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Bounds for the Riemann–Stieltjes integral via s-convex integrand or integrator

Mohammad Wajeeh Alomari

Abstract. Several bounds in approximating the Riemann–Stieltjes integral in terms of s-convex integrands or integrator are given.

1. Introduction

A function $f : \mathbb{R}^+ \to \mathbb{R}$, where $\mathbb{R}^+ = [0, \infty)$, is said to be s-convex in the second sense if

$$f(\alpha x + \beta y) \le \alpha^s f(x) + \beta^s f(y)$$

for all $x, y \in [0, \infty)$, $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$ and for some fixed $s \in (0, 1]$. This class of functions is denoted by K_s^2. It can be easily seen that for $s = 1$, s-convexity reduces to the ordinary convexity of functions defined on $[0, \infty)$ (see [6]).

In [2], Cerone and Dragomir have proved some error bounds in approximating the Riemann–Stieltjes integral in terms of some moments of the integrand. Among others, they proved the following result.

Theorem 1. *Let u be p-convex with $p > 0$, f be monotonically increasing on $[a, b]$ and such that the Riemann–Stieltjes integral $\int_a^b f(t)\, du(t)$ and the Riemann integrals $\int_a^b (t-a)^{p-1} f(t)\, dt$, $\int_a^b (b-t)^{p-1} f(t)\, dt$ exist. Then*

$$\int_a^b f(t)\, du(t) \ge \frac{p}{(b-a)^p} \left[u(b) \int_a^b (t-a)^{p-1} f(t)\, dt - u(a) \int_a^b (b-t)^{p-1} f(t)\, dt \right].$$

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For other results concerning different bounds for the Riemann–Stieltjes integral under various assumptions on f and u, see the recent papers [1]–[5] and the references therein.

In this paper, several inequalities for the Riemann–Stieltjes integral $\int_a^b f(x)\,dg(x)$ are proved. Namely, the integrand f is assumed to be s-convex (s-concave) and the integrator g is monotonically increasing, bounded and s-convex (s-concave).

2. Inequalities for s-convex integrands or integrators

We may start with the following result.

Theorem 2. *Let $f, g : [a,b] \subseteq \mathbb{R}^+ \to \mathbb{R}$ be such that f is s-convex on $[a,b]$, g is monotonically increasing on $[a,b]$ and the Riemann–Stieltjes integral $\int_a^b f(x)\,dg(x)$ and the Riemann integrals $\int_a^b (x-a)^{s-1} g(x)\,dx$, $\int_a^b (b-x)^{s-1} g(x)\,dx$ exist. Then we have the inequalities*

$$\begin{aligned}\int_a^b f(x)\,dg(x) &\le \frac{f(b)}{(b-a)^s}\left[(b-a)^s g(b) - s\int_a^b g(x)(x-a)^{s-1}\,dx\right]\\ &\quad + \frac{f(a)}{(b-a)^s}\left[-(b-a)^s g(a) + s\int_a^b g(x)(b-x)^{s-1}\,dx\right]\\ &\le [g(b)-g(a)][f(a)+f(b)]. \qquad (2.1)\end{aligned}$$

Proof. Since f is s-convex on $[a,b]$ and by using the integration by parts formula for Riemann–Stieltjes integral, we have

$$\begin{aligned}\int_a^b f(x)\,dg(x) &\le \int_a^b \left[\left(\frac{x-a}{b-a}\right)^s f(b) + \left(\frac{b-x}{b-a}\right)^s f(a)\right] dg(x)\\ &= f(b)\int_a^b \left(\frac{x-a}{b-a}\right)^s dg(x) + f(a)\int_a^b \left(\frac{b-x}{b-a}\right)^s dg(x)\\ &= \frac{f(b)}{(b-a)^s}\int_a^b (x-a)^s\,dg(x) + \frac{f(a)}{(b-a)^s}\int_a^b (b-x)^s\,dg(x)\\ &= \frac{f(b)}{(b-a)^s}\left[(b-a)^s g(b) - s\int_a^b g(x)(x-a)^{s-1}\,dx\right] \qquad (2.2)\\ &\quad + \frac{f(a)}{(b-a)^s}\left[-(b-a)^s g(a) + s\int_a^b g(x)(b-x)^{s-1}\,dx\right],\end{aligned}$$

which proves the first inequality in (2.1). To prove the second inequality in (2.1), using the monotonicity of g on $[a,b]$, we get

$$\int_a^b (x-a)^{s-1} g(x)\,dx \ge g(a)\int_a^b (x-a)^{s-1}\,dx = \frac{1}{s} g(a)(b-a)^s$$

and

$$\int_a^b (b-x)^{s-1} g(x)\,dx \le g(b)\int_a^b (b-x)^{s-1}\,dx = \frac{1}{s} g(b)(b-a)^s .$$

Therefore by (2.2), we get

$$\begin{aligned}\int_a^b f(x)\,dg(x) &\le \frac{f(b)}{(b-a)^s}\left[(b-a)^s g(b) - s\int_a^b g(x)(x-a)^{s-1}\,dx\right] \\ &\quad + \frac{f(a)}{(b-a)^s}\left[-(b-a)^s g(a) + s\int_a^b g(x)(b-x)^{s-1}\,dx\right] \\ &\le \frac{f(b)}{(b-a)^s}\left[(b-a)^s g(b) - g(a)(b-a)^s\right] \\ &\quad + \frac{f(a)}{(b-a)^s}\left[-(b-a)^s g(a) + g(b)(b-a)^s\right] \\ &= \left[g(b)-g(a)\right]\left[f(a)+f(b)\right],\end{aligned}$$

which proves the second inequality in (2.1). □

The following result holds.

Theorem 3. *Let $g : [a,b] \subseteq \mathbb{R}^+ \to \mathbb{R}$ be a monotonically increasing function on $[a,b]$.*

(1) *If $f : [a,b] \to \mathbb{R}^+$ is convex on $[a,b]$, then we have the inequality*

$$\begin{aligned}\int_a^b f(x)\,dg(x) \le \min\Bigg\{ &\frac{1}{2}\left[f(a)+f(b)+\left|f(a)-f(b)\right|\right]\left[g(b)-g(a)\right], \\ &\left[\frac{g(b)-g(a)}{2} + \left|\frac{g(a)+g(b)}{2} - \frac{1}{b-a}\int_a^b g(x)\,dx\right|\right]\left[f(a)+f(b)\right]\Bigg\}. \end{aligned} \tag{2.3}$$

(2) *If f is concave, then we have the inequality*

$$\begin{aligned}\int_a^b f(x)\,dg(x) \ge \max\Bigg\{ &\frac{1}{2}\left[f(a)+f(b)-\left|f(a)-f(b)\right|\right]\left[g(b)-g(a)\right], \\ &\left[\frac{g(b)-g(a)}{2} - \left|\frac{g(a)+g(b)}{2} - \frac{1}{b-a}\int_a^b g(x)\,dx\right|\right]\left[f(a)+f(b)\right]\Bigg\} \end{aligned} \tag{2.4}$$

provided that the Riemann–Stieltjes integral $\int_a^b f(x)\,dg(x)$ exists.

Proof. (1) In (2.2), set $s=1$, then we get

$$\begin{aligned}
&\int_a^b f(x)\,dg(x) \le \int_a^b \left[\frac{x-a}{b-a} f(b) + \frac{b-x}{b-a} f(a)\right] dg(x) \\
&= \frac{f(b)}{b-a}\left[(b-a)\,g(b) - \int_a^b g(x)\,dx\right] + \frac{f(a)}{b-a}\left[-(b-a)\,g(a) + \int_a^b g(x)\,dx\right] \\
&= f(b)\left[g(b) - \frac{1}{b-a}\int_a^b g(x)\,dx\right] + f(a)\left[\frac{1}{b-a}\int_a^b g(x)\,dx - g(a)\right].
\end{aligned}$$

Thus

$$\begin{aligned}
\int_a^b f(x)\,dg(x) &\le \max\{f(a), f(b)\}\,[g(b) - g(a)] \\
&= \frac{1}{2}\left[f(a) + f(b) + |f(a) - f(b)|\right][g(b) - g(a)]
\end{aligned}$$

and also

$$\begin{aligned}
\int_a^b f(x)\,dg(x) \le \max\Bigg\{&\left[g(b) - \frac{1}{b-a}\int_a^b g(x)\,dx\right], \\
&\left[\frac{1}{b-a}\int_a^b g(x)\,dx - g(a)\right]\Bigg\}\,[f(a) + f(b)] \\
= \left[\frac{g(b)-g(a)}{2} + \left|\frac{g(a)+g(b)}{2} - \frac{1}{b-a}\int_a^b g(x)\,dx\right|\right]&[f(a) + f(b)]
\end{aligned}$$

which proves (2.3).

(2) If f is concave, then we have

$$\int_a^b f(x)\,dg(x) \ge \int_a^b \left[\frac{x-a}{b-a} f(b) + \frac{b-x}{b-a} f(a)\right] dg(x).$$

So, similarly to the proof of (1),

$$\int_a^b f(x)\,dg(x) \ge \min\{f(a), f(b)\}\,[g(b) - g(a)]$$

and

$$\begin{aligned}
\int_a^b f(x)\,dg(x) \ge \min\Bigg\{&\left[g(b) - \frac{1}{b-a}\int_a^b g(x)\,dx\right], \\
&\left[\frac{1}{b-a}\int_a^b g(x)\,dx - g(a)\right]\Bigg\}\,[f(a) + f(b)]
\end{aligned}$$

which proves (2.4). □

Theorem 4. *Let $f, g : [a,b] \subseteq \mathbb{R}^+ \to \mathbb{R}$ be, respectively, s_1-, s_2-convex functions on $[a,b]$, $s_1, s_2 \in (0,1]$. Then we have the inequality*

$$\int_a^b f(x)\,dg(x) \leq \frac{s_2}{s_1+s_2}\left[f(b)\,g(b) - f(a)\,g(a)\right] + s_1\beta(s_2+1, s_1)\left[f(a)\,g(b) - f(b)\,g(a)\right] \tag{2.5}$$

provided that the Riemann–Stieltjes integral $\int_a^b f(x)\,dg(x)$ exists. If f, g are s_1-, s_2-concave, then the inequality (2.5) is reversed.

Proof. Since f is s_1-convex on $[a,b]$, by using the integration by parts formula for Riemann–Stieltjes integral, we have

$$\begin{aligned}
\int_a^b f(x)\,dg(x) &\leq \int_a^b \left[\left(\frac{x-a}{b-a}\right)^{s_1} f(b) + \left(\frac{b-x}{b-a}\right)^{s_1} f(a)\right] dg(x) \\
&= f(b)\int_a^b \left(\frac{x-a}{b-a}\right)^{s_1} dg(x) + f(a)\int_a^b \left(\frac{b-x}{b-a}\right)^{s_1} dg(x) \\
&= \frac{f(b)}{(b-a)^{s_1}}\int_a^b (x-a)^{s_1}\,dg(x) + \frac{f(a)}{(b-a)^{s_1}}\int_a^b (b-x)^{s_1}\,dg(x) \\
&= \frac{f(b)}{(b-a)^{s_1}}\left[(b-a)^{s_1} g(b) - s_1\int_a^b g(x)\,(x-a)^{s_1-1}\,dx\right] \\
&\quad + \frac{f(a)}{(b-a)^{s_1}}\left[-(b-a)^{s_1} g(a) + s_1\int_a^b g(x)\,(b-x)^{s_1-1}\,dx\right].
\end{aligned} \tag{2.6}$$

Since $g(x)$ is s_2-convex on $[a,b]$, we have

$$g(x) \leq \left[\left(\frac{x-a}{b-a}\right)^{s_2} g(b) + \left(\frac{b-x}{b-a}\right)^{s_2} g(a)\right],$$

which, by (2.6), gives

$$\begin{aligned}
&\int_a^b f(x)\,dg(x) \\
&\leq \frac{f(b)}{(b-a)^{s_1}}\left[(b-a)^{s_1} g(b)\right. \\
&\quad \left. -s_1\int_a^b \left(\left(\frac{x-a}{b-a}\right)^{s_2} g(b) + \left(\frac{b-x}{b-a}\right)^{s_2} g(a)\right)(x-a)^{s_1-1}\,dx\right] \\
&\quad + \frac{f(a)}{(b-a)^{s_1}}\left[-(b-a)^{s_1} g(a)\right. \\
&\quad \left. +s_1\int_a^b \left(\left(\frac{x-a}{b-a}\right)^{s_2} g(b) + \left(\frac{b-x}{b-a}\right)^{s_2} g(a)\right)(b-x)^{s_1-1}\,dx\right]
\end{aligned}$$

$$= \frac{f(b)}{(b-a)^{s_1}} \left[(b-a)^{s_1} g(b) - s_1 \frac{g(b)}{(b-a)^{s_2}} \int_a^b (x-a)^{s_1+s_2-1} dx \right.$$
$$\left. - s_1 \frac{g(a)}{(b-a)^{s_2}} \int_a^b (b-x)^{s_2} (x-a)^{s_1-1} dx \right]$$
$$+ \frac{f(a)}{(b-a)^{s_1}} \left[-(b-a)^{s_1} g(a) + s_1 \frac{g(b)}{(b-a)^{s_2}} \int_a^b (x-a)^{s_2} (b-x)^{s_1-1} dx \right.$$
$$\left. + s_1 \frac{g(a)}{(b-a)^{s_2}} \int_a^b (b-x)^{s_1+s_2-1} dx \right].$$

Simple calculations yield that

$$\int_a^b (b-x)^{s_2} (x-a)^{s_1-1} dx = (b-a)^{s_1+s_2} \beta(s_1, s_2+1),$$

and

$$\int_a^b (x-a)^{s_2} (b-x)^{s_1-1} dx = (b-a)^{s_1+s_2} \beta(s_2+1, s_1),$$

where,

$$\int_a^b (x-a)^p (b-x)^q dx = (b-a)^{p+q+1} \int_0^1 (1-t)^p t^q dt$$
$$= (b-a)^{p+q+1} \beta(p+1, q+1)$$

and $\beta(\cdot,\cdot)$ is the Euler Beta function.

It follows that

$$\int_a^b f(x)\, dg(x)$$
$$\leq \frac{f(b)}{(b-a)^{s_1}} \left[(b-a)^{s_1} g(b) - s_1 \frac{g(b)}{(b-a)^{s_2}} \frac{(b-a)^{s_1+s_2}}{s_1+s_2} \right.$$
$$\left. - s_1 \frac{(b-a)^{s_1+s_2} g(a)}{(b-a)^{s_2}} \beta(s_1, s_2+1) \right]$$
$$+ \frac{f(a)}{(b-a)^{s_1}} \left[-(b-a)^{s_1} g(a) + s_1 \frac{(b-a)^{s_1+s_2} g(b)}{(b-a)^{s_2}} \beta(s_2+1, s_1) \right.$$
$$\left. + s_1 \frac{g(a)}{(b-a)^{s_2}} \frac{(b-a)^{s_1+s_2}}{s_1+s_2} \right]$$
$$= f(b) g(b) - \frac{s_1}{s_1+s_2} f(b) g(b) - s_1 \beta(s_1, s_2+1) f(b) g(a)$$
$$- f(a) g(a) + s_1 \beta(s_2+1, s_1) f(a) g(b) + \frac{s_1}{s_1+s_2} f(a) g(a)$$
$$= \frac{s_2}{s_1+s_2} [f(b) g(b) - f(a) g(a)] + s_1 \beta(s_2+1, s_1) [f(a) g(b) - f(b) g(a)],$$

since $\beta(s_1, s_2+1) = \beta(s_2+1, s_1)$, which proves (2.5). □

Corollary 1. *In Theorem 4, if $s_1 = s_2 = 1$, i.e., f, g are two convex functions on $[a,b]$, then we have*

$$\int_a^b f(x)\,dg(x) \le \frac{f(a)+f(b)}{2}\left[g(b)-g(a)\right].$$

Corollary 2. *In Theorem 4, if f is convex and g is s-convex, then we have*

$$\int_a^b f(x)\,dg(x) \le f(b)\left[\frac{sg(b)-g(a)}{s+1}\right] + f(a)\left[\frac{g(b)-sg(a)}{s+1}\right] \tag{2.7}$$

provided that the Riemann–Stieltjes integral $\int_a^b f(x)\,dg(x)$ exists. If f is concave and g is s-concave, then the inequality (2.7) is reversed.

Theorem 5. *Let $f, g : [a,b] \subseteq \mathbb{R}^+ \to \mathbb{R}$ be such that g satisfies $\phi \le g(t) \le \Phi$ for all $t \in [a,b]$.*

(1) *If f is s-convex on $[a,b]$, then we have the inequality*

$$\int_a^b f(x)\,dg(x) \le f(b)\left[g(b)-\phi\right] + f(a)\left[\Phi - g(a)\right]. \tag{2.8}$$

(2) *If f is s-concave on $[a,b]$, then we have the inequality*

$$\int_a^b f(x)\,dg(x) \ge f(b)\left[g(b)-\Phi\right] + f(a)\left[\phi - g(a)\right] \tag{2.9}$$

provided that the Riemann–Stieltjes integral $\int_a^b f(x)\,dg(x)$ exists.

Proof. (1) From (2.2), we get

$$\begin{aligned}
\int_a^b f(x)\,dg(x) &\le \frac{f(b)}{(b-a)^s}\left[(b-a)^s g(b) - s\phi\int_a^b (x-a)^{s-1}\,dx\right] \\
&\quad + \frac{f(a)}{(b-a)^s}\left[-(b-a)^s g(a) + s\Phi\int_a^b (b-x)^{s-1}\,dx\right] \\
&\le \frac{f(b)}{(b-a)^s}\left[(b-a)^s g(b) - \phi(b-a)^s\right] \\
&\quad + \frac{f(a)}{(b-a)^s}\left[-(b-a)^s g(a) + \Phi(b-a)^s\right] \\
&= f(b)\left[g(b)-\phi\right] + f(a)\left[\Phi - g(a)\right]
\end{aligned}$$

which proves (2.8).

(2) If f is s-concave, then, similarly,

$$\begin{aligned}\int_a^b f(x)\,dg(x) &\geq \frac{f(b)}{(b-a)^s}\left[(b-a)^s g(b) - s\Phi\int_a^b (x-a)^{s-1}\,dx\right]\\ &\quad + \frac{f(a)}{(b-a)^s}\left[-(b-a)^s g(a) + s\phi\int_a^b (b-x)^{s-1}\,dx\right]\\ &\geq \frac{f(b)}{(b-a)^s}\left[(b-a)^s g(b) - \Phi(b-a)^s\right]\\ &\quad + \frac{f(a)}{(b-a)^s}\left[-(b-a)^s g(a) + \phi(b-a)^s\right]\\ &= f(b)\left[g(b)-\Phi\right] + f(a)\left[\phi - g(a)\right]\end{aligned}$$

which proves (2.9). □

Remark 1. Define the function $g : [a,b] \to \mathbb{R}^+$, $g(t) = \int_a^t u(s)ds$. Then g is differentiable on (a,b) and $g'(t) = u(t)$. And we have

$$\int_a^b f(x)\,dg(x) = \int_a^b f(x)\,u(x)\,dx.$$

Therefore, we can point out some results for the Riemann integral of a product.

(1) Under the assumptions of Theorem 2, we have

$$\int_a^b f(x)\,u(x)\,dx \leq [f(a)+f(b)]\int_a^b u(x)dx.$$

(2) Under the assumptions of Theorem 3, we have the following.

(a) If $f : [a,b] \to \mathbb{R}^+$ is convex on $[a,b]$, then

$$\begin{aligned}\int_a^b f(x)\,u(x)dx \leq \min\Bigg\{&\frac{1}{2}\left[f(a)+f(b)+|f(a)-f(b)|\right]\int_a^b u(x)dx,\\ &\frac{1}{2}\left[\int_a^b u(x)dx + \left|\int_a^b u(x)dx\right.\right.\\ &\left.\left.-\frac{2}{b-a}\int_a^b\int_a^x u(t)dtdx\right|\right]\left[f(a)+f(b)\right]\Bigg\}.\end{aligned}$$

(b) If f is concave, then

$$\int_a^b f(x)\,u(x)dx \geq \max\left\{\frac{1}{2}\left[f(a)+f(b)-|f(a)-f(b)|\right]\int_a^b u(x)dx,\right.$$
$$\frac{1}{2}\left[\int_a^b u(x)dx - \left|\int_a^b u(x)dx\right.\right.$$
$$\left.\left.\left. -\frac{2}{b-a}\int_a^b\int_a^x u(t)dtdx\right|\right]\left[f(a)+f(b)\right]\right\}.$$

(3) Under the assumptions of Theorem 4, we have

$$\int_a^b f(x)\,u(x)\,dx \leq \left[\frac{s_2}{s_1+s_2}f(b)+s_1\beta(s_2+1,s_1)\,f(a)\right]\int_a^b u(x)\,dx.$$

(4) Under the assumptions of Theorem 5, we have the following.

(a) If f is s-convex on $[a,b]$, then

$$\int_a^b f(x)\,u(x)\,dx \leq f(b)\left[\int_a^b u(x)\,dx-\phi\right]+\Phi f(a)\,.$$

(b) If f is s-concave, then

$$\int_a^b f(x)\,u(x)\,dx \geq f(b)\left[\int_a^b u(x)\,dx-\Phi\right]+\phi f(a)\,.$$

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Department of Mathematics, Faculty of Science, Jerash University, 26150 Jerash, Jordan

E-mail address: mwomath@gmail.com