

## Sharp theorems on multipliers in harmonic function spaces in higher dimension

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ABSTRACT. We present new sharp results concerning multipliers in various spaces of harmonic functions on the unit ball of  $\mathbb{R}^n$ .

### 1. Introduction and preliminaries

The aim of this paper is to describe spaces of multipliers between certain spaces of harmonic functions on the unit ball. We note that so far there are no results in this direction in the multidimensional case, where the use of spherical harmonics is a natural substitute for power series expansion. In fact, even the case of the unit disc has not been extensively studied in this context. We refer the reader to [6], where multipliers between harmonic Bergman type classes were considered, and to [4] and [3] for the case of harmonic Hardy classes. Most of our results are present in these papers in the special case of the unit disc.

Let  $\mathbb{B}$  be the open unit ball in  $\mathbb{R}^n$ ,  $\mathbb{S} = \partial\mathbb{B}$  is the unit sphere in  $\mathbb{R}^n$ , for  $x \in \mathbb{R}^n$  we have  $x = rx'$ , where  $r = |x| = \sqrt{\sum_{j=1}^n x_j^2}$  and  $x' \in \mathbb{S}$ . The normalized Lebesgue measure on  $\mathbb{B}$  is denoted by  $dx = dx_1 \dots dx_n = r^{n-1} dr dx'$ , so that  $\int_{\mathbb{B}} dx = 1$ . We denote the space of all harmonic functions in an open set  $\Omega$  by  $h(\Omega)$ . In this paper letter  $C$  designates a positive constant, which can change its value even in the same chain of inequalities.

For  $0 < p < \infty$ ,  $0 \leq r < 1$  and  $f \in h(\mathbb{B})$  we set

$$M_p(f, r) = \left( \int_{\mathbb{S}} |f(rx')|^p dx' \right)^{1/p},$$

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with the usual modification to cover the case  $p = \infty$ . Weighted Hardy spaces are defined, for  $\alpha \geq 0$  and  $0 < p \leq \infty$ , by

$$H_\alpha^p(\mathbb{B}) = H_\alpha^p = \{f \in h(\mathbb{B}) : \|f\|_{p,\alpha} = \sup_{r < 1} M_p(f,r)(1-r)^\alpha < \infty\}.$$

For  $\alpha = 0$  the space  $H_\alpha^p$  is denoted simply by  $H^p$ .

For  $0 < p \leq \infty$ ,  $0 < q \leq \infty$  and  $\alpha > 0$  we consider mixed (quasi-)norms  $\|f\|_{p,q;\alpha}$  defined by

$$\|f\|_{p,q;\alpha} = \left( \int_0^1 M_q(f,r)^p (1-r^2)^{\alpha p - 1} r^{n-1} dr \right)^{1/p}, \quad f \in h(\mathbb{B}), \quad (1)$$

again with the usual interpretation for  $p = \infty$ , and the corresponding spaces

$$B_\alpha^{p,q}(\mathbb{B}) = B_\alpha^{p,q} = \{f \in h(\mathbb{B}) : \|f\|_{p,q;\alpha} < \infty\}.$$

It is not hard to show that these spaces are complete metric spaces and that for  $\min(p, q) \geq 1$  they are Banach spaces. These spaces include weighted Bergman spaces  $A_\beta^p(\mathbb{B}) = A_\beta^p = B_{\frac{\beta+1}{p}}^{p,p}$ , where  $\beta > -1$  and  $0 < p < \infty$ . We set  $A_\beta^\infty = B_\beta^{\infty,\infty}$  for  $\beta > 0$ .

Note that  $A_\alpha^\infty = H_\alpha^\infty$  for  $\alpha \geq 0$  and  $B_\alpha^{\infty,q} = H_\alpha^q$  for  $0 < q \leq \infty$ ,  $\alpha > 0$ . We also have, for  $0 < p_0 \leq p_1 \leq \infty$ ,  $B_\alpha^{p_0,1} \subset B_\alpha^{p_1,1}$  (see [1]).

Next we need certain facts on spherical harmonics and the Poisson kernel (see [6] for a detailed exposition). Let  $Y_j^{(k)}$  be the spherical harmonics of order  $k$ ,  $1 \leq j \leq d_k$ , on  $\mathbb{S}$ . Let

$$Z_{x'}^{(k)}(y') = \sum_{j=1}^{d_k} Y_j^{(k)}(x') \overline{Y_j^{(k)}(y')}$$

be the zonal harmonics of order  $k$ . Note that the spherical harmonics  $Y_j^{(k)}$ , ( $k \geq 0$ ,  $1 \leq j \leq d_k$ ) form an orthonormal basis of  $L^2(\mathbb{S}, dx')$ . Every  $f \in h(\mathbb{B})$  has an expansion

$$f(x) = f(rx') = \sum_{k=0}^{\infty} r^k b_k \cdot Y^k(x'),$$

where  $b_k = (b_k^1, \dots, b_k^{d_k})$ ,  $Y^k = (Y_1^{(k)}, \dots, Y_{d_k}^{(k)})$  and  $b_k \cdot Y^k$  is interpreted in the scalar product sense:  $b_k \cdot Y^k = \sum_{j=1}^{d_k} b_k^j Y_j^{(k)}$ . To stress dependence on a function  $f \in h(\mathbb{B})$ , we often write  $b_k = b_k(f)$  and  $b_k^j = b_k^j(f)$ , in fact, we have linear functionals  $b_k^j$ ,  $k \geq 0$ ,  $1 \leq j \leq d_k$  on  $h(\mathbb{B})$ .

We denote the Poisson kernel for the unit ball by  $P(x, y')$ , it is defined by

$$\begin{aligned} P(x, y') &= P_{y'}(x) = \sum_{k=0}^{\infty} r^k \sum_{j=1}^{d_k} Y_j^{(k)}(y') Y_j^{(k)}(x') \\ &= \frac{1}{n\omega_n} \frac{1 - |x|^2}{|x - y'|^n}, \quad x = rx' \in \mathbb{B}, \quad y' \in \mathbb{S}, \end{aligned}$$

where  $\omega_n$  is the volume of the unit ball in  $\mathbb{R}^n$ . We are also going to use a Bergman kernel for  $A_\beta^p$  spaces, namely, the function

$$Q_\beta(x, y) = 2 \sum_{k=0}^{\infty} \frac{\Gamma(\beta + 1 + k + n/2)}{\Gamma(\beta + 1)\Gamma(k + n/2)} r^k \rho^k Z_{x'}^{(k)}(y'), \quad x = rx', \quad y = \rho y' \in \mathbb{B}. \quad (2)$$

For details on this kernel we refer to [1], where the following theorem can be found.

**Theorem 1** (see [1]). *Let  $p \geq 1$  and  $\beta \geq 0$ . Then for every  $f \in A_\beta^p$  and  $x \in \mathbb{B}$  we have*

$$f(x) = \int_0^1 \int_{\mathbb{S}^{n-1}} Q_\beta(x, y) f(\rho y') (1 - \rho^2)^\beta \rho^{n-1} d\rho dy', \quad y = \rho y'.$$

The following lemma provides estimates for the kernel  $Q_\beta$  (see [1], [2]).

**Lemma 1.** 1) *Let  $\beta > 0$ . Then for  $x = rx', y = \rho y' \in \mathbb{B}$  we have*

$$|Q_\beta(x, y)| \leq \frac{C}{|\rho x - y'|^{n+\beta}}.$$

2) *Let  $\beta > -1$ . Then*

$$\int_{\mathbb{S}^{n-1}} |Q_\beta(rx', y)| dx' \leq \frac{C}{(1 - r\rho)^{1+\beta}}, \quad |y| = \rho, \quad 0 \leq r < 1.$$

3) *Let  $\beta > n - 1$ ,  $0 \leq r < 1$  and  $y' \in \mathbb{S}^{n-1}$ . Then*

$$\int_{\mathbb{S}^{n-1}} \frac{dx'}{|rx' - y'|^\beta} \leq \frac{C}{(1 - r)^{\beta-n+1}}.$$

**Lemma 2** (see [1]). *Let  $\alpha > -1$  and  $\lambda > \alpha + 1$ . Then*

$$\int_0^1 \frac{(1 - r)^\alpha}{(1 - r\rho)^\lambda} dr \leq C(1 - \rho)^{\alpha+1-\lambda}, \quad 0 \leq \rho < 1.$$

**Lemma 3.** *Let  $G(r)$ ,  $0 \leq r < 1$ , be a positive increasing function. Then for  $\alpha > -1$ ,  $\beta > -1$ ,  $\gamma \geq 0$  and  $0 < q \leq 1$  we have*

$$\left( \int_0^1 G(r) \frac{(1 - r)^\beta}{(1 - \rho r)^\gamma} r^\alpha dr \right)^q \leq C \int_0^1 G(r)^q \frac{(1 - r)^{\beta q + q - 1}}{(1 - \rho r)^{q\gamma}} r^\alpha dr, \quad 0 \leq \rho < 1. \quad (3)$$

A special case of the above lemma appears in [5]. For reader's convenience we present a proof.

*Proof.* We use a subdivision of  $I = [0, 1)$  into subintervals  $I_k = [r_k, r_{k+1})$ ,  $k \geq 0$ , where  $r_k = 1 - 2^{-k}$ . Since  $1 - \rho r_k \asymp 1 - \rho r_{k+1}$ ,  $0 \leq \rho < 1$ , we have

$$\begin{aligned}
J &= \left( \int_0^1 G(r) \frac{(1-r)^\beta}{(1-\rho r)^\gamma} r^\alpha dr \right)^q = \left( \sum_{k \geq 0} \int_{I_k} G(r) \frac{(1-r)^\beta}{(1-\rho r)^\gamma} r^\alpha dr \right)^q \\
&\leq \sum_{k \geq 0} \left( \int_{I_k} G(r) \frac{(1-r)^\beta}{(1-\rho r)^\gamma} r^\alpha dr \right)^q \leq C \sum_{k \geq 0} 2^{-kq\beta} G^q(r_{k+1}) \left( \int_{I_k} \frac{r^\alpha dr}{(1-\rho r)^\gamma} \right)^q \\
&\leq C \sum_{k \geq 0} 2^{-kq\beta} G^q(r_{k+1}) 2^{-kq} (1 - \rho r_{k+1})^{-q\gamma} \\
&\leq C \sum_{k \geq 0} 2^{-kq\beta} G^q(r_{k+1}) 2^{-kq} (1 - \rho r_k)^{-q\gamma} \\
&\leq C \sum_{k \geq 0} G^q(r_{k+1}) \int_{I_{k+1}} \frac{(1-r)^{\beta q + q - 1} r^\alpha dr}{(1-\rho r)^{q\gamma}} \\
&\leq C \int_0^1 G(r)^q \frac{(1-r)^{\beta q + q - 1}}{(1-\rho r)^{q\gamma}} r^\alpha dr.
\end{aligned}$$

□

**Lemma 4.** For  $\delta > -1$ ,  $\gamma > n + \delta$  and  $\beta > 0$  we have

$$\int_{\mathbb{B}} |Q_\beta(x, y)|^{\frac{\gamma}{n+\beta}} (1 - |y|)^\delta dy \leq C(1 - |x|)^{\delta - \gamma + n}, \quad x \in \mathbb{B}.$$

*Proof.* Using Lemma 1 and Lemma 2 we obtain:

$$\begin{aligned}
\int_{\mathbb{B}} |Q_\beta(x, y)|^{\frac{\gamma}{n+\beta}} (1 - |y|)^\delta dy &\leq C \int_{\mathbb{B}} \frac{(1 - |y|)^\delta}{|\rho r x' - y'|^\gamma} dy \\
&\leq C \int_0^1 (1 - \rho)^\delta \int_{\mathbb{S}} \frac{dy'}{|\rho r x' - y'|^\gamma} dy' d\rho \\
&\leq C \int_0^1 (1 - \rho)^\delta (1 - r\rho)^{n - \gamma - 1} d\rho \\
&\leq C(1 - r)^{n + \delta - \gamma}.
\end{aligned}$$

□

**Lemma 5** (see [1]). For real  $s$  and  $t$  such that  $s > -1$  and  $2t + n > 0$  we have

$$\int_0^1 (1 - r^2)^s r^{2t + n - 1} dr = \frac{1}{2} \frac{\Gamma(s + 1) \Gamma(n/2 + t)}{\Gamma(s + 1 + n/2 + t)}.$$

## 2. Multipliers on spaces of harmonic functions

In this section we present our results on multipliers between spaces of harmonic functions on the unit ball. To formulate these theorems the following definitions are needed.

**Definition 1.** We consider the double indexed sequence of complex numbers

$$c = \{c_k^j : k \geq 0, 1 \leq j \leq d_k\}$$

and a harmonic function  $f(rx') = \sum_{k=0}^{\infty} \sum_{j=1}^{d_k} r^k b_k^j(f) Y_j^{(k)}(x')$ . We define

$$(c * f)(rx') = \sum_{k=0}^{\infty} \sum_{j=1}^{d_k} r^k c_k^j b_k^j(f) Y_j^{(k)}(x'), \quad rx' \in \mathbb{B},$$

if the series converges in  $\mathbb{B}$ . Similarly we define the convolution of  $f, g \in h(\mathbb{B})$  by

$$(f * g)(rx') = \sum_{k=0}^{\infty} \sum_{j=1}^{d_k} r^k b_k^j(f) b_k^j(g) Y_j^{(k)}(x'), \quad rx' \in \mathbb{B}.$$

It is easily seen that  $f * g$  is defined and harmonic in  $\mathbb{B}$ .

**Definition 2.** For  $t > 0$  and a harmonic function  $f(x) = \sum_{k=0}^{\infty} b_k(f) Y^k(x')$  on the unit ball we define a fractional derivative of order  $t$  of  $f$  by the formula

$$(\Lambda_t f)(x) = \sum_{k=0}^{\infty} r^k \frac{\Gamma(k + n/2 + t)}{\Gamma(k + n/2)\Gamma(t)} b_k(f) \cdot Y^k(x'), \quad x = rx' \in \mathbb{B}.$$

Clearly, for  $f \in h(\mathbb{B})$  and  $t > 0$  the function  $\Lambda_t h$  is also harmonic in  $\mathbb{B}$ .

**Definition 3.** Let  $X$  and  $Y$  be subspaces of  $h(\mathbb{B})$ . We say that a double indexed sequence  $c$  is a multiplier from  $X$  to  $Y$  if  $c * f \in Y$  for every  $f \in X$ . The vector space of all multipliers from  $X$  to  $Y$  is denoted by  $M_H(X, Y)$ .

Clearly, every multiplier  $c \in M_H(X, Y)$  induces a linear map  $M_c : X \rightarrow Y$ . If, in addition,  $X$  and  $Y$  are (quasi-)normed spaces such that all functionals  $b_k^j$  are continuous on both spaces  $X$  and  $Y$ , then the map  $M_c : X \rightarrow Y$  is continuous, as is easily seen using the Closed Graph Theorem. We note that this holds for all spaces which we consider in this paper:  $A_\alpha^p$ ,  $B_\alpha^{p,q}$  and  $H_\alpha^p$ .

**Lemma 6.** Let  $f, g \in h(\mathbb{B})$  have the expansions

$$f(rx') = \sum_{k=0}^{\infty} r^k \sum_{j=1}^{d_k} b_k^j Y_j^{(k)}(x'), \quad g(rx') = \sum_{l=0}^{\infty} r^l \sum_{i=1}^{d_l} c_l^i Y_i^{(l)}(x').$$

Then we have

$$\int_{\mathbb{S}} (g * P_{y'}) (rx') f(\rho x') dx' = \sum_{k=0}^{\infty} r^k \rho^k \sum_{j=1}^{d_k} b_k^j c_k^j Y_j^{(k)}(y'), \quad y' \in \mathbb{S}, \quad 0 \leq r, \rho < 1.$$

Moreover, for every  $m > -1$ ,  $y' \in \mathbb{S}$  and  $0 \leq r, \rho < 1$  we have

$$\int_{\mathbb{S}} (g * P_{y'})(rx') f(\rho x') dx' = 2 \int_0^1 \int_{\mathbb{S}} \Lambda_{m+1}(g * P_{y'})(rRx') f(\rho Rx') (1 - R^2)^m R^{n-1} dx' dR.$$

*Proof.* The first assertion of this lemma easily follows from the orthogonality relations for spherical harmonics  $Y_j^{(k)}$ . Using Lemma 5 and the orthogonality relations we have

$$\begin{aligned} I &= 2 \int_0^1 \int_{\mathbb{S}} \Lambda_{m+1}(g * P_{y'})(rRx') f(\rho Rx') (1 - R^2)^m R^{n-1} dx' dR \\ &= 2 \int_0^1 \sum_{k=0}^{\infty} r^k \rho^k R^{2k+n-1} (1 - R^2)^m \frac{\Gamma(k + n/2 + m + 1)}{\Gamma(k + n/2)\Gamma(m + 1)} \sum_{j=1}^{d_k} b_k^j c_k^j Y_j^{(k)} dR \\ &= \sum_{k=0}^{\infty} r^k \rho^k \sum_{j=1}^{d_k} b_k^j c_k^j Y_j^{(k)}(y'), \end{aligned}$$

which proves the second assertion. □

We note that

$$(g * P_{y'})(rx') = (g * P_{x'})(ry')$$

and

$$\Lambda_t(g * P_{y'})(x) = (\Lambda_t g * P_{y'})(x);$$

these easy-to-prove formulae are often used in our proofs.

In this section  $f_{m,y}$  stands for the harmonic function  $f_{m,y}(x) = Q_m(x, y)$ ,  $y \in \mathbb{B}$ . We often write  $f_y$  instead of  $f_{m,y}$ . Let us collect some norm estimates for  $f_y$ .

**Lemma 7.** For  $0 < p \leq \infty$  and  $m > 0$  we have

$$M_{\infty}(f_{m,y}, r) \leq C(1 - |y|r)^{-n-m}. \tag{4}$$

$$M_1(f_{m,y}, r) \leq C(1 - |y|r)^{-1-m}. \tag{5}$$

$$\|f_{m,y}\|_{B_{\alpha}^{p,1}} \leq C(1 - |y|)^{\alpha-1-m}, \quad m > \alpha - 1, \quad \alpha > 0. \tag{6}$$

$$\|f_{m,y}\|_{B_{\alpha}^{p,\infty}} \leq C(1 - |y|)^{\alpha-n-m}, \quad m > \alpha - n, \quad \alpha > 0. \tag{7}$$

$$\|f_{m,y}\|_{A_{\alpha}^1} \leq C(1 - |y|)^{\alpha-m}, \quad m > \alpha > -1. \tag{8}$$

$$\|f_{m,y}\|_{H_{\alpha}^1} \leq C(1 - |y|)^{\alpha-1-m}, \quad m > \alpha - 1, \quad \alpha \geq 0. \tag{9}$$

*Proof.* Using Lemma 1 we obtain

$$M_{\infty}(f_{m,y}, r) = \max_{x' \in \mathbb{S}} |Q_m(y, rx')| \leq \max_{x' \in \mathbb{S}} \frac{C}{|\rho x' - y'|^{n+m}} = C(1 - r|y|)^{-n-m},$$

which gives (4). Estimate (5) follows from Lemma 1. Estimates (6), for finite  $p$ , and (8) follow from Lemma 2 and (5). Similarly, for finite  $p$ , (7) follows from (4) and Lemma 2. Next, using (5),

$$\|f_{m,y}\|_{H_\alpha^1} \leq C \sup_{0 \leq r < 1} (1-r)^\alpha (1-r\rho)^{-m-1}, \quad \rho = |y|.$$

The function  $\phi(r) = (1-r)^\alpha (1-r\rho)^{-m-1}$  attains its maximum on  $[0, 1]$  at

$$r_0 = 1 - (1-\rho) \frac{\alpha}{\rho(1+m-\alpha)},$$

as is readily seen by a simple calculus. This suffices to establish (9) and therefore (6) for  $p = \infty$ . Finally, (7) directly follows from Lemma 1.  $\square$

In this section we are looking for sufficient and/or necessary conditions for a double indexed sequence  $c$  to be in  $M_H(X, Y)$  for certain spaces  $X$  and  $Y$  of harmonic functions. With such a sequence  $c$  we associate a harmonic function

$$g_c(x) = g(x) = \sum_{k \geq 0} r^k \sum_{j=1}^{d_k} c_k^j Y_j^{(k)}(x'), \quad x = rx' \in \mathbb{B}, \quad (10)$$

and express our conditions in terms of  $g_c$ . Our main results provide conditions in terms of fractional derivatives of  $g_c$ . However, it is possible to obtain some results on the basis of the following formula, contained in Lemma 6:

$$(c * f)(r^2 x') = \int_{\mathbb{S}} (g * P_{y'})(rx') f(ry') dy'. \quad (11)$$

Using the continuous form of Minkowski's inequality, or more generally Young's inequality, this formula immediately yields the following proposition.

**Proposition 1.** *Let  $c = \{c_k^j : k \geq 0, 1 \leq j \leq d_k\}$  be a double indexed sequence and let  $g(x) = \sum_{k \geq 0} r^k \sum_{j=1}^{d_k} c_k^j Y_j^{(k)}(x')$  be the corresponding harmonic function. If*

$$\int_{\mathbb{S}} |(g * P_{y'})(rx')|^p dx' \leq C, \quad y' \in \mathbb{S}, \quad 0 \leq r < 1$$

for some  $1 \leq p < \infty$ , then  $c \in M_H(H^1, H^p)$ . An analogous statement is true for  $p = \infty$ .

More generally, if  $1/q + 1/p = 1 + 1/r$ , where  $1 \leq p, q, r \leq \infty$ ,  $\alpha + \gamma = \beta$ ,  $\alpha, \beta, \gamma \geq 0$  and  $g \in H_\gamma^p$ , then  $c \in M_H(H_\alpha^q, H_\beta^r)$ .

**Lemma 8.** *Let  $0 < p, q \leq \infty$ ,  $1 \leq s \leq \infty$  and  $m > \alpha - 1$ . Assume a double indexed sequence  $c = \{c_k^j : k \geq 0, 1 \leq j \leq d_k\}$  is a multiplier from*

$B_\alpha^{p,1}$  to  $B_\beta^{q,s}$  and  $g = g_c$  is defined by (10). Then the following condition is satisfied:

$$N_s(g) = \sup_{0 \leq \rho < 1} \sup_{y' \in \mathbb{S}} (1 - \rho)^{m+1-\alpha+\beta} \left( \int_{\mathbb{S}} |\Lambda_{m+1}(g * P_{x'}) (\rho y')|^s dx' \right)^{1/s} < \infty, \tag{12}$$

where the case  $s = \infty$  requires the usual modification.

Also, let  $0 < p \leq \infty$ ,  $1 \leq s \leq \infty$  and  $m > \alpha - 1$ . If a double indexed sequence  $c = \{c_k^j : k \geq 0, 1 \leq j \leq d_k\}$  is a multiplier from  $B_\alpha^{p,1}$  to  $H_\beta^s$ , then the function  $g$  defined above satisfies condition (12).

*Proof.* Let  $c \in M_H(B_\alpha^{p,1}, B_\beta^{q,s})$  and assume that both  $p$  and  $q$  are finite (the infinite cases require only small modifications). We have  $\|M_c f\|_{B_\beta^{q,s}} \leq C \|f\|_{B_\alpha^{p,1}}$  for  $f$  in  $B_\alpha^{p,1}$ . Set  $h_y = M_c f_y$ , then we have

$$h_y(x) = \sum_{k \geq 0} r^k \rho^k \sum_{j=1}^k \frac{\Gamma(k + n/2 + m + 1)}{\Gamma(k + n/2)\Gamma(m + 1)} c_k^j Y_j^{(k)}(y') Y_j^{(k)}(x'), \quad x = rx' \in \mathbb{B}, \tag{13}$$

moreover,

$$\|h_y\|_{B_\beta^{q,s}} \leq C \|f_y\|_{B_\alpha^{p,1}}. \tag{14}$$

This estimate and Lemma 7 imply

$$\|h_y\|_{B_\beta^{q,s}} \leq C(1 - |y|)^{\alpha-m-1}, \quad y \in \mathbb{B}. \tag{15}$$

Note that  $h_y(x) = \Lambda_{m+1}(g * P_{y'}) (\rho x)$ . Using the monotonicity of  $M_s(h_y, r)$  we obtain:

$$\begin{aligned} I_{y'}(\rho^2) &= \left( \int_{\mathbb{S}} |\Lambda_{m+1}(g * P_{x'}) (\rho^2 y')|^s dx' \right)^{1/s} = \left( \int_{\rho}^1 (1 - r)^{\beta q - 1} r^{n-1} dr \right)^{-1/q} \\ &\quad \times \left( \int_{\rho}^1 (1 - r)^{\beta q - 1} r^{n-1} \left( \int_{\mathbb{S}} |\Lambda_{m+1}(g * P_{y'}) (\rho^2 x')|^s dx' \right)^{q/s} dr \right)^{1/q} \\ &\leq C(1 - \rho)^{-\beta} \left( \int_{\rho}^1 (1 - r)^{\beta q - 1} r^{n-1} M_s^q(h_y, r) dr \right)^{1/q} \\ &\leq C(1 - \rho)^{-\beta} \|h_y\|_{B_\beta^{q,s}}. \end{aligned} \tag{16}$$

Combining (16) and (15) we get

$$\left( \int_{\mathbb{S}} |\Lambda_{m+1}(g * P_{x'}) (\rho^2 y')|^s dx' \right)^{1/s} \leq C(1 - \rho)^{\alpha-\beta-m-1},$$

which is equivalent to (12). The case  $s = \infty$  is treated similarly.



Next we consider  $c \in M_H(B_\alpha^{p,1}, H_\beta^s)$ , assuming  $0 < p \leq \infty$ . Set  $h_y = M_c f_y = g * f_y$ . We have, by Lemma 7,

$$\|f_y\|_{B_\alpha^{p,1}} \leq C(1 - |y|)^{\alpha-m-1}, \quad y \in \mathbb{B},$$

and, by the continuity of  $M_c$ ,  $\|h_y\|_{H_\beta^s} \leq C\|f_y\|_{B_\alpha^{p,1}}$ . Therefore

$$\|h_y\|_{H_\beta^s} \leq C(1 - |y|)^{\alpha-m-1}, \quad y \in \mathbb{B}.$$

Setting  $y = \rho y'$  we have

$$\begin{aligned} I_{y'}(\rho^2) &= \left( \int_{\mathbb{S}} |\Lambda_{m+1}(g * P_{x'}) (\rho^2 y')|^s dx' \right)^{1/s} \\ &= \left( \int_{\mathbb{S}} |\Lambda_{m+1}(g * P_y) (\rho x')|^s dx' \right)^{1/s} \\ &= M_s(h_y, \rho) \leq (1 - |y|)^{-\beta} \|h_y\|_{H_\beta^s}. \end{aligned}$$

The last two estimates yield

$$\left( \int_{\mathbb{S}} |\Lambda_{m+1}(g * P_{x'}) (\rho^2 y')|^s dx' \right)^{1/s} \leq C(1 - |y|)^{\alpha-\beta-m-1}, \quad |y| = \rho,$$

which is equivalent to (12). □

One of the main results of this paper is a characterization of the multiplier space  $M_H(B_\alpha^{p,1}, B_\beta^{q,1})$  for  $0 < p \leq q \leq \infty$ . The following theorem treats the case  $p > 1$ , while Theorem 5 below covers the case  $0 < p \leq 1$ .

**Theorem 2.** *Let  $1 < p \leq q \leq \infty$  and  $m > \alpha - 1$ . Then for a double indexed sequence  $c = \{c_k^j : k \geq 0, 1 \leq j \leq d_k\}$  the following conditions are equivalent:*

- 1)  $c \in M_H(B_\alpha^{p,1}, B_\beta^{q,1})$ .
- 2) The function  $g(x) = \sum_{k \geq 0} r^k \sum_{j=1}^{d_k} c_k^j Y_j^{(k)}(x')$  is harmonic in  $\mathbb{B}$  and

$$N_1(g) < \infty. \tag{17}$$

*Proof.* Since the necessity of (17) follows from Lemma 8, we prove the sufficiency of condition (17). We assume that  $p$  and  $q$  are finite (the remaining cases can be treated in a similar manner). Take  $f \in B_\alpha^{p,1}$  and set  $h = M_c f$ . Applying the operator  $\Lambda_{m+1}$  to both sides of equation (11) we obtain

$$\Lambda_{m+1} h(rx) = \int_{\mathbb{S}} \Lambda_{m+1}(g * P_{y'})(x) f(ry') dy'. \tag{18}$$

Now we estimate the  $L^1$  norm of the above function on  $|x| = r$ :

$$\begin{aligned} M_1(\Lambda_{m+1}h, r^2) &\leq \int_{\mathbb{S}} M_1(\Lambda_{m+1}(g * P_{y'}), r) |f(ry')| dy' \\ &\leq M_1(f, r) \sup_{y' \in \mathbb{S}} \int_{\mathbb{S}} |\Lambda_{m+1}(g * P_{y'})(rx')| dx' \\ &\leq M_1(f, r) N_1(g) (1-r)^{\alpha-\beta-m-1}. \end{aligned} \tag{19}$$

Since

$$\int_0^1 M_1^p(h, r^2) (1-r)^{\beta p-1} r^{n-1} dr \leq C \int_0^1 (1-r)^{p(m+1)} M_1^p(\Lambda_{m+1}h, r^2) (1-r)^{\beta p-1} r^{n-1} dr$$

(see [1]), we have

$$\begin{aligned} \|h\|_{B_\beta^{p,1}}^p &\leq C \int_0^1 (1-r)^{p(m+1)} M_1^p(\Lambda_{m+1}h, r^2) (1-r)^{\beta p-1} r^{n-1} dr \\ &\leq C N_1^p(g) \int_0^1 M_1^p(f, r) (1-r)^{\alpha p-1} r^{n-1} dr \\ &= C N_1^p(g) \|f\|_{B_\alpha^{p,1}}^p, \end{aligned}$$

and therefore  $\|h\|_{B_\beta^{p,1}} \leq \|f\|_{B_\alpha^{p,1}}$ . Since  $\|h\|_{B_\beta^{q,1}} \leq C \|h\|_{B_\beta^{p,1}}$ , the proof is complete.  $\square$

Next we consider the multipliers from  $B_\alpha^{p,1}$  to  $H_\beta^s$ . In the case  $0 < p \leq 1$  we obtain a characterization of the corresponding space.

**Theorem 3.** *Let  $\beta \geq 0$ ,  $0 < p \leq 1$ ,  $s \geq 1$  and  $m > \alpha - 1$ . Then for a double indexed sequence  $c = \{c_k^j : k \geq 0, 1 \leq j \leq d_k\}$  the following two conditions are equivalent:*

- 1)  $c \in M_H(B_\alpha^{p,1}, H_\beta^s)$ .
- 2) The function  $g(x) = \sum_{k \geq 0} r^k \sum_{j=1}^{d_k} c_k^j Y_j^{(k)}(x')$  is harmonic in  $\mathbb{B}$  and

$$N_s(g) < \infty. \tag{20}$$

*Proof.* The necessity of condition (20) follows from Lemma 8. Now we turn to the sufficiency of (20). We choose  $f \in B_\alpha^{p,1}$  and set  $h = c * f$ . Then, by Lemma 6,

$$h(r^2x') = 2 \int_0^1 \int_{\mathbb{S}} \Lambda_{m+1}(g * P_\xi)(rRx') f(rR\xi) (1-R^2)^m R^{n-1} d\xi dR. \tag{21}$$

This allows us to obtain the following estimate:

$$\begin{aligned}
M_s(h, r^2) &\leq 2 \int_0^1 (1-R^2)^m R^{n-1} \left\| \int_{\mathbb{S}} \Lambda_{m+1}(g * P_\xi)(rRx') f(rR\xi) d\xi \right\|_{L^s(\mathbb{S}, dx')} dR \\
&\leq 2 \int_0^1 (1-R^2)^m R^{n-1} M_1(f, rR) \sup_{\xi \in \mathbb{S}} \|\Lambda_{m+1}(g * P_\xi)(rRx')\|_{L^s(\mathbb{S}, dx')} dR \\
&\leq CN_s(g) \int_0^1 (1-R)^m M_1(f, rR) (1-rR)^{\alpha-\beta-m-1} dR \\
&\leq CN_s(g) (1-r)^{-\beta} \int_0^1 M_1(f, rR) (1-R)^m (1-rR)^{\alpha-m-1} dR.
\end{aligned}$$

Note that  $M_1(f, rR)$  is increasing in  $0 \leq R < 1$ . Therefore, we can combine Lemma 3 and the above estimate to obtain:

$$\begin{aligned}
M_s^p(h, r^2) &\leq CN_s^p(g) (1-r)^{-\beta p} \int_0^1 M_1^p(f, rR) \frac{(1-R)^{mp+p-1}}{(1-rR)^{pm-\alpha p+p}} dR \\
&\leq CN_s^p(g) (1-r)^{-p\beta} \int_0^1 M_1^p(f, R) (1-R)^{\alpha p-1} dR \\
&\leq CN_s^p(g) (1-r)^{-p\beta} \|f\|_{B_\alpha^{p,1}}^p
\end{aligned}$$

Hence,  $M_s(h, r^2) \leq CN_s(g) (1-r)^{-\beta} \|f\|_{B_\alpha^{p,1}}$ , which completes the proof.  $\square$

The omitted case  $p = \infty$  is treated in our next theorem, which gives a characterization of the space  $M_H(H_\alpha^1, H_\beta^p)$ .

**Theorem 4.** *Let  $\alpha \geq 0$ ,  $\beta > 0$ ,  $1 \leq p \leq \infty$  and  $m > \alpha - 1$ . Then for a double indexed sequence  $c = \{c_k^j : k \geq 0, 1 \leq j \leq d_k\}$  the following conditions are equivalent:*

- 1)  $c \in M_H(H_\alpha^1, H_\beta^p)$ .
- 2) *The function  $g(x) = \sum_{k \geq 0} r^k \sum_{j=1}^{d_k} c_k^j Y_j^{(k)}(x')$  is harmonic in  $\mathbb{B}$  and*

$$N_p(g) < \infty. \quad (22)$$

*In the case  $p = \infty$  condition (22) is interpreted in the usual manner.*

*Proof.* Let us assume  $c \in M_H(H_\alpha^1, H_\beta^p)$  and set  $h_y = M_c f_y$  for  $y \in \mathbb{B}$ . Then by the continuity of  $M_c$  and by Lemma 7 we have

$$\|h_y\|_{H_\beta^p} \leq C \|f_y\|_{H_\alpha^1} \leq C(1-|y|)^{\alpha-m-1}.$$

On the other hand,

$$\|h_y\|_{H_\beta^p} \geq (1-\rho)^\beta M_p(h_y, \rho) \geq (1-\rho)^\beta \left( \int_{\mathbb{S}} |\Lambda_{m+1}(g * P_{x'}) (\rho^2 y)|^p dx' \right)^{1/p}, \quad (23)$$

and the above estimates imply (22). Now we prove the sufficiency of condition (22). Choose  $f \in H_\alpha^1$  and set  $h = c * f$ . We apply the continuous form of Minkowski's inequality to (18) and obtain

$$\begin{aligned} M_p(\Lambda_{m+1}h, r^2) &\leq M_1(f, r) \sup_{y' \in \mathbb{S}} M_p(\Lambda_{m+1}(g * P_{y'}), r) \\ &\leq N_p(g)(1-r)^{\alpha-\beta-m-1} M_1(f, r). \end{aligned}$$

Therefore  $\sup_{r < 1} (1-r)^{m+1+\beta} M_p(\Lambda_{m+1}h, r) \leq C \|f\|_{H_\alpha^1}$ . It follows (see [1]) that  $\sup_{r < 1} (1-r)^\beta M_p(h, r) \leq C \|f\|_{H_\alpha^1}$ , as required. The case  $p = \infty$  is treated in the same way.  $\square$

Since  $H_\beta^\infty = A_\beta^\infty$ , the case  $p = \infty$  of this theorem gives a complete description of the space  $M_H(H_\alpha^1, A_\beta^\infty)$ . The next proposition provides necessary conditions for  $c$  to be in  $M_H(X, A_\beta^\infty)$  for some spaces  $X$ .

**Proposition 2.** *Let  $m > \alpha$ . Consider the following conditions for a double indexed sequence  $c = \{c_k^j : k \geq 0, 1 \leq j \leq d_k\}$ :*

- 1)  $c \in M_H(A_\alpha^1, A_\beta^\infty)$ .
- 2)  $c \in M_H(B_\alpha^{p,1}, A_\beta^\infty)$ .
- 3) *The function  $g(x) = \sum_{k \geq 0} r^k \sum_{j=1}^{d_k} c_k^j Y_j^{(k)}(x')$  is harmonic in  $\mathbb{B}$  and*

$$M_t(g) = \sup_{0 \leq \rho < 1} \sup_{x', y' \in \mathbb{S}} (1-\rho)^t |\Lambda_{m+1}(g * P_{x'})(\rho y')| < \infty. \tag{24}$$

*Then we have: 1)  $\Rightarrow$  3) with  $t = m + \beta - \alpha$  and 2)  $\Rightarrow$  3) with  $t = m + 1 + \beta - \alpha$ .*

*Proof.* Let  $X$  be one of the spaces  $A_\alpha^1, B_\alpha^{p,1}$ . As in the previous theorems, we choose a multiplier  $c$  from  $X$  to  $A_\beta^\infty$  and note that  $\|c * f\|_{A_\beta^\infty} \leq C \|f\|_X$ . We apply this inequality to  $f_y, y = \rho y' \in \mathbb{B}$ , with  $h_y = c * f_y$ , and obtain the estimate

$$\|h_y\|_{A_\beta^\infty} \leq C \|f_y\|_X.$$

Next,

$$\begin{aligned} \|h_y\|_{A_\beta^\infty} &\geq (1-\rho)^\beta M_\infty(h_y, \rho) = (1-\rho)^\beta \sup_{x' \in \mathbb{S}} |h_y(\rho x')| \\ &= (1-\rho)^\beta \sup_{x' \in \mathbb{S}} |\Lambda_{m+1}(g * P_{x'})(\rho^2 y')|. \end{aligned}$$

Now both implications follow from Lemma 7.  $\square$

Theorem 5 below complements Theorem 2. Its less general form appeared in [5]. For the completeness of the exposition and with permission of the authors we present a proof.

**Theorem 5.** *Let  $0 < p \leq 1$ ,  $m > \alpha - 1$  and  $p \leq q \leq \infty$ . Then for a double indexed sequence  $c = \{c_k^j : k \geq 0, 1 \leq j \leq d_k\}$  the following conditions are equivalent:*

1)  $c \in M_H(B_\alpha^{p,1}, B_\beta^{q,1})$ .

2) *The function  $g(x) = \sum_{k \geq 0} r^k \sum_{j=1}^{d_k} c_k^j Y_j^{(k)}(x')$  is harmonic in  $\mathbb{B}$  and*

$$N_1(g) < \infty. \tag{25}$$

*Proof.* The necessity of the condition (25) follows from Lemma 8. Now we prove the sufficiency of condition (25). Let  $f \in B_\alpha^{p,1}(\mathbb{B})$  and set  $h = c * f$ . Then, using Lemma 6, we have

$$\begin{aligned} \int_{\mathbb{S}} |h(r\rho x')| dx' &\leq \int_0^1 \int_{\mathbb{S}} \int_{\mathbb{S}} |\Lambda_{m+1}(g * P_{x'})(rR\xi)| |f(\rho R\xi)| \\ &\quad (1 - R^2)^m R^{n-1} d\xi dx' dR \\ &\leq C \int_0^1 \left( \sup_{\xi \in \mathbb{S}} \int_{\mathbb{S}} |\Lambda_{m+1}(g * P_{x'})(rR\xi)| dx' \right) \int_{\mathbb{S}} |f(\rho R\xi)| d\xi \\ &\quad (1 - R^2)^m R^{n-1} dR. \end{aligned}$$

Letting  $\rho \rightarrow 1$  in the above inequality yields

$$\begin{aligned} \int_{\mathbb{S}} |h(rx')| dx' &\leq C \int_0^1 \left( \sup_{\xi \in \mathbb{S}} \int_{\mathbb{S}} |\Lambda_{m+1}(g * P_{x'})(rR\xi)| dx' \right) \int_{\mathbb{S}} |f(R\xi)| d\xi \\ &\quad (1 - R^2)^m R^{n-1} dR. \end{aligned}$$

Since for each fixed  $\xi \in \mathbb{S}$  the function  $u_\xi(x) = |\Lambda_{m+1}(g * P_\xi)(rx)|$  is subharmonic, we see that

$$\psi_\xi(R) = \int_{\mathbb{S}} |\Lambda_{m+1}(g * P_{x'})(rR\xi)| dx' = \int_{\mathbb{S}} |\Lambda_{m+1}(g * P_\xi)(rRx')| dx'$$

is increasing for  $0 \leq R < 1$ . Therefore the function

$$G_r(R) = \left( \sup_{\xi \in \mathbb{S}} \int_{\mathbb{S}} |\Lambda_{m+1}(g * P_{x'})(rR\xi)| dx' \right) \int_{\mathbb{S}} |f(R\xi)| d\xi, \quad 0 \leq R < 1,$$

is increasing and we can apply Lemma 3 to obtain

$$\begin{aligned} \left( \int_{\mathbb{S}} |h(rx')| dx' \right)^p &\leq C \left( \int_0^1 G_r(R) (1 - R^2)^m R^{n-1} dR \right)^p \\ &\leq C \int_0^1 G_r(R)^p (1 - R)^{mp+p-1} R^{n-1} dR. \end{aligned}$$

Since  $G_r(R) \leq N_1(g)M_1(f, R)(1-rR)^{\alpha-\beta-m-1}$  for  $0 \leq r < 1$ , using Lemma 2 we get

$$\begin{aligned} \|h\|_{B_\beta^{p,1}}^p &= \int_0^1 \left( \int_{\mathbb{S}} |h(rx')| dx' \right)^p (1-r)^{p\beta-1} r^{n-1} dr \\ &\leq CN_1(g)^p \int_0^1 M_1(f, R)^p (1-R)^{mp+p-1} R^{n-1} \int_0^1 \frac{(1-r)^{p\beta-1} r^{n-1} dr}{(1-rR)^{p(m+1+\beta-\alpha)}} dR \\ &\leq CN_1(g)^p \int_0^1 M_1(f, R)^p (1-R)^{p\alpha-1} dR = C\|f\|_{B_\alpha^{p,1}}^p. \end{aligned}$$

Hence,  $\|h\|_{B_\beta^{p,1}} \leq C\|f\|_{B_\alpha^{p,1}}$ . This, together with the inequality  $\|h\|_{B_\beta^{q,1}} \leq C\|h\|_{B_\beta^{p,1}}$ , finishes the proof.  $\square$

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## References

- [1] M. Djrbashian and F. Shamoian, *Topics in the Theory of  $A_\alpha^p$  Classes*, Teubner-Texte zur Mathematik **105**, BSB B. G. Teubner Verlagsgesellschaft, Leipzig, 1988.
- [2] M. Jevtić and M. Pavlović, *Harmonic Bergman functions on the unit ball in  $\mathbb{R}^n$* , Acta Math. Hungar. **85** (1999), 81–96.
- [3] M. Pavlović, *Convolution in the harmonic Hardy class  $h^p$  with  $0 < p < 1$* , Proc. Amer. Math. Soc. **109** (1990), 129–134.
- [4] M. Pavlović, *Multipliers of the vanishing Hardy classes*, Publ. Inst. Math. (Beograd) (N.S.) **52(66)** (1992), 34–36.
- [5] R. F. Shamoyan and A. Abkar, *A note on multipliers of spaces of harmonic functions in the unit ball of  $\mathbb{R}^n$* , preprint, 2011.
- [6] A. Shields and D. Williams, *Bounded projections, duality and multipliers in spaces of harmonic functions*, J. Reine Angew. Math. **299/300** (1978). 256–279.
- [7] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, Princeton, 1971.

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