ 32 & 39 & 30 & 37 & 25 & 34 & 27 & 36 \\ 57 & 2 & 59 & 4 & 64 & 7 & 62 & 5 \\ 56 & 15 & 54 & 13 & 49 & 10 & 51 & 12 \\ 41 & 18 & 43 & 20 & 48 & 23 & 46 & 21 \\ 40 & 31 & 38 & 29 & 33 & 26 & 35 & 28 \end{pmatrix},$$
$$\mathbf{U}_2 = \begin{pmatrix} 17 & 113 \\ 33 & 97 \\ 49 & 81 \\ 89 & 41 \\ 113 & 17 \\ 97 & 33 \\ 81 & 49 \\ 41 & 89 \end{pmatrix}, \quad (\mathbf{U}')_2 = \begin{pmatrix} 59 & 71 \\ 61 & 69 \\ 63 & 67 \\ 68 & 62 \\ 71 & 59 \\ 69 & 61 \\ 67 & 63 \\ 62 & 68 \end{pmatrix}. \tag{2.2.12}$$

The row totals of $\mathbf{U}_2$ and $(\mathbf{U}')_2$ are all equal to $m/2 = 130$ (but $\mathbf{U}_2 \neq (\mathbf{U}')_2$). The rows of $\mathbf{RU}$ and of $\mathbf{RU}'$ are the sums of successive pairs of rows of $\mathbf{U}$ and of $\mathbf{U}'$ (with wrap-around). These sums "alternate" and are given in "couplets" (pairs) in the corresponding rows of the $n \times 2$ matrices $\mathbf{U}_2$ and $(\mathbf{U}')_2$.

When the magic matrix $\mathbf{M}$ is 4-pac, then from (2.2.1) we see that $\text{rank}(\mathbf{RM}) \leq 2$ with equality when $\mathbf{M}$ is classic. Moreover, from Sylvester's Law of Nullity we find that

$$\text{rank}(\mathbf{M}) \leq \text{rank}(\mathbf{RM}) - \text{rank}(\mathbf{P}) + n = \text{rank}(\mathbf{RM}) + 1 \leq 3 \tag{2.2.13}$$

since $\text{rank}(\mathbf{R}) = n - 1$ ($n \geq 4$). When $\mathbf{M}$ is classic, then from Drury [93] we know that $\text{rank}(\mathbf{M}) \geq 3$, and so we have proved the following theorem.

Theorem 2.2.3. *Let* $\mathbf{M}$ *denote an* $n \times n$ *magic matrix with* $n = 4k$ *doubly-even. If* $\mathbf{M}$ *is 4-pac, then rank*$(\mathbf{M}) \leq 3$. *When* $\mathbf{M}$ *is also classic, then rank*$(\mathbf{M}) = 3$.

The converse of Theorem 2.2.3 does not hold. For example, the classic magic matrix $\mathbf{M}_0$ (2.2.14) generated by Matlab has rank 3 but does not have the alternate couplets property and is not 4-ply, but is $\mathbf{F}$-associated.

$$\mathbf{M}_0 = \begin{pmatrix} 64 & 2 & 3 & 61 & 60 & 6 & 7 & 57 \\ 9 & 55 & 54 & 12 & 13 & 51 & 50 & 16 \\ 17 & 47 & 46 & 20 & 21 & 43 & 42 & 24 \\ 40 & 26 & 27 & 37 & 36 & 30 & 31 & 33 \\ 32 & 34 & 35 & 29 & 28 & 38 & 39 & 25 \\ 41 & 23 & 22 & 44 & 45 & 19 & 18 & 48 \\ 49 & 15 & 14 & 52 & 53 & 11 & 10 & 56 \\ 8 & 58 & 59 & 5 & 4 & 62 & 63 & 1 \end{pmatrix},$$
$$\mathbf{M}_0^* = \begin{pmatrix} 64 & 2 & 61 & 3 & 60 & 6 & 57 & 7 \\ 9 & 55 & 12 & 54 & 13 & 51 & 16 & 50 \\ 40 & 26 & 37 & 27 & 36 & 30 & 33 & 31 \\ 17 & 47 & 20 & 46 & 21 & 43 & 24 & 42 \\ 32 & 34 & 29 & 35 & 28 & 38 & 25 & 39 \\ 41 & 23 & 44 & 22 & 45 & 19 & 48 & 18 \\ 8 & 58 & 5 & 59 & 4 & 62 & 1 & 63 \\ 49 & 15 & 52 & 14 & 53 & 11 & 56 & 10 \end{pmatrix}. \tag{2.2.14}$$

If, however, we switch rows and columns 3 and 4, and rows and columns 7 and 8, then $\mathbf{M}_0$ becomes $\mathbf{M}_0^*$, which has the alternate couplets property and is 4-ply, is pandiagonal, and is $\mathbf{V}$-associated with $\mathbf{V} = \mathbf{F}_4 \otimes \mathbf{I}_2$, but $\mathbf{M}_0^*$ is not $\mathbf{H}$-associated. As we will show below (Theorem 2.2.4), a 4-pac magic matrix is necessarily pandiagonal.

Definition 2.2.4 (McClintock [31, (1897), §16, pp. 110–111])**.** We define an $n \times n$ magic matrix $\mathbf{M}$ with $n = 4k$ doubly-even to be *most-perfect*, or *complete* or *complete most-perfect* or *most-perfect pandiagonal* (MMPM) [78, 79, 80, 82, 90, 108], whenever it is

(1) pandiagonal,

(2) $\mathbf{H}$-associated,

(3) 4-ply,

(4) and has the alternate-couplets property.

We have already shown that properties (3) and (4) in Definition 2.2.4 are equivalent and we introduced the term 4-pac for this. We have also shown that when n is even then condition (2) implies (1). In our next theorem we show that condition (3) implies (1). It follows, therefore, that an $n \times n$ magic matrix $\mathbf{M}$ with $n = 4k$ doubly-even is "most-perfect" whenever it is 4-pac and $\mathbf{H}$-associated. The 8×8 matrix $\mathbf{M}_0^*$ (2.2.14) is 4-pac but not $\mathbf{H}$-associated and the matrix

$$\mathbf{M}_0^{**} = \begin{pmatrix} 1 & 62 & 5 & 59 & 2 & 61 & 12 & 58 \\ 57 & 14 & 50 & 48 & 9 & 18 & 45 & 19 \\ 10 & 27 & 34 & 25 & 54 & 33 & 36 & 41 \\ 49 & 30 & 26 & 23 & 52 & 21 & 22 & 37 \\ 63 & 4 & 53 & 7 & 64 & 3 & 60 & 6 \\ 56 & 47 & 20 & 46 & 8 & 51 & 15 & 17 \\ 11 & 32 & 29 & 24 & 55 & 38 & 31 & 40 \\ 13 & 44 & 43 & 28 & 16 & 35 & 39 & 42 \end{pmatrix} \tag{2.2.15}$$

given by Setsuda [111] is $\mathbf{H}$-associated but not 4-pac, and so neither $\mathbf{M}_0^*$ nor $\mathbf{M}_0^{**}$ is most-perfect.

Theorem 2.2.4 (Pickover [86, p. 73])**.** *A 4-ply magic matrix is necessarily pandiagonal.*

To prove Theorem 2.2.4 we note first that the $n \times n$ magic matrix $\mathbf{M}$ with magic sum m is pandiagonal whenever

$$\mathrm{tr}\mathbf{S}^p\mathbf{M} = \mathrm{tr}\mathbf{S}^p\mathbf{F}\mathbf{M} = m = \mathrm{tr}\mathbf{M} = \mathrm{tr}\mathbf{F}\mathbf{M}, \quad p = 1, 2, \ldots, n-1. \tag{2.2.16}$$

where the *one-step forwards shift matrix*

$$\mathbf{S} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{2.2.17}$$

To prove Theorem 2.2.4, we now use the fact that $\mathbf{S} + \mathbf{I} = \mathbf{R}$, the *couplets-summing matrix* (2.2.3), and Theorem 2.2.1.

2.3. *Keyed* magic matrices and the *magic key*. When at most 3 eigenvalues of the magic matrix are nonzero, then there are two eigenvalues, in addition to the magic-sum eigenvalue which add to 0. This observation leads to

Definition 2.3.1 [96, 97]**.** We define the $n \times n$ magic matrix $\mathbf{M}$ with magic sum m to be *keyed* whenever its characteristic polynomial is of the form

$$\det(\lambda\mathbf{I} - \mathbf{M}) = \lambda^{n-3}(\lambda - m)(\lambda^2 - \kappa), \tag{2.3.1}$$

where the *magic key*

$$\kappa = \tfrac{1}{2}\big(\mathrm{tr}\mathbf{M}^2 - m^2\big) \tag{2.3.2}$$

may be positive, negative or zero.

Definition 2.3.2. The $n \times n$ matrix $\mathbf{A}$ has *index* 1 whenever $\mathrm{rank}(\mathbf{A}^2) = \mathrm{rank}(\mathbf{A})$.

Theorem 2.3.1 [96, 97]**.** *Suppose that the $n \times n$ keyed magic matrix $\mathbf{M}$ has index 1 with magic sum $m \neq 0$ and magic key κ. Then*

$$\kappa \neq 0 \;\Leftrightarrow\; \mathit{rank}(\mathbf{M}) = 3. \tag{2.3.3}$$

Definition 2.3.3. The $n \times n$ index-1 matrix $\mathbf{A}$ has a *group inverse* $\mathbf{A}^{\#}$ which satisfies the three conditions

$$\mathbf{A}\mathbf{A}^{\#}\mathbf{A} = \mathbf{A}, \quad \mathbf{A}^{\#}\mathbf{A}\mathbf{A}^{\#} = \mathbf{A}^{\#}, \quad \mathbf{A}\mathbf{A}^{\#} = \mathbf{A}^{\#}\mathbf{A}. \tag{2.3.4}$$

Theorem 2.3.2 [88, Ex. 11, p. 58]**.** *When $\mathbf{A}$ has index 1, then the group inverse*

$$\mathbf{A}^{\#} = \mathbf{A}(\mathbf{A}^3)^{+}\mathbf{A}, \tag{2.3.5}$$

where $(\mathbf{A}^3)^{+}$ is the Moore–Penrose inverse of $\mathbf{A}^3$.

Theorem 2.3.3 [96, 97]**.** *Let the magic matrix $\mathbf{M}$ with magic sum $m \neq 0$ be keyed with magic key $\kappa \neq 0$ and index 1. Then $\mathbf{M}$ has rank 3 and all odd powers are "linear in the parent" in that*

$$\mathbf{M}^{2p+1} = \kappa^p\mathbf{M} + m(m^{2p} - \kappa^p)\bar{\mathbf{E}}; \quad p = 1, 2, 3, \ldots; \tag{2.3.6}$$

here each element of the $n \times n$ matrix $\bar{\mathbf{E}}$ is equal to $1/n$.

Moreover, the group inverse

$$\mathbf{M}^{\#} = \frac{1}{\kappa}\mathbf{M} + m\left(\frac{1}{m^2} - \frac{1}{\kappa}\right)\bar{\mathbf{E}} \tag{2.3.7}$$

is also linear in the parent. The right-hand side of (2.3.7) is the right-hand side of (2.3.6) with $p = -1$.

2.4. EP matrices. We now consider EP matrices. We believe that the term EP was introduced by Schwerdtfeger[20] [57, p. 130 (1950)]:

> An n-matrix $\mathbf{A}$ may be called an EP$_r$-matrix if it is a P_r-matrix and the linear relations existing among its rows are the same as those among the columns. The $n \times n$ matrix $\mathbf{A}$ is said to be a "P_r" matrix [57, Th. 18.1, p. 130] whenever there is an r-rowed principal submatrix $\mathbf{A}_r$ of rank r.

Ben-Israel & Greville [88, p. 157] call a complex EP matrix "range-Hermitian" and cite Schwerdtfeger [57, (1950)] and Pearl [61, (1959)]. Campbell & Meyer [67, p. 74 (1979)] define a square matrix $\mathbf{A}$ to be EP whenever

$$\mathbf{A}\mathbf{A}^{+} = \mathbf{A}^{+}\mathbf{A}, \tag{2.4.1}$$

i.e., whenever $\mathbf{A}$ commutes with its Moore–Penrose inverse [88, Ex. 16, p. 159]. Oskar Maria Baksalary [110, (2011)] notes that the property (2.4.1) may be called "Equal Projectors", which has the initial letters EP. See also Baksalary, Styan & Trenkler [100, (2009)].

Definition 2.4.1 [67, p. 74]**.** We define a square matrix to be EP whenever the "Equal Projectors" property (2.4.1) holds.

Theorem 2.4.1 [88, Th. 4, p. 157]**.** *The $n \times n$ matrix $\mathbf{A}$ is* EP *if and only if the group and Moore–Penrose inverses coincide, i.e.,* $\mathbf{A}^{\#} = \mathbf{A}^{+}$.

Theorem 2.4.2. *The $n \times n$ matrix $\mathbf{A}$ with index 1 is* EP *if and only if* $\mathbf{A}\mathbf{A}^{+}\mathbf{A}' = \mathbf{A}'$.

Theorem 2.4.3. *Suppose that the magic matrix $\mathbf{M}$ with magic sum $m \neq 0$ has rank 3 and index 1. Then $\mathbf{M}$ is* EP *if and only if $\mathbf{M}^2$ is symmetric.*

Proof. From (2.3.7) we know that the group inverse $\mathbf{M}^{\#}$ is linear in the parent,

$$\mathbf{M}^{\#} = \frac{1}{\kappa}\mathbf{M} + m\left(\frac{1}{m^2} - \frac{1}{\kappa}\right)\bar{\mathbf{E}}, \tag{2.4.2}$$

where the matrix $\bar{\mathbf{E}}$ has every entry equal to $1/n$. The matrix $\mathbf{M}$ is EP if and only if $\mathbf{M}^{\#} = \mathbf{M}^{+}$ (Theorem 2.4.1) or equivalently if and only if the

[20] Hans Wilhelm Eduard Schwerdtfeger (1902–1990) was Professor of Mathematics at McGill University from 1960–1983.

projectors $\mathbf{M}\mathbf{M}^{\#}$ and $\mathbf{M}^{\#}\mathbf{M}$ are symmetric, and the result follows at once from (2.4.2). □

Theorem 2.4.4. *Suppose that the* $n \times n$ *magic matrix* $\mathbf{M}$ *with magic sum* $m \neq 0$ *has rank* 3 *and that* $\mathbf{M}^2$ *is symmetric. Then* $\mathbf{M}$ *has index* 1 *and is* EP (*at least when* $n = 4$).

Proof. If $\mathbf{M}^2$ is symmetric then

$$3 = \operatorname{rank}(\mathbf{M}) \geq \operatorname{rank}(\mathbf{M}^2) = \operatorname{rank}(\mathbf{M}^3) = \cdots \tag{2.4.3}$$

and so $\mathbf{M}^2$ has rank $1, 2$ or 3. Let $\mathbf{M}$ have magic key κ. Then $\mathbf{M}^2$ has 2 eigenvalues equal to κ and the magic eigenvalue $m^2 \neq 0$; the other $n - 3$ eigenvalues are 0. If $\kappa \neq 0$, then $\operatorname{rank}(\mathbf{M}^2) = 3$ and $\mathbf{M}$ has index 1 and is EP (Theorem 2.4.3) and our theorem is established. □

If, however, $\kappa = 0$, then $\operatorname{rank}(\mathbf{M}^2) = 1$. This is impossible when $n = 4$ since from Sylvester's Law of Nullity

$$3 = \operatorname{rank}(\mathbf{M}) \geq \operatorname{rank}(\mathbf{M}^2) \geq 2\operatorname{rank}(\mathbf{M}) - n = 6 - n = 2 \tag{2.4.4}$$

when $n = 4$. More generally for $n \geq 4$

$$\operatorname{rank}(\mathbf{M}^2) = 1 \Rightarrow \mathbf{M}^2 = m^2\bar{\mathbf{E}}. \tag{2.4.5}$$

Open question 2.4.4. Let the $n \times n$ magic matrix $\mathbf{M}$ have magic sum $m \neq 0$ and rank 3. Does there exist such a matrix $\mathbf{M}$ with magic key $\kappa = 0$ and $n \geq 5$ such that $\mathbf{M}^2$ satisfies (2.4.5)?

The squared Ursus matrix $\mathbf{U}^2$ is

$$\mathbf{U}^2 = \begin{pmatrix} 9570 & 8674 & 9122 & 8226 & 8002 & 7554 & 8450 & 8002 \\ 8674 & 9186 & 8354 & 8866 & 7554 & 8386 & 7874 & 8706 \\ 9122 & 8354 & 8930 & 8162 & 8450 & 7874 & 8642 & 8066 \\ 8226 & 8866 & 8162 & 8802 & 8002 & 8706 & 8066 & 8770 \\ 8002 & 7554 & 8450 & 8002 & 9570 & 8674 & 9122 & 8226 \\ 7554 & 8386 & 7874 & 8706 & 8674 & 9186 & 8354 & 8866 \\ 8450 & 7874 & 8642 & 8066 & 9122 & 8354 & 8930 & 8162 \\ 8002 & 8706 & 8066 & 8770 & 8226 & 8866 & 8162 & 8802 \end{pmatrix} \tag{2.4.6}$$

and since $\mathbf{U}$ has rank 3 and index 1, and is keyed with magic key $\kappa = 2688 \neq 0$, it follows, using Theorem 2.4.3, that $\mathbf{U}$ is EP since $\mathbf{U}^2$ is symmetric. And we note that $\mathbf{U}^2$ is block-Latin, which also follows since $\mathbf{U}$ is $\mathbf{H}$-associated (Theorem 2.1.5).

Moreover, using Theorem 2.3.1, we find that the odd powers are all *linear in the parent*

$$\mathbf{U}^{2p+1} = 2688^p\mathbf{U} + 260(260^{2p} - 2688^p)\bar{\mathbf{E}}; \quad p = -1, +1, +2, \ldots \tag{2.4.7}$$

where the 8×8 matrix $\bar{\mathbf{E}}$ has every entry equal to 1/8. Since $\mathbf{U}$ is EP, we see, using Theorem 2.3.2, that the group inverse $\mathbf{U}^{\#}$ coincides with its Moore–Penrose inverse $\mathbf{U}^{+}$:

$$\mathbf{U}^{\#} = \mathbf{U}^{+} = \frac{1}{2688}\mathbf{U} - \frac{4057}{43680}\bar{\mathbf{E}}, \tag{2.4.8}$$

which is (2.4.7) with $p = -1$. We note also that the Ursus matrix $\mathbf{U}$ is 4-pac and $\mathbf{H}$-associated.

Theorem 2.4.5. *Suppose that the magic matrix* $\mathbf{M}$ *is* $\mathbf{F}$*-associated and* EP. *Then the row-flipped matrix* $\mathbf{FM}$ *is* $\mathbf{F}$*-associated and* EP *if and only if the column-flipped matrix* $\mathbf{MF}$ *is* $\mathbf{F}$*-associated and* EP, *and* $\mathbf{M}^2$ *is bisymmetric, i.e., symmetric and centrosymmetric.*

Proof. The row-flipped

$$\begin{aligned}
\mathbf{FM} \text{ is EP} &\Leftrightarrow \mathbf{FM}(\mathbf{FM})^{+}(\mathbf{FM})' = (\mathbf{FM})' && (2.4.9)\\
&\Leftrightarrow \mathbf{FM}(\mathbf{FM})^{+}\mathbf{M}' = \mathbf{M}' && (2.4.10)\\
&\Leftrightarrow (2m\bar{\mathbf{E}} - \mathbf{MF})\big((2/m)\bar{\mathbf{E}} - \mathbf{FM}^{+}\big)\mathbf{M}' = \mathbf{M}' && (2.4.11)\\
&\Leftrightarrow \mathbf{MM}^{+}\mathbf{M}' = \mathbf{M}' && (2.4.12)
\end{aligned}$$

and $\mathbf{M}$ is EP, using Theorems 2.1.4 and 2.4.3. If follows at once from (2.4.1) that $\mathbf{M}$ is EP if and only if $\mathbf{FMF}$ is EP and the result then follows since $\mathbf{F}(\mathbf{FM})\mathbf{F} = \mathbf{MF}$. □

Let $\mathbf{M}_{\mathrm{A}}$ denote the Agrippa "Mercury" magic matrix [2], [74, p. 738][21]. See also [115]. Then $\mathbf{M}_{\mathrm{A}}$ and the column-flipped $\mathbf{M}_{\mathrm{A}}\mathbf{F}$ ([92, p. 49, Fig. 3]) are:

$$\begin{aligned}
\mathbf{M}_{\mathrm{A}} &= \begin{pmatrix}
8 & 58 & 59 & 5 & 4 & 62 & 63 & 1 \\
49 & 15 & 14 & 52 & 53 & 11 & 10 & 56 \\
41 & 23 & 22 & 44 & 45 & 19 & 18 & 48 \\
32 & 34 & 35 & 29 & 28 & 38 & 39 & 25 \\
40 & 26 & 27 & 37 & 36 & 30 & 31 & 33 \\
17 & 47 & 46 & 20 & 21 & 43 & 42 & 24 \\
9 & 55 & 54 & 12 & 13 & 51 & 50 & 16 \\
64 & 2 & 3 & 61 & 60 & 6 & 7 & 57
\end{pmatrix}, \\
\mathbf{M}_{\mathrm{A}}\mathbf{F} &= \begin{pmatrix}
1 & 63 & 62 & 4 & 5 & 59 & 58 & 8 \\
56 & 10 & 11 & 53 & 52 & 14 & 15 & 49 \\
48 & 18 & 19 & 45 & 44 & 22 & 23 & 41 \\
25 & 39 & 38 & 28 & 29 & 35 & 34 & 32 \\
33 & 31 & 30 & 36 & 37 & 27 & 26 & 40 \\
24 & 42 & 43 & 21 & 20 & 46 & 47 & 17 \\
16 & 50 & 51 & 13 & 12 & 54 & 55 & 9 \\
57 & 7 & 6 & 60 & 61 & 3 & 2 & 64
\end{pmatrix}
\end{aligned} \tag{2.4.13}$$

and are both $\mathbf{F}$-associated and EP since

$$\mathbf{M}_{\mathrm{A}}^2 = \begin{pmatrix}
7330 & 9346 & 9122 & 8002 & 8226 & 8450 & 8226 & 8898 \\
9346 & 7714 & 7874 & 8866 & 8706 & 8354 & 8514 & 8226 \\
9122 & 7874 & 7970 & 8834 & 8738 & 8258 & 8354 & 8450 \\
8002 & 8866 & 8834 & 8098 & 8130 & 8738 & 8706 & 8226 \\
8226 & 8706 & 8738 & 8130 & 8098 & 8834 & 8866 & 8002 \\
8450 & 8354 & 8258 & 8738 & 8834 & 7970 & 7874 & 9122 \\
8226 & 8514 & 8354 & 8706 & 8866 & 7874 & 7714 & 9346 \\
8898 & 8226 & 8450 & 8226 & 8002 & 9122 & 9346 & 7330
\end{pmatrix}, \tag{2.4.14}$$

$$(\mathbf{M}_{\mathrm{A}}\mathbf{F})^2 = \begin{pmatrix}
9570 & 7554 & 7778 & 8898 & 8674 & 8450 & 8674 & 8002 \\
7554 & 9186 & 9026 & 8034 & 8194 & 8546 & 8386 & 8674 \\
7778 & 9026 & 8930 & 8066 & 8162 & 8642 & 8546 & 8450 \\
8898 & 8034 & 8066 & 8802 & 8770 & 8162 & 8194 & 8674 \\
8674 & 8194 & 8162 & 8770 & 8802 & 8066 & 8034 & 8898 \\
8450 & 8546 & 8642 & 8162 & 8066 & 8930 & 9026 & 7778 \\
8674 & 8386 & 8546 & 8194 & 8034 & 9026 & 9186 & 7554 \\
8002 & 8674 & 8450 & 8674 & 8898 & 7778 & 7554 & 9570
\end{pmatrix} \tag{2.4.15}$$

are both bisymmetric (and, maybe surprisingly, are quite different).

In his pamphlet [11, (1845)] describing *A New Method of Ascertaining Interest and Discount*, Israel Newton includes *A Few Magic Squares of a Singular Quantity*. One of these magic squares is 16×16 and dated "September 28, 1844, in the 82nd year of his age" and "containing 4 squares of

[21]From Swetz [87, p.117], we note that the Agrippa "Mercury" magic square [74, p. 738], see also Paracelsus [4], is called the "Jupiter" magic square by Girolamo Cardano (1501–1576) [3].

8 and 16 squares of 4"[22]. We define this 16×16 magic square as the magic Newton matrix[23] $\mathbf{N}$ by

$$\begin{pmatrix}
1 & 254 & 255 & 4 & 121 & 252 & 8 & 133 & 118 & 11 & 247 & 138 & 244 & 113 & 15 & 142 \\
128 & 131 & 130 & 125 & 134 & 7 & 251 & 122 & 137 & 248 & 12 & 117 & 14 & 143 & 241 & 116 \\
132 & 127 & 126 & 129 & 135 & 6 & 250 & 123 & 140 & 245 & 9 & 120 & 141 & 16 & 114 & 243 \\
253 & 2 & 3 & 256 & 124 & 249 & 5 & 136 & 119 & 10 & 246 & 139 & 115 & 242 & 144 & 13 \\
159 & 30 & 100 & 225 & 25 & 230 & 103 & 156 & 106 & 150 & 235 & 23 & 145 & 148 & 111 & 110 \\
97 & 228 & 158 & 31 & 104 & 155 & 26 & 229 & 107 & 151 & 234 & 22 & 112 & 109 & 146 & 147 \\
226 & 99 & 29 & 160 & 154 & 101 & 232 & 27 & 152 & 108 & 21 & 233 & 18 & 19 & 240 & 237 \\
32 & 157 & 227 & 98 & 231 & 28 & 153 & 102 & 149 & 105 & 24 & 236 & 239 & 238 & 17 & 20 \\
33 & 164 & 95 & 222 & 168 & 37 & 217 & 92 & 86 & 215 & 41 & 172 & 84 & 209 & 46 & 175 \\
96 & 221 & 34 & 163 & 91 & 218 & 38 & 167 & 43 & 170 & 88 & 213 & 174 & 47 & 212 & 81 \\
162 & 35 & 224 & 93 & 90 & 219 & 39 & 166 & 216 & 85 & 171 & 42 & 211 & 82 & 173 & 48 \\
223 & 94 & 161 & 36 & 165 & 40 & 220 & 89 & 169 & 44 & 214 & 87 & 45 & 176 & 83 & 210 \\
191 & 68 & 65 & 190 & 200 & 59 & 186 & 69 & 53 & 203 & 182 & 76 & 208 & 51 & 77 & 178 \\
194 & 61 & 64 & 195 & 185 & 70 & 199 & 60 & 202 & 56 & 73 & 183 & 177 & 78 & 52 & 207 \\
62 & 193 & 196 & 63 & 71 & 188 & 57 & 198 & 75 & 181 & 204 & 54 & 50 & 205 & 179 & 80 \\
67 & 192 & 189 & 66 & 58 & 197 & 72 & 187 & 184 & 74 & 55 & 201 & 79 & 180 & 206 & 49
\end{pmatrix}. \tag{2.4.16}$$

Both the Newton matrix $\mathbf{N}$ (2.4.16) and its column-flipped partner $\mathbf{NF}$ are (surprisingly) EP but $\mathbf{N}$ is not $\mathbf{F}$-associated (in fact $\mathbf{N}^+$ is not fully-magic). The 16×16 Newton matrix $\mathbf{N}$ has rank 13 and index 1 and each of the 4 magic 8×8 submatrices and its 4 column-flipped partners are EP (all with rank 7 and index 1).

2.5. Checking that an 8×8 magic matrix is CSP2-magic. To check that an 8×8 magic matrix $\mathbf{M}$ is CSP2-magic (regular-knight-magic) we check that four 4×4 related matrices are $\mathbf{H}$-associated.

Theorem 2.5.1. *The 32 CSP2-paths in the 8×8 magic matrix $\mathbf{M}$ are all magic if and only if the four 4×4 matrices*

$$\mathbf{J}_1'\mathbf{K}_2\mathbf{MJ},\ \mathbf{J}_2'\mathbf{K}_2\mathbf{MJ},\ \mathbf{J}_1'\mathbf{K}_2\mathbf{M}'\mathbf{J},\ \mathbf{J}_2'\mathbf{K}_2\mathbf{M}'\mathbf{J}, \tag{2.5.1}$$

are all $\mathbf{H}$-associated (and hence pandiagonal). Here the 8×4 matrices

$$\mathbf{J}_1 = \begin{pmatrix} \mathbf{I}_4 \\ \mathbf{0} \end{pmatrix}, \quad \mathbf{J}_2 = \begin{pmatrix} \mathbf{0} \\ \mathbf{I}_4 \end{pmatrix}, \quad \mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2 = \begin{pmatrix} \mathbf{I}_4 \\ \mathbf{I}_4 \end{pmatrix}, \tag{2.5.2}$$

[22] There are also 5 more magic squares: 3 are 8×8 and 2 are 4×4.

[23] We have corrected several typos in the original given by Newton [11].

where $\mathbf{I}_4$ *is the* 4×4 *identity matrix, and the knight-selection matrix*

$$\mathbf{K}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.5.3}$$

The *knight-selection matrix* $\mathbf{K}_2$ is almost a regular-knight's move (CSP2) matrix but the "move" (with wrap-around) from row 4 to row 5 is that of a special-knight (CSP3) rather than that of the usual knight (CSP2) in chess.

We recall the Ursus matrix

$$\mathbf{U} = \begin{pmatrix} \mathbf{1} & 58 & 3 & 60 & \mathbf{8} & 63 & 6 & 61 \\ 16 & 55 & 14 & 53 & 9 & 50 & 11 & 52 \\ 17 & \mathbf{42} & 19 & 44 & 24 & \mathbf{47} & 22 & 45 \\ 32 & 39 & 30 & 37 & 25 & 34 & 27 & 36 \\ 57 & 2 & \mathbf{59} & 4 & 64 & 7 & \mathbf{62} & 5 \\ 56 & 15 & 54 & 13 & 49 & 10 & 51 & 12 \\ 41 & 18 & 43 & \mathbf{20} & 48 & 23 & 46 & \mathbf{21} \\ 40 & 31 & 38 & 29 & 33 & 26 & 35 & 28 \end{pmatrix}, \tag{2.5.4}$$

where we have identified a CSP2-magic path in red. We find that

$$\begin{aligned} \mathbf{K}_2\mathbf{U} &= \begin{pmatrix} \mathbf{1} & 58 & 3 & 60 & \mathbf{8} & 63 & 6 & 61 \\ 17 & \mathbf{42} & 19 & 44 & 24 & \mathbf{47} & 22 & 45 \\ 57 & 2 & \mathbf{59} & 4 & 64 & 7 & \mathbf{62} & 5 \\ 41 & 18 & 43 & \mathbf{20} & 48 & 23 & 46 & \mathbf{21} \\ 16 & 55 & 14 & 53 & 9 & 50 & 11 & 52 \\ 32 & 39 & 30 & 37 & 25 & 34 & 27 & 36 \\ 56 & 15 & 54 & 13 & 49 & 10 & 51 & 12 \\ 40 & 31 & 38 & 29 & 33 & 26 & 35 & 28 \end{pmatrix}, \\ \mathbf{K}_2\mathbf{U}' &= \begin{pmatrix} 1 & 16 & 17 & 32 & 57 & 56 & 41 & 40 \\ 3 & 14 & 19 & 30 & 59 & 54 & 43 & 38 \\ 8 & 9 & 24 & 25 & 64 & 49 & 48 & 33 \\ 6 & 11 & 22 & 27 & 62 & 51 & 46 & 35 \\ 58 & 55 & 42 & 39 & 2 & 15 & 18 & 31 \\ 60 & 53 & 44 & 37 & 4 & 13 & 20 & 29 \\ 63 & 50 & 47 & 34 & 7 & 10 & 23 & 26 \\ 61 & 52 & 45 & 36 & 5 & 12 & 21 & 28 \end{pmatrix}, \end{aligned} \tag{2.5.5}$$

and hence

$$\mathbf{J}_1'\mathbf{K}_2\mathbf{UJ} = \begin{pmatrix} \mathbf{9} & 121 & 9 & 121 \\ 41 & \mathbf{89} & 41 & 89 \\ 121 & 9 & \mathbf{121} & 9 \\ 89 & 41 & 89 & \mathbf{41} \end{pmatrix}, \quad \mathbf{J}_2'\mathbf{K}_2\mathbf{UJ} = \begin{pmatrix} 25 & 105 & 25 & 105 \\ 57 & 73 & 57 & 73 \\ 105 & 25 & 105 & 25 \\ 73 & 57 & 73 & 57 \end{pmatrix}, \tag{2.5.6}$$

$$\mathbf{J}_1'\mathbf{K}_2\mathbf{U}'\mathbf{J} = \begin{pmatrix} 58 & 72 & 58 & 72 \\ 62 & 68 & 62 & 68 \\ 72 & 58 & 72 & 58 \\ 68 & 62 & 68 & 62 \end{pmatrix}, \quad \mathbf{J}_2'\mathbf{K}_2\mathbf{U}'\mathbf{J} = \begin{pmatrix} 60 & 70 & 60 & 70 \\ 64 & 66 & 64 & 66 \\ 70 & 60 & 70 & 60 \\ 66 & 64 & 66 & 64 \end{pmatrix} \tag{2.5.7}$$

are indeed all $\mathbf{H}$-associated and pandiagonal and so all 32 regular knight's (CSP2) paths are magic.

2.6. Checking that an 8×8 magic matrix is CSP3-magic. To check that an 8×8 magic matrix $\mathbf{M}$ is CSP3-magic (special-knight magic), we compute $\mathbf{K}_3\mathbf{M}$, where the 8×8 CSP3 (special knight) selection matrix

$$\mathbf{K}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \tag{2.6.1}$$

is symmetric and magic, and commutes with $\mathbf{H}$. This leads to

Theorem 2.6.1. *The 8×8 magic matrix $\mathbf{M}$ is* CSP3-*magic if and only if $\mathbf{K}_3\mathbf{M}$ is pandiagonal, where 8×8* CSP3 (*special knight*) *selection matrix $\mathbf{K}_3$ is defined in* (2.6.1). *If $\mathbf{M}$ is also $\mathbf{H}$-associated, then so is $\mathbf{K}_3\mathbf{M}$ and hence $\mathbf{K}_3\mathbf{M}$ is pandiagonal and $\mathbf{M}$ is* CSP3-*magic.*

Acknowledgements. Special thanks go to Miguel Angel Amela, Oskar Maria Baksalary, Christian Boyer, Ka Lok Chu, S. W. Drury, Tõnu Kollo, Daniel J. H. Rosenthal, and Götz Trenkler for their help.

We are also most grateful to two anonymous referees and to Katherine M. Ammerlaan, Nicolas C. Ammerlaan, Thomas W. Ammerlaan, Victoria E. Bain, Philip V. Bertrand, Robert E. Bradley, Eva Brune, Yogendra P. Chaubey, Pavel Chebotarev, Linda Rose Childs, Richard Lee Childs, Elizabeth Dunkley, Harvey D. Heinz, Jeffrey J. Hunter, Henry G. Kitts, Peter D. Loly, Owen S. Martin, Amir Memartoluie, Simo Puntanen, Janice Simpkins, Evelyn Matheson Styan, and Jani A. Virtanen.

We found several hard-to-find references, including [36, 40, 62, 64, 68, 69], in the Bibliotheca Van Der Linde–Niemeijeriana, the Chess Collection at the Royal Library in The Hague in July 2011. This research was supported in part by the Natural Sciences and Engineering Research Council of Canada.

References

[1] [1356/1942-Nārāyaṇa] *The Gaṇita Kaumudī by Nārāyaṇa Paṇḍita (Part II)*, in Sanskrit, edited and with introduction in English by Pandit Padmākara Dvivedī Jyautishāchārya, Princess of Wales Saraswvati Bhavana Texts, no. 57, Government Sanskrit College, Benares, 1942: online at Internet Archive. Reprinted as *The Ganitha Koumudi Part-2,* by Narayana Pandita, in Sanskrit, edited and with introduction in English by Pandit Padmākara Dvivedī Jyautishāchārya, Nabu Public Domain Reprint, 2011; first published in 1356. [See also Kusuba [73], Shukla [72], Singh [85],] 110, 111, 113, 140

[2] [1533-Agrippa] *De occulta philosophia libri tres,* by Henrici Cornelii Agrippae ab Nettesheym [Heinrich Cornelius Agrippa von Nettesheim (1486–1535)], pub. Soter, Köln [Cologne], July 1533. Reprinted 1550 online at Swiss electronic library, Universitätsbibliothek Basel. 130

[3] [1539-Cardano] *Practica arithmetice et mensurandi singularis : in qua que preter alias coninentur, versa pagina demonstrabit,* by Hieronimi C. Cardani [Gerolamo Cardano (1501–1576)] medico mediolanonsis. pub. Mediolani: Io. Antonins Castellioneus medidani imprimebat, impensis Bernardini Calusci, 312 pp., 1539: online (Google eBook). 130

[4] [1567-Paracelsus] *Archidoxa Magica,* attributed to Paracelsus [Theophrastus Bombastus von Hohenheim (1493–1541)], pub. 1567. 130

[5] [1691-La Loubère/*royaume*] *Du royaume de Siam, Tome Second: Contenant plusieurs Piéces détachées,* by [Simon] de La Loubère, Envoyé extraordinaire du ROY auprés du Roy de Siam en 1687 & 1688, pub. Chez La Veuve de Jean Baptiste Coignard, Imprimeur & Libraire ordinaire du Roy, et Jean Baptiste Coignard, Imprimeur & Libraire ordinaire du Roy, Paris, 1691. 134

[6] [1693-La Loubère/*Kingdom*] *The Kingdom of Siam,* by Simon de La Loubère, in English translated from French [5, (1691)], reprint edition, pub. Oxford University Press, 1986. [See vol. 2, pp. 227–247.] 108

[7] [1763-Jones/Caïssa] "Caissa, or The game at chess, a poem" by Sir William Jones, written in the year 1763, In Sir William Jones's *Poems* [8, pp. 149–170]. 99, 134, 140, 142

[8] [1772-Jones/*Poems*] *Poems, consisting chiefly of translations from the Asiatick languages, to which are added two essays: I. On the poetry of the Eastern nations, II. On the arts, commonly called imitative,* by Sir William Jones, pub. Oxford University Press, 1772: online (Google eBook). See also [7, 103, 77]. 134, 140, 142

[9] [1810-Jones/*Works*] *The Poetical Works of Sir William Jones : with the life of the author,* by Richard Dagley, pub. J. Nichols & Son, London, 1810. [Frontispiece (Fig. 1.3.1 (right panel): online at *Chess Notes* by Edward Winter.] 100

[10] [1844-*Scotland*] *Travels in Scotland,* by J. G. Kohl [Johann Georg Kohl (1808–1878)], translated from the German *Reisen in Schottland* into English by John Kesson[24] (d. 1876), With notes by the translator, in correction or elucidation of Mr. Kohl's observations, pub. Bruce and Wyld, London, 1844: online (Google eBook). 103

[11] [1844/1845-Newton] *A New Method of Ascertaining Interest and Discount, at various rates, both simple and compound, and of interest on indorsements: also a few magic squares of a singular quality,* by Israel Newton, of Norwich, Vt., pub. E. P. Walton and Sons, printers, Montpelier VT, 16 pp., 1845. 130, 131

[12] [1846-*Childhood*] *The Childhood of King Erik Menved: An historical romance,* by B. S. Ingemann [Bernhard Severin Ingemann (1789–1862)], translated from the Danish *Erik Menveds Bamdom* into English by John Kesson (d. 1876), Library of Foreign Romance and Nobel Newspaper, vol. VII, pub. Bruce and Wyld, London, 1846: online (Google eBook). 103

[13] [1854-Kesson/*Cross*] *The Cross and the Dragon; or, The Fortunes of Christianity in China: with notices of the Christian missions and missionaries, and some account of the Chinese Secret Societies,* by John Kesson (of the British Museum), pub. Smith, Elder, and Co., London, 1854: online (Google eBook). 103

[14] [1865-Frost] "Invention of magic cubes, and construction of magic squares possessing additional properties", by Rev. A. H. Frost (communicated by Rev. Percival Frost), *The Quarterly Journal of Pure and Applied Mathematics,* vol. 7, no. 25 (February 1865), pp. 92–96 & vol. 7, no. 26 (June 1865), pp. 97–102: complete *Quarterly Journal* vol. 7 (1865/1866): online (Google eBook). 98, 112

[15] [1866-Frost] "Supplementary note on Nasik cubes", by Andrew Frost, *The Quarterly Journal of Pure and Applied Mathematics,* vol. 8, no. 29 (1866), p. 74: complete *Quarterly Journal* vol. 8 (1866/1867): online (Google eBook). 112

[16] [1877a-Frost] "On the general properties of Nasik squares", by A. H. Frost, *The Quarterly Journal of Pure and Applied Mathematics,* vol. 15, no. 57 (June 1877), pp. 34–49: complete *Quarterly Journal* vol. 15 (1877/1878): online (Google eBook). 98, 101, 110, 111, 112

[17] [1877b-Frost] "On the general properties of Nasik cubes", by A. H. Frost, *The Quarterly Journal of Pure and Applied Mathematics,* vol. 15, no. 57 (June 1877), pp. 93–96 & vol. 15, no. 58 (November 1877), pp. 97–123: complete *Quarterly Journal* vol. 15 (1878): online (Google eBook). 98

[18] [1877c-Frost] "Description of Plates 3 to 9", by A. H. Frost, *The Quarterly Journal of Pure and Applied Mathematics,* vol. 15, no. 60 (June 1878), pp. 366–368: complete *Quarterly Journal* vol. 15 (1878): online (Google eBook). [Plates 2–5 at end of no. 58 (November 1877); Plates 6–9 at end no. 59 (April 1878).] 98

[19] [1879-*Penzance*] *The Pirates of Penzance,* a comic opera in two acts, with music by Arthur Sullivan and libretto by W. S. Gilbert. [The opera's official première was at the Fifth Avenue Theatre in New York City on 31 December 1879.] 101

[20] [1879-Ursus] "Magic squares and circles", by "Ursus", *The Queen, The Lady's Newspaper & Court Chronicle,* vol. 65, p. 577 (June 21, 1879), vol. 66, pp. 4–5 (July 5, 1879), vol. 66, p. 66 [sic] (July 19, 1879), vol. 66, [within pp. 165–186: missing in McGill microfilm] (August 23, 1879) & vol. 66, p. 215 (6 September 6, 1879). 103

[21] [1880-Ursus] "Trees in rows", by "Ursus", *The Queen, The Lady's Newspaper & Court Chronicle,* vol. 67, p. 298 (April 3, 1880), vol. 67, pp. 350–351 (April 17,

[24] We conjecture that Henry James Kesson was born c. 1844 and that the translator John Kesson (d. 1876) was his father.

1880), vol. 67, p. 474 (May 22, 1880), vol. 67, pp. 500–501 (May 29, 1880), vol. 68, p. 54 (July 17, 1880), vol. 68, p. 127 (July 24, 1880). 103

[22] [1881-Ursus] "Caïssan magic squares", by "Ursus", *The Queen: The Lady's Newspaper & Court Chronicle,* vol. 70, p. 142 (August 6, 1881), vol. 70, pp. 276–277 (September 10, 1881), vol. 70, p. 391 (October 15, 1881): online at McGill. 97, 99, 101, 105, 106, 109, 111, 112

[23] [1882-Frost/EB9] "Magic square", by A. H. F., In *The Encyclopædia Britannica: A Dictionary of Arts, Sciences, and General Literature,* Ninth Edition, vol. 15, pp. 213–216, Popular Reprint, pub. Henry G. Allen, New York, 1882 & Charles Scribner's Sons, New York, 1883. 112

[24] [1892-Falkener] *Games Ancient and Oriental and How to Play Them: Being the Games of the Ancient Egyptians, the Hiera Gramme of the Greeks, the Ludus Latrunclulorum of the Romans and the Oriental Games of Chess, Draughts, Backgammon and Magic Squares,* by Edward Falkener [Edward Lyon Falkener (1814–1896], pub. Longmans, Green and Company, London, [5] + 366 pp., 1892: online (Google eBook). "Unabridged and corrected republication" by Dover, New York, 1961. [See also review [25, (1998)].] 109, 136

[25] [1892/1998-Falkener/Finkel-review] "Edward Falkener: old board games for new", by Irving L. Finkel, review of *Games Ancient and Oriental and How to Play Them* [24], In *Board Games Studies,* vol. 1, pp. 104–108 (1998): online. 99, 136

[26] [1894-Cavendish] *Recreations with Magic Squares: The eight queens' problem solved by magic squares and domino squares,* by "Cavendish" [Henry Jones (1831–1899)] pub. Thomas de la Rue, London, 1894: online (Google eBook). [Reprinted: Kessinger Legacy Reprints, 2010.] 108, 112

[27] [1894-*Blacksmiths*/Hancock1] *Merry Blacksmiths,* A School Song, in Tonic Sol-fa and Staff Notations, music by Benjamin John W. Hancock, words by H. J. Kesson, pub. Beal & Co., London, 1894. [(Founded on Handel's Harmonious Blacksmith.) 105

[28] [1894-*Peasants*/Hancock2] *The Peasants,* action song, in Staff and Tonic sol-fa Notations, music by Benjamin John W. Hancock, words by H. J. Kesson, pub. Beal & Co., London, 1894. 105

[29] [c. 1894-*Shoe*/Hancock3] *The Magic Shoe,* an operetta-cantata for young people (Tonic sol-fa), music by Benjamin John W. Hancock, words by H. J. Kesson, pub. J. Curwen & Sons, London, c. 1894. 105

[30] [1896-Frost] "The construction of Nasik squares of any order", by Rev. A. H. Frost, *Proceedings of the London Mathematical Society,* vol. 27, no. 1, pp. 487–518: online at Oxford University Press Journals Digital Archive & Highwire Press Oxford University Press Archive. 98, 99

[31] [1897-McClintock] "On the most perfect forms of magic squares, with methods for their production", by Emory McClintock [John Emory McClintock (1840–1916)], *American Journal of Mathematics,* vol. 19, no. 2, pp. 99–120 (1897): online at JSTOR. 112, 118, 120, 121, 124

[32] [1899-Percival/obit] "Dr. Percival Frost", by Dr. George Bruce Halsted, *The American Mathematical Monthly,* vol. 6, nos. 8–9, pp. 189–191 (August–September 1899): online at JSTOR. [Includes a portrait (p. 190).] 98

[33] [1900-Planck/n-queens] "The n queens problem," by C. Planck, *The British Chess Magazine,* vol. 20, no. 3, pp. 94–97 (March 1900). 108, 112, 114, 121

[34] [1901-*Boating*/Hancock4] *A Boating Song,* music by Benjamin John W. Hancock, words by H. J. Kesson, Choruses for equal Voices no. 389, pub. J. Curwen & Sons, London (printed in Leipzig), 1901. 105

[35] [1901-*Railway*/Hancock5] *The Railway Journey,* a school song, music by Benjamin John W. Hancock, words by H. J. Kesson [For S.S.C.], Choruses for equal Voices, no. 390, pub. J. Curwen & Sons, London (printed in Leipzig), 1901. 105

[36] [1902a-MacMahon] "Magic squares and other problems on a chess-board", by Major P. A. MacMahon [Percy Alexander MacMahon (1854–1929)], *Proceedings of the Royal Institution of Great Britain,* vol. 17, pp. 50–63 (14 February 1902). [Reprinted as [37].] 134, 137

[37] [1902b-MacMahon] "Magic squares and other problems upon a chess-board: A discourse delivered at the Royal Institution on Friday evening, February 14", by Major P. A. MacMahon, F.R.S., *Nature,* vol. 65, no. 1689, pp. 447–452 (March 13, 1902): online at Nature Publishing. [Reprint of [36].] 137

[38] [1904-LincolnImp] *The Legend of the Lincoln Imp,* by H. J. Kesson (Ursus), pub. J. W. Ruddock & Sons, Lincoln, [16] p. 15 cm., 1904. [Reprinted 1907, 1911, 1919, 1922, 1923, 1925, 1927, 1930. Apparently also reprinted 1965. Book is listed in "Curwen's 1895 Publications" as by "Ursus", price 4d., postage 2d. Dedicated "To my friend E. B. K. D." Booklet signed by "H. J. Kesson (Ursus)" on title page.]105

[39] [1905-Planck/PathNasiks] *The Theory of Path Nasiks,* by C. Planck, printed [apparently for private circulation] by A. J. Lawrence, Printer, Rugby [England], [1] + 18 pp. (1905). Copy from the British Library, 106

[40] [1907-Cashmore] "XI. Chess magic squares", by M. Cashmore, M.P.S. *Report of the South African Association for the Advancement of Science, Cape Town,* vol. 3, pp. 83–90 (1907). [Typewritten copy (apparently typed by Pavle Bidev, c. 1975) at KBH KSR 564, 8 pp. unpaginated. See also Bidev [68, (1981)].] 115, 134, 139

[41] [1913-Murray] *A History of Chess,* by H. J. R. Murray [Harold James Ruthven Murray (1868–1955)], pub. Oxford University Press, 1913; reprinted 1962; for Oxbow Books, Oxford [2002?] & by Benjamin Press, 1985. [For Caïssa, see pp. 793, 874.] 103

[42] [1916-Woodruff/Editor] "Four-ply pandiagonal associated magic squares", by "Editor" [Paul Carus], *The Monist: A Quarterly Magazine Devoted to the Philosophy of Science,* vol. 26, pp. 315–316 (1916). ["Frederic A. Woodruff has sent us three original magic squares, ...; apparently not reoriented in *Magic Squares and Cubes* WSA-1917.] 106

[43] [1916a-Planck/*Monist*] "General rule for constructing ornate magic squares of orders $\equiv 0 \pmod{4}$" by C. Planck, *The Monist: A Quarterly Magazine Devoted to the Philosophy of Science,* vol. 26, pp. 463–470 (1916): online at the Internet Archive. Article reset and republished (signed C. P.) in *Magic Squares and Cubes* [45, in ch. XV: Ornate magic squares, pp. 376–383]. 106

[44] [1916b-Planck/*Monist*] "Ornate magic squares of composite odd orders", by C. Planck (article signed "C. Planck" [*sic*]), *The Monist: A Quarterly Magazine Devoted to the Philosophy of Science,* vol. 26, pp. 470–476 (1916). 106

[45] [1917-Andrews] *Magic Squares and Cubes,* by W. S. Andrews [William Symes Andrews (1847–1929)], with chapters by other writers [H. M. Kingery, D. F. Savage, Harry A. Sayles, Frederic A. Woodruff (b. 1855) & John Worthington], 2nd edition, revised and enlarged, pub. Open Court, Chicago, 1917: Reprinted by Dover (1960), Cosimo Classics (2004): online at David Moews's home page. 106, 108, 109, 137, 138

[46] [1917-Woodruff/Andrews] "The construction of ornate magic squares of orders $8, 12$ and 16 by tables", by "F. A. W." [Frederic A. Woodruff], In *Magic Squares and Cubes* [45, pp. 390–404]. [Apparently not published in *The Monist.*] 106

[47] [1917/2012-KohtzBrunner] *Das Magische Quadrat und die Erfindung des Schachspiels*, by Johannes Kohtz & Erich Brunner, handwritten manuscript,

ca. 100 pp., section by Kohtz dated April 1917, Sächsische Landesbibliothek – Staats- und Universitätsbibliothek Dresden (SLUB), pdf file Mscr.Dresd.App.157 (2012): online in Dresden. 106

[48] [1918-Foster] "Puzzles of arrangement: mental calistenics for the T.B.M. and the W.R.W. (War-Racked-Woman)", by R. F. Foster, *Vanity Fair,* vol. 9, no. 5, pp. 65 & 90 (March 1918). 106

[49] [1918-Kohtz] Unpublished investigation by Johannes Kohtz cited by [66]. 106

[50] [1919-Planck/*Monist*] "Pandiagonal magics of orders 6 and 10 with minimal numbers", by C. Planck (article signed C. P.), *The Monist: Devoted to the Philosophy of Science,* vol. 29, pp. 307–316 (1919): online at the Internet Archive. [Article published in 1919 and not reprinted in Andrews [45, 2nd ed., 1917].] 119

[51] [1934-JainaHymns] "A note on Jaina hymns and magic squares", by Hiralal R. Kapadia, *The Indian Historical Quarterly,* vol. 10, pp. 148–153 (1934).111

[52] [1940-Marder1] *The Magic Squares of Benjamin Franklin: the first of a series of four papers describing the technique of Leonhard Euler applied to the Lahireian method of forming magic squares of all sizes under the general title "The Intrinsic Harmony of Number", Three entirely new methods of producing "The Bent Diagonals of Benjamin Franklin",* by Clarence C. Marder, pub. Edmond Byrne Hackett, The Brick Row Book Shop, New York, 1940. [See also [53, 54, 55].] 106, 108

[53] [1941-Marder2] *Magic squares of the Fifth and Seventh Orders, a new application of the method used by Claude Gaspar Bachet de Mezeriac : the second of a series of four papers describing the technique of Leonhard Euler applied to the Lahireian method of forming magic squares of all sizes under the general title "The Intrinsic Harmony of Number",* by Clarence C. Marder, pub. Edmond Byrne Hackett, The Brick Row Book Shop, New York. 1941. 138

[54] [1941-Marder3] *Magic squares of the Orders Three, Six, Nine and Twelve : the third of a series of papers describing the technique of Leonhard Euler applied to the Lahireian method of forming magic squares of all sizes under the general title "The Intrinsic Harmony of Number",* by Clarence C. Marder, pub. Edmond Byrne Hackett, The Brick Row Book Shop, New York. 1941. 138

[55] [1941-Marder4] *The Auxiliary Square: used with four or more identical magic squares to construct the larger sizes from* 10×10 *to* 64×64 *: the last of a series of four papers under the general title "The Intrinsic Harmony of Number",* by Clarence C. Marder, pub. Edmond Byrne Hackett, The Brick Row Book Shop, New York. 1941. 138

[56] [1941-Vijayaraghavan] "On Jaina magic squares", by T. Vijayaraghavan, *The Mathematics Student,* vol. 9, pp. 97–102. 111

[57] [1950/1961-Schwerdtfeger] *Introduction to Linear Algebra and the Theory of Matrices,* by Hans Schwerdtfeger, pub. P. Noordhoff N. V., Groningen, 280 pp., 1950; 2nd edition: 289 pp., 1961. [For EP matrices see pp. 130–131.] 127

[58] [1956-Fox1] Untitled problem: 4707, "A magic matrix is one whose elements are the numbers of a magic square, ..." , by Charles Fox, *The American Mathematical Monthly,* vol. 63, no. 8, p. 584 (October 1956): online at JSTOR. [Solution by F. D. Parker [60]. See also [59].] 98, 138

[59] [1956-Fox2] "[Mathematical Note] 2617: Magic matrices", by Charles Fox [Charles Fox (1897–1977)], *The Mathematical Gazette,* vol. 40, no. 333, pp. 209–211 (October 1956): online at JSTOR. [See also [58, 60].] 98, 138

[60] [1957-FoxParker] "4707: Magic matrix", problem proposed by Charles Fox [58], with solution by F. D. Parker, *The American Mathematical Monthly,* vol. 64, no. 8, p. 584 (October 1957): online at JSTOR. 98, 138

[61] [1959-Pearl] "On normal and EPr matrices", by Martin Pearl, *The Michigan Mathematical Journal,* vol. 6, no. 1, pp. 1–5 (1959): online at Project Euclid. 127

[62] [1969-Rudin] *Ot magičeskogo kvadrata k šakhmatam* [*From Magic Squares to Chess*], by N. M. Rudin, in Russian, pub. "Prošveščenie", Moscow, 46 pp., 1969. [See also Fig. 1.6.5.] 115, 116, 134

[63] [1974/1997-Horn] *The Victorian Country Child,* by Pamela Horn, pub. The Roundwood Press, Kineton, 1974. Reprinted: Illustrated History Paperback Series, pub. Alan Sutton (September 1997). 104

[64] [1977-Bidev/*Bonus*] "Vom genetischen Code-Algorithmus des indischen Schach im magischen Quadrat der Acht – Neueste Erkenntnisse" [in German: "On genetic code-algorithms of Indian chess in 8×8 magic squares – Newest findings"] by Pavle Bidev, In *Bonus Socius: Bijdragen tot de cultuurgeschiedenis van het schaakspel en andere bordspelen, jubileumuitgave voor Meindert Niemeijer ter gelegenheid van zijn 75ste verjaardag*[25]*,* samengesteld door [compiled by] C. Reedijk [Cornelis Reedijk (1921–2000)] & K. W. Kruijswijk [Karel Wendel Kruijswijk (b. 1921)] pub. Onder auspiciën van de Koninklijke Bibliotheek, 's-Gravenhage, 296 pp., 1977. 134

[65] [1977-Golombek] *The Encyclopedia of Chess,* edited by Harry Golombek (1911–1995), pub. Batsford, London, 1977. ["Caissa: The muse or goddess of chess", p. 54; see our Fig. 1.3.3 (left panel).] 102

[66] [1978-Whyld] "Quotes & Queries, no. 3942: Arising from Professor Bidev's article [**?**] ...", by K. Whyld [Kenneth Whyld (1926–2003)], *The British Chess Magazine,* vol. 98, pp. 521–522 (1978). [See our Fig. 1.5.1.] 107, 108, 109, 138

[67] [1979/1991/2008-Campbell/Meyer] *Generalized Inverses of Linear Transformations,* by S. L. Campbell & C. D. Meyer, Jr. [Stephen L. Campbell & Carl D. Meyer,], pub. Pitman, London, xi + 272 pp., 1979. [Reprinted by Dover, New York, 1991; Classics in Applied Mathematics vol. 56, SIAM, Philadelphia, 2008.] 127

[68] [1981-Bidev/Cashmore] "Bedenken gegen Anzahlbestimmung von Schach-Nasiks 8×8: Kritik der Arbeit von M. Cashmore, Capetown 1907: Chess magic squares" [40, (1907)], by Pavle Bidev, typewritten manuscript (in German), 16 unnumbered pages, issued "Igalo YU', c. 1981. ["Thoughts against the total counts of 8×8 Nasik chess magic squares: critique of the work [40, 1907)].] 115, 134, 137

[69] [1986-Bidev/*Stammt*] *Stammt Schach aus Alt Indien oder China? Teil I, Deutsch, Teil II, Englisch = Did chess originate in China or India: chess and magic squares* by Pavle Bidev, Selbstverlag, Igalo 1986, iii, [i], 304, 88 pp. 302 pages in German and 88 pages in English. ["Expounds theories about origin of chess related to 8×8 magic squares."] 134

[70] [1986-Singh] "Nārāyaṇa's treatment of magic squares", by Parmanand Singh, *Indian Journal of History of Science,* vol. 21, no. 2, pp. 123–130 (1986). 111

[71] [1987-Hayashi] "Varāhamihira's pandiagonal magic square of the order four", by Takao Hayashi, *Historia Mathematica,* vol. 14, pp. 159–166 (1987): online at ScienceDirect. [See also [95, 98].] 141

[72] [1992-Datta+Singh/Shukla] "Magic squares in India," by Bibhutibhusan Datta & Awadhesh Narayan Singh (revised by Kripa Shankar Shukla), *Indian Journal of History of Science,* vol. 72, no. 1, pp. 51–120 (1992). 111, 113, 134

[25] *Bonus Socius: Contributions to the Cultural History of Chess and Other Board Games, Festschrift for Meindert Niemeijer on the occasion of his 75th birthday.* Meindert Niemeijer (1902–1987) was a Dutch chess composer, generous patron and collector of chess literature. His library formed the basis of one of the greatest collections of chess books in the world."

[73] [1993-Kusuba] *Combinatorics and Magic Squares in India: A Study of Nārāyaṇa Paṇḍita's "Gaṇitakaumudīi', Chapters 13–14,* by Takanori Kusuba, Ph.D. dissertation, Department of History of Mathematics, Brown University, May 1993. 111, 134, 140

[74] [1993-Agrippa/Tyson] *Three Books of Occult Philosophy,* written by Henry Cornelius Agrippa of Nettesheim [Heinrich Cornelius Agrippa von Nettesheim (1486–1535)], completely annotated, with modern commentary, and The Foundation Book of Western Occultism, translated by James Freake; edited by Donald Tyson. Llewellyns Source Book Series, Llewellyn Publications, St. Paul, Minnesota 1993. GHB [Appendix V: Magic squares, pp. 733–761.] 130

[75] [1994-Calvo/lightning] "Some flashes of lightning into the impenetrable darkness of chess origins", by Ricardo Calvo [Ricardo Calvo Mínguez (1943–2002)], lecture delivered at a meeting of Chess Collectors International, St. Petersburg, June 1994, 15 pp. 106

[76] [1994-Thompson] "Odd magic powers", by A. C. Thompson [Anthony Charles Thompson (b. 1937)], *The American Mathematical Monthly*, vol. 101, no. 4, pp. 339–342 (April 1994): online at JSTOR. 119

[77] [1995-Caissa/Who] "Who is Caissa? Caissa is the "patron goddess" of chess players: online © 1995–2010 Caissa's Web and Valdina LLC. See also [7, 8]. 134

[78] [1998-OllerenshawBrée/book] *Most-Perfect Pandiagonal Magic Squares: Their Construction and Enumeration,* by [Dame] Kathleen Ollerenshaw & David S. Brée, with a foreword by [Sir] Hermann Bondi, pub. The Institute of Mathematics and its Applications, Southend-on-Sea, Essex, England, 1998. 118, 124

[79] [1998-BréeOllerenshaw/*Today*] "Pandiagonal magic squares from mixed auxiliary squares", by David Brée & [Dame] Kathleen Ollerenshaw, *Mathematics Today: Bulletin of the Institute of Mathematics and its Applications,* vol. 34, no. 4, pp. 105–110 (1998). 124

[80] [1998-OllerenshawBrée/*Today*] "Most-perfect pandiagonal magic squares", [Dame] Kathleen Ollerenshaw & David S. Brée *Mathematics Today: Bulletin of the Institute of Mathematics and its Applications*, vol. 34, no. 5, pp. 139–143 (1998). 124

[81] [2000-*Lexicon*] *Magic Square Lexicon: Illustrated,* by Harvey D. Heinz & John R. Hendricks, pub. HDH, Surrey, British Columbia, 2000. 118

[82] [2001-TrenklerTrenkler/*Image*] "Magic squares, melancholy and the Moore–Penrose inverse", by Dietrich Trenkler & Götz Trenkler, *Image: The Bulletin of the International Linear Algebra Society,* no. 27, pp. 3–10 (2001): online at the International Linear Algebra Society. 124

[83] [2001-TianStyan] "How to establish universal block-matrix factorizations", by Yongge Tian & George P. H. Styan, *The Electronic Journal of Linear Algebra,* vol. 8, pp. 115–127 (2001): online at the International Linear Algebra Society. 119

[84] [2002-AbuJeib] "Centrosymmetric matrices: properties and an alternative approach", by Iyad T. Abu-Jeib, *Canadian Applied Mathematics Quarterly,* vol. 10, no. 4, pp. 429—445 (Winter 2002). 118

[85] [2002-Singh/Narayana] "The *Gaṇitakaumudī* by Nārāyaṇa Paṇḍita", by P. Singh [Parmanand Singh], *Gaṇita Bhāratī,* vol. 24, pp. 35–98 (2002). [See also [1, 73].] 134

[86] [2002/2003-Pickover] *The Zen of Magic Squares, Circles, and Stars: An Exhibition of Surprising Structures across Dimensions*, by Clifford A. Pickover, 2nd printing and first paperback printing, pub. Princeton University Press, 2003. [2nd printing (2003) has section on "Updates and Breakthroughs", pp. 395–401. Original version: 2002.] 125

[87] [2002/2008-Swetz] *Legacy of the Luoshu: The 4,000 Year Search for the Meaning of the Magic Square of Order Three,* 2nd Edition, by Frank J. Swetz, pub. A K Peters, Wellesley MA. [2nd edition apparently just a reprinting of the original version: Open Court, Chicago, 2002.] 130

[88] [2003-BIG2] *Generalized Inverses: Theory and Applications,* by Adi Ben-Israel & Thomas N. E. Greville, Second edition. CMS Books in Mathematics/Ouvrages de mathématiques de la SMC, pub. Springer, New York, 2003. [Original version: Wiley, New York, 1974; revised printing: Robert E. Krieger, 1980.] 126, 127

[89] [2004-AbuJeib] "Centrosymmetric and skew-centrosymmetric matrices and regular magic squares", by Iyad T. Abu-Jeib, *New Zealand Journal of Mathematics,* vol. 33, pp. 105–112 (2004). 118

[90] [2004-TrenklerTrenkler/IJMEST] "Most-perfect pandiagonal magic squares and their Moore-Penrose inverse", by Dietrich Trenkler & Götz Trenkler, *International Journal of Mathematical Education in Science and Technology*, vol. 35, no. 5, pp. 697–701 (2004). 124

[91] [2004-Watkins] *Across the Board: The Mathematics of Chessboard Problems,* by John J. Watkins, pub. Princeton University Press, 2004.

[92] [2005-Hankin] Robin K. S. Hankin (2005). "Recreational mathematics with *R*: introducing the "magic" package. *R News: The Newsletter of the R Project,* vol. 5, no. 1, pp. 48–51 (May 2005): online at *The R Journal*. 130

[93] [2007-Drury/no rank2] "There are no [classic] magic squares of rank 2", by S. W. Drury, Personal communication, August 2007. 123

[94] [2007-Derksen+2] "Multimagic squares", by Harm Derksen, Christian Eggermont, Arno van den Essen, *The American Mathematical Monthly,* vol. 114, no. 8, pp. 703–713 (October 2007): online at JSTOR. 114

[95] [2007-Hayashi] "Magic squares in Indian mathematics", by Takao Hayashi, In *Encyclopaedia of Classical Indian Sciences: Natural Science, Technology and Medicine,* edited by Helaine Selin & Roddam Narasimha, pub. Universities Press, Hyderabad, pp. 222–233 (2007). [See also [71, 98].] 139, 141

[96] [2007-Styan/SSC1] "An illustrated (philatelic) introduction to magic "matrices and statistics", with special emphasis on magic matrices having 3 nonzero eigenvalues", by George P. H. Styan, Invited talk presented in the session on Matrices and Statistics at the SSC Annual Meeting, St. John's, Newfoundland, June 13, 2007: beamer file (148 pp.): online at McGill, Shortened version: [97]. 118, 125, 126, 141

[97] [2007-Styan/SSC2] "Some comments on magic matrices with at most 3 non-zero eigenvalues", by George P. H. Styan, Revised (and shortened) version of the invited talk [96], beamer file (42 pp.): online at McGill, September 18, 2007. 118, 125, 126, 141

[98] [2008-Hayashi/Selin2] "Magic squares in Indian mathematics", by Takao Hayashi, In *Encyclopaedia of the History of Science, Technology, and Medicine in Non-Western Cultures,* 2nd edition, edited by Helaine Selin, pub. Springer-Verlag Berlin, [vol. 2, part 13] pp. 1252–1259 (2008): online at SpringerLink. [See also [71, 95].] 139, 141

[99] [c. 2009-Frost/bio-Boyer]"Andrew Hollingworth Frost (1819–1907), Biographical sketch" by Christian Boyer: online at the Multimagic squares website by Christian Boyer. Undated but c. 2009. 98

[100] [2009-BST] "On a matrix decomposition of Hartwig and Spindelböck", by Oskar Maria Baksalary, George P. H. Styan & Götz Trenkler, *Linear Algebra and its Applications,* vol. 430, no. 10, pp. 2798–2812 (1 May 2009): online at ScienceDirect. 127

[101] [2009-PJS/CPS] "Some comments on philatelic Latin squares from Pakistan", by Ka Lok Chu, Simo Puntanen & George P. H. Styan, *Pakistan Journal of Statistics,* vol. 25, no. 4, pp. 427–471: online. 119

[102] [2010-CDST] "Magic generalized inverses, with special emphasis on involution-associated magic matrices: Preliminary version", by Ka Lok Chu, S. W. Drury, George P. H. Styan & Götz Trenkler, Report 2010-02, Dept. of Mathematics and Statistics, McGill University, Montréal, 33 pp., May 25, 2010. 117

[103] [2010-Caissa/ChessDryad] "CAISSA (pronounced ky-eé-sah) or The Game at Chess; a Poem" (written in the year 1763, by Sir William Jones): online at Chess-Dryad/California Chess History website, November 2010. ["And fair Caissa was the damsel nam'd. Mars saw the maid; with deep surprize he gaz'd, ... And still he press'd, yet still Caissa frown'd; ..." See also [7, 8].] 134

[104] [2010-Styan/Pisa] "Magic generalized inverses, with special emphasis on involution-associated magic matrices", by George P. H. Styan, invited paper presented in the Minisymposium on Generalized Inverses and Applications at The 16th Conference of the International Linear Algebra Society, Pisa, Italy, 21 June 2010: beamer file online at McGill, December 15, 2010. 118

[105] [2011-Planck/biobib] "A bio-bibliography for Charles Planck *père et fils*", by George P. H. Styan, Report 2011-02, Dept. of Mathematics and Statistics, McGill University, Montréal. 106

[106] [2011-Drury/46080] "List of 46080 Caïssan magic squares: pandiagonal and CSP2- and CSP3-magic", Personal communication from S. W. Drury to George P. H. Styan, 6 February 2011. 114

[107] [2011-Jelliss/LinksKTN] "Knight's Tour Notes Index", compiled by George Peter Jelliss, 2000–2011: online. [Website table of contents with links to many articles by Jelliss related to the knight's tour. Transferred to this 'Mayhematics' site February–March 2011.]

[108] [2011-Trenkler/MPPM] "Most-perfect pandiagonal magic squares of order four", Personal communication from Götz Trenkler to George P. H. Styan: 14 pp. pdf file, 1 April 2011. 124

[109] [2011-seven] "Yesterday, in Germany, I had dinner with seven courses!", "Wow, and what did you have?", "Oh, a sixpack and a hamburger!" Personal communication from Götz Trenkler to George P. H. Styan, 26 March 2011. 122

[110] [2011-EP/OMB] "EP matrices have Equal Projectors", Personal communication from Oskar Maria Baksalary, via Götz Trenkler, to George P. H. Styan, 2011. 127

[111] [2011-Setsuda] "8×8 Symmetrical and/or Pan-Magic Squares by Mr. [Kanji] Setsuda", on Mutsumi Suzuki's Magic Squares online at Drexel University, Philadelphia, accessed through Suzuki [112] on 2 May 2011. 125

[112] [2011-Suzuki] "Magic squares", by Mutsumi Suzuki: online at Drexel University, Philadelphia, accessed on 2 May 2011. ["These pages were written by Mutsumi Suzuki and until his retirement in 2001, were available through his site in Japan. It is the Math Forum's pleasure to host these pages so that the mathematical community can continue to enjoy all of the information presented by Mr. Suzuki on the topic of magic squares."] 142

[113] [2011-Tartu] "Caïssan squares: the magic of chess", by George P. H. Styan, Special talk presented at The 9th Tartu Conference on Multivariate Statistics & The 20th International Workshop on Matrices and Statistics, Tartu, Estonia, 26 June–1 July 2011: beamer file online at McGill, talk based on Report 2011-05 from the McGill Department of Mathematics and Statistics: online at McGill. 97, 115

[114] [2012-*Scott*] *Scott Standard Postage Stamp Catalogue* (James E. Kloetzel, ed.), published annually (currently in 6 volumes on paper) by Scott Publishing, Sidney, Ohio. 104

[115] [2012-Manipal-Workshop] "An introduction to Yantra magic squares and Agrippa–Cardano type magic matrices", by George P. H. Styan, invited talk prepared for the International Workshop on Combinatorial Matrix Theory and Generalized Inverses of Matrices, Manipal University, Manipal (Karnataka), India, 2–7 January 2012: website. Beamer file online at McGill and video online at McGill. 130

[116] [2012-Amela1] "Caïssan squares: report for George Styan", by Miguel Angel Amela, 27 January 2012. 111

[117] [2012-Amela2] "Narayana/Frost Caïssan squares: report 2 for George Styan", by Miguel Angel Amela, 30 January 2012. 111

[118] [2012-Amela3] "Gift to George Styan: in memory of Nārāyaṇa Paṇḍita", by Miguel Angel Amela, 31 January 2012. 111, 113

McGill University, Department of Mathematics and Statistics, Burnside Hall 1005, 805 ouest rue Sherbrooke, Montréal (Québec), Canada H3A 0B9

E-mail address: george.styan@mcgill.ca