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On a topological simple Warne extension of a semigroup

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Abstract. In the paper we introduce topological $\mathbb{Z}$-Bruck–Reilly and topological $\mathbb{Z}$-Bruck extensions of (semi)topological monoids, which are generalizations of topological Bruck–Reilly and topological Bruck extensions of (semi)topological monoids, and study their topologizations. The sufficient conditions under which the topological $\mathbb{Z}$-Bruck–Reilly ($\mathbb{Z}$-Bruck) extension admits only the direct sum topology and conditions under which the direct sum topology can be coarsened are given. Also, topological characterizations for some classes of I-bisimple (semi)topological semigroups are given.

1. Introduction and preliminaries

In this paper all topological spaces are assumed to be Hausdorff. We shall follow the terminology of [12, 13, 18, 38]. If Y is a subspace of a topological space X and $A \subseteq Y$, then by $\mathrm{cl}_Y(A)$ we shall denote the topological closure of A in Y. By $\mathbb{N}$ we denote the set of positive integers. Also, for a map $\theta\colon X \to Y$ and a positive integer n we denote by $\theta^{-1}(A)$ and $\theta^n(B)$ the full preimage of a set $A \subseteq Y$ and the n-power image of a set $B \subseteq X$, respectively, i.e., $\theta^{-1}(A) = \{x \in X \colon \theta(x) \in A\}$ and $\theta^n(B) = \{(\underbrace{\theta \circ \ldots \circ \theta}_{n \text{ times}})(x) \colon x \in B\}$.

A semigroup S is *regular* if $x \in xSx$ for every $x \in S$. A semigroup S is called *inverse* if for any element $x \in S$ there exists a unique $x^{-1} \in S$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. The element x^{-1} is called the *inverse of* $x \in S$. If S is an inverse semigroup, then the function $\mathrm{inv}\colon S \to S$ which assigns to every element x of S its inverse element x^{-1} is called the *inversion*. An inverse semigroup S is said to be *Clifford* if $x \cdot x^{-1} = x^{-1} \cdot x$ for all $x \in S$.

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If S is a semigroup, then we shall denote the subset of idempotents in S by $E(S)$. If S is an inverse semigroup, then $E(S)$ is closed under multiplication and we shall refer to $E(S)$ as a *band* (or the *band of* S). If the band $E(S)$ is a non-empty subset of S, then the semigroup operation on S determines the following partial order $\leqslant$ on $E(S)$: $e \leqslant f$ if and only if $ef = fe = e$. This order is called the *natural partial order* on $E(S)$. A *semilattice* is a commutative semigroup of idempotents. A semilattice E is called *linearly ordered* or a *chain* if its natural order is a linear order. If E is a semilattice and $e \in E$, then we denote ${\downarrow}e = \{f \in E \mid f \leqslant e\}$ and ${\uparrow}e = \{f \in E \mid e \leqslant f\}$.

If S is a semigroup, then by $\mathscr{R}$, $\mathscr{L}$, $\mathscr{J}$, $\mathscr{D}$ and $\mathscr{H}$ we shall denote the Green relations on S (see [13, Section 2.1]). A semigroup S is called *simple* if S does not contain any proper two-sided ideals and *bisimple* if S has only one $\mathscr{D}$-class.

A *semitopological* (respectively, *topological*) *semigroup* is a Hausdorff topological space together with a separately (respectively, jointly) continuous semigroup operation [12, 38]. An inverse topological semigroup with continuous inversion is called a *topological inverse semigroup*. A topology τ on a (inverse) semigroup S which turns S into a topological (inverse) semigroup is called a *semigroup* (*inverse*) *topology* on S. A *semitopological group* is a Hausdorff topological space together with a separately continuous group operation [38], and a *topological group* is a Hausdorff topological space together with a jointly continuous group operation and inversion [12].

The bicyclic semigroup $\mathscr{C}(p,q)$ is the semigroup with the identity 1 generated by elements p and q subjected only to the condition $pq = 1$. The bicyclic semigroup is bisimple and each of its congruences is either trivial or a group congruence. Moreover, for every non-annihilating homomorphism h of the bicyclic semigroup either h is an isomorphism or the image of $\mathscr{C}(p,q)$ under h is a cyclic group (see [13, Corollary 1.32]). The bicyclic semigroup plays an important role in algebraic theory of semigroups and in the theory of topological semigroups. For example, the well-known Andersen's result [6] states that a (0–)simple semigroup is completely (0–)simple if and only if it does not contain the bicyclic semigroup. The bicyclic semigroup admits only the discrete semigroup topology, and a topological semigroup S can contain the bicyclic semigroup $\mathscr{C}(p,q)$ as a dense subsemigroup only as an open subset [16]. Bertman and West in [10] proved that the bicyclic semigroup as a Hausdorff semitopological semigroup also admits only the discrete topology. The problem of the embedding of the bicycle semigroup into compact-like topological semigroups was solved in the papers [7, 8, 9, 26, 27], and the closure of the bicyclic semigroup in topological semigroups was studied in [16].

The properties of the bicyclic semigroup were extended to the following two directions: bicyclic-like semigroups which are bisimple and bicyclic-like extensions of semigroups. In the first case such are inverse bisimple semigroups with well-ordered subset of idempotents: ω^n-bisimple semigroups

[28], ω^α-bisimple semigroups [29] and an α-bicyclic semigroup, and bisimple inverse semigroups with linearly ordered subsets of idempotents which are isomorphic to either $[0,\infty)$ or $(-\infty,\infty)$ as subsets of the real line: $B^1_{[0,\infty)}$, $B^2_{[0,\infty)}$, $B^1_{(-\infty,\infty)}$ and $B^2_{(-\infty,\infty)}$. Ahre [1, 2, 3, 4, 5] and Korkmaz [33, 34] studied Hausdorff semigroup topologizations of the semigroups $B^1_{[0,\infty)}$, $B^2_{[0,\infty)}$, $B^1_{(-\infty,\infty)}$, $B^2_{(-\infty,\infty)}$ and their closures in topological semigroups. Annie Selden [42] and Hogan [30] proved that the only locally compact Hausdorff topology which turns an α-bicyclic semigroup into a topological semigroup is the discrete topology. In [31] Hogan studied Hausdorff inverse semigroup topologies on an α-bicyclic semigroup. There he constructed a non-discrete Hausdorff inverse semigroup topology on an α-bicyclic semigroup.

Let $\mathbb{Z}$ be the additive group of integers. On the Cartesian product $\mathscr{C}_{\mathbb{Z}} = \mathbb{Z}\times\mathbb{Z}$ we define the semigroup operation as follows:

$$(a,b)\cdot(c,d) = \begin{cases} (a-b+c,d), & \text{if } b<c,\\ (a,d), & \text{if } b=c,\\ (a,d-c+b), & \text{if } b>c, \end{cases} \tag{1}$$

for $a,b,c,d\in\mathbb{Z}$. The set $\mathscr{C}_{\mathbb{Z}}$ equipped with this operation is called the *extended bicyclic semigroup* [44]. It is obvious that the extended bicyclic semigroup is an extension of the bicyclic semigroup. The extended bicyclic semigroup admits only the discrete topology as a semitopological semigroup [19]. Also the problem of the closure of $\mathscr{C}_{\mathbb{Z}}$ in a topological semigroup was studied in [19].

The concept of Bruck–Reilly extensions originates from the Bruck paper [11], where he constructed an embedding of semigroups into simple monoids. Reilly in [37] generalized the Bruck construction to what is nowadays called the Bruck–Reilly construction and, using it, described the structure of ω-bisimple semigroups. Annie Selden in [39, 40, 41] described the structure of locally compact topological inverse ω-bisimple semigroups and their closures in topological semigroups.

The disquisition of topological Bruck–Reilly extensions of topological and semitopological semigroups was started in the papers [22, 24] and continued in [35, 25]. Using the ideas of the paper [22] Gutik in [23] constructed an embedding of an arbitrary topological (inverse) semigroup into a simple path-connected topological (inverse) monoid.

Let G be a linearly ordered group and let S be any semigroup. Let $\alpha\colon G^+\to \operatorname{End}(S^1)$ be a homomorphism from the positive cone G^+ into the semigroup of all endomorphisms of S^1. By $\mathscr{B}(S,G,\alpha)$ we denote the set $G\times S^1\times G$ with the following binary operation

$$\begin{aligned}&(g_1,s_1,h_1)\cdot(g_2,s_2,h_2) =\\ &= (g_1(h_1\wedge g_2)^{-1}g_2,\ \alpha[e\vee h_1^{-1}g_2](s_1)\cdot\alpha[e\vee g_2^{-1}h_1](s_2),\ h_2(h_1\wedge g_2)^{-1}h_1).\end{aligned} \tag{2}$$

This binary operation is associative and the set $\mathscr{B}(S,G^+,\alpha)=G^+\times S^1\times G^+$ with the semigroup operation induced from $\mathscr{B}(S,G,\alpha)$ is a subsemigroup of $\mathscr{B}(S,G,\alpha)$ [20].

Now we let $G=\mathbb{Z}$ be the additive group of integers with the usual order $\leq$ and let S be any semigroup. Let $\alpha\colon \mathbb{Z}^+\to \operatorname{End}(S^1)$ be a homomorphism from the positive cone $\mathbb{Z}^+$ into the semigroup of all endomorphisms of S^1. Then formula (2) determines the following semigroup operation on $\mathscr{B}(S,\mathbb{Z},\alpha)$:

$$(i,s,j)\cdot(m,t,n)=$$
$$(i+m-\min\{j,m\},\alpha[m-\min\{j,m\}](s)\cdot\alpha[j-\min\{j,m\}](t),j+n-\min\{j,m\}),$$

where $s,t\in S^1$ and $i,j,m,n\in\mathbb{Z}$.

Let $\theta\colon S^1\to H(1_S)$ be a homomorphism from the monoid S^1 into the group of units $H(1_S)$ of S^1. Then we put $\alpha[n](s)=\theta^n(s)$ for a positive integer n and let $\theta^0\colon S^1\to S^1$ be the identity map of S^1. The semigroup $\mathscr{B}(S,\mathbb{Z},\alpha)$ with such a homomorphism α will be denoted by $\mathscr{B}(S,\mathbb{Z},\theta)$ or, when the homomorphism $\theta\colon S^1\to H(1_S)$ is defined by the formula

$$\theta^n(s)=\begin{cases} 1_S, & \text{if } n>0,\\ s, & \text{if } n=0,\end{cases}$$

simply by $\mathscr{B}(S,\mathbb{Z})$. We observe that the semigroup operation on $\mathscr{B}(S,\mathbb{Z},\theta)$ is defined by the formula

$$(i,s,j)\cdot(m,t,n)=\begin{cases} (i-j+m,\theta^{m-j}(s)\cdot t,n), & \text{if } j<m,\\ (i,s\cdot t,n), & \text{if } j=m,\\ (i,s\cdot\theta^{j-m}(t),n-m+j), & \text{if } j>m,\end{cases} \tag{3}$$

for $i,j,m,n\in\mathbb{Z}$ and $s,t\in S^1$. We shall call the semigroup $\mathscr{B}(S,\mathbb{Z},\theta)$ the *$\mathbb{Z}$-Bruck–Reilly extension of the semigroup S* and $\mathscr{B}(S,\mathbb{Z})$ the *$\mathbb{Z}$-Bruck extension of the semigroup S*, respectively. We also observe that if S is a trivial semigroup, then the semigroups $\mathscr{B}(S,\mathbb{Z},\theta)$ and $\mathscr{B}(S,\mathbb{Z})$ are isomorphic to the extended bicyclic semigroup (see [44]).

Proposition 1.1. *Let S^1 be a monoid and $\theta\colon S^1\to H(1_S)$ be a homomorphism from S^1 into the group of units $H(1_S)$ of S^1. Then the following statements hold:*

(i) *$\mathscr{B}(S,\mathbb{Z},\theta)$ and $\mathscr{B}(S,\mathbb{Z})$ are simple semigroups;*
(ii) *$\mathscr{B}(S,\mathbb{Z},\theta)$ ($\mathscr{B}(S,\mathbb{Z})$) is an inverse semigroup if and only if S^1 is an inverse semigroup;*
(iii) *$\mathscr{B}(S,\mathbb{Z},\theta)$ ($\mathscr{B}(S,\mathbb{Z})$) is a regular semigroup if and only if S^1 is a regular semigroup.*

The proofs of the statements of Proposition 1.1 are similar to corresponding theorems of [13, Section 8.5] and [32, Theorem 5.6.6].

Also, we remark that the descriptions of Green's relations on the semigroups $\mathscr{B}(S,\mathbb{Z},\theta)$ and $\mathscr{B}(S,\mathbb{Z})$ are similar to those on the Bruck–Reilly and

Bruck extensions of S^1 (see [13, Lemma 8.46] and [32, Theorem 5.6.6(2)]). Hence the semigroup $\mathscr{B}(S,\mathbb{Z},\theta)$ (respectively, $\mathscr{B}(S,\mathbb{Z})$) is bisimple if and only if S^1 is bisimple.

Remark 1.2. Formula (3) implies that if $(i,s,j)\cdot(m,t,n)=(k,d,l)$ in the semigroup $\mathscr{B}(S,\mathbb{Z},\theta)$, then $k-l=i-j+m-n$.

For every $m,n\in\mathbb{Z}$ and $A\subseteq S$ we define $S_{m,n}=\{(m,s,n)\colon s\in S\}$ and $A_{m,n}=\{(m,s,n)\colon s\in A\}$.

In this paper we introduce the topological $\mathbb{Z}$-Bruck–Reilly and the topological $\mathbb{Z}$-Bruck extensions of (semi)topological monoids, which are generalizations of topological Bruck–Reilly and topological Bruck extensions of (semi)topological monoids, and study their topologizations. The sufficient conditions under which the topological $\mathbb{Z}$-Bruck–Reilly ($\mathbb{Z}$-Bruck) extension admits only the direct sum topology and conditions under which the direct sum topology can be coarsened are given. Also, topological characterizations for some classes of I-bisimple (semi)topological semigroups are given.

2. On topological $\mathbb{Z}$-Bruck–Reilly extensions

Let S be a monoid and let $H(1_S)$ be its group of units. Obviously if one of the following conditions holds:

1) $H(1_S)$ is a trivial group,
2) S is congruence-free and S is not a group,
3) S has zero,

then every homomorphism $\theta\colon S\to H(1_S)$ is annihilating. Also, many topological properties of a (semi)topological semigroup S guarantee the triviality of θ. For example, such is the following: $H(1_S)$ is a discrete subgroup of S and S has a minimal ideal $K(S)$ which is a connected subgroup of S.

On the other side, there exist many conditions on a (semitopological, topological) semigroup S which ensure the existence of a non-annihilating (continuous) homomorphism $\theta\colon S^1\to H(1_S)$ from S into the non-trivial group of units $H(1_S)$. For example, such conditions are the following:

1) the (semitopological, topological) semigroup S has a minimal ideal $K(S)$ which is a non-trivial group and there exists a non-annihilating (continuous) homomorphism $h\colon K(S)\to H(1_S)$;
2) S is an inverse semigroup and there exists a non-annihilating homomorphism $h\colon S/\sigma\to H(1_S)$, where σ is the least group congruence on S (see [36, Section III.5]).

Let (S,τ) be a semitopological monoid and let 1_S be the identity of S. If S does not contain an identity, then without loss of generality we can assume that S is a semigroup with an isolated adjoined identity. We shall also assume that the homomorphism $\theta\colon S^1\to H(1_S)$ is continuous.

Let $\mathcal{B}$ be a base of the topology τ on S. According to [22] the topology $\tau_{\mathbf{BR}}$ on $\mathscr{B}(S,\mathbb{Z},\theta)$ generated by the base

$$\mathcal{B}_{BR} = \{(i,U,j)\colon U\in\mathcal{B}, i,j\in\mathbb{Z}\}$$

is called the *direct sum topology* on $\mathscr{B}(S,\mathbb{Z},\theta)$. We shall denote it by $\tau^{\mathbf{ds}}_{\mathbf{BR}}$. We observe that the topology $\tau^{\mathbf{ds}}_{\mathbf{BR}}$ is the product topology on $\mathscr{B}(S,\mathbb{Z},\theta) = \mathbb{Z}\times S\times\mathbb{Z}$.

Proposition 2.1. *Let (S,τ) be a semitopological (respectively, topological, topological inverse) semigroup, and let $\theta\colon S^1\to H(1_S)$ be a continuous homomorphism from S into the group of units $H(1_S)$ of S. Then $\left(\mathscr{B}(S,\mathbb{Z},\theta),\tau^{\mathbf{ds}}_{\mathbf{BR}}\right)$ is a semitopological (respectively, topological, topological inverse) semigroup.*

The proof of Proposition 2.1 is similar to the proof of [22, Theorem 1].

Definition 2.2. Let $\mathfrak{S}$ be some class of semitopological semigroups and $(S,\tau)\in\mathfrak{S}$. If $\tau_{\mathbf{BR}}$ is a topology on $\mathscr{B}(S,\mathbb{Z},\theta)$ such that the homomorphism $\theta\colon S^1\to H(1_S)$ is a continuous map, $(\mathscr{B}(S,\mathbb{Z},\theta),\tau_{\mathbf{BR}})\in\mathfrak{S}$ and $\tau_{\mathbf{BR}}|_{S_{m,m}} = \tau$ for some $m\in\mathbb{Z}$, then the semigroup $(\mathscr{B}(S,\mathbb{Z},\theta),\tau_{\mathbf{BR}})$ is called a *topological $\mathbb{Z}$-Bruck–Reilly extension* of the semitopological semigroup (S,τ) in the class $\mathfrak{S}$. In the case when $\theta(s)=1_S$ for all $s\in S^1$, the semigroup $(\mathscr{B}(S,\mathbb{Z}),\tau_{\mathbf{BR}})$ is called a *topological $\mathbb{Z}$-Bruck extension* of the semitopological semigroup (S,τ) in the class $\mathfrak{S}$.

Proposition 2.1 implies that for every semitopological (respectively, topological, topological inverse) semigroup (S,τ) there exists a *topological $\mathbb{Z}$-Bruck–Reilly extension* $\left(\mathscr{B}(S,\mathbb{Z},\theta),\tau^{\mathbf{ds}}_{\mathbf{BR}}\right)$ of the semitopological (respectively, topological, topological inverse) semigroup (S,τ) in the class of semitopological (respectively, topological, topological inverse) semigroups. It is natural to ask: *when is $\left(\mathscr{B}(S,\mathbb{Z},\theta),\tau^{\mathbf{ds}}_{\mathbf{BR}}\right)$ unique for the semigroup (S,τ)?*

Proposition 2.3. *Let $(\mathscr{B}(S,\mathbb{Z},\theta),\tau_{\mathbf{BR}})$ be a semitopological semigroup. Then the following conditions hold:*

(i) *for every $i,j,k,l\in\mathbb{Z}$ the topological subspaces $S_{i,j}$ and $S_{k,l}$ are homeomorphic; moreover, $S_{i,i}$ and $S_{k,k}$ are topologically isomorphic subsemigroups in $(\mathscr{B}(S,\mathbb{Z},\theta),\tau_{\mathbf{BR}})$;*

(ii) *for every $(i,s,j)\in\mathscr{B}(S,\mathbb{Z},\theta)$ there exists an open neighbourhood $U_{(i,s,j)}$ of the point (i,s,j) in $(\mathscr{B}(S,\mathbb{Z},\theta),\tau_{\mathbf{BR}})$ such that*

$$U_{(i,s,j)}\subseteq\bigcup\left\{S_{i-k,j-k}\colon k=0,1,2,3,\ldots\right\}.$$

Proof. (i) For every $i,j,k,l\in\mathbb{Z}$ the map $\phi^{k,l}_{i,j}\colon\mathscr{B}(S,\mathbb{Z},\theta)\to\mathscr{B}(S,\mathbb{Z},\theta)$ defined by the formula $\phi^{k,l}_{i,j}(x) = (k,1_S,i)\cdot x\cdot(j,1_S,l)$ is continuous as a composition of left and right translations in the semitopological semigroup $(\mathscr{B}(S,\mathbb{Z},\theta),\tau_{\mathbf{BR}})$. Since $\phi^{i,j}_{k,l}(\phi^{k,l}_{i,j}(s)) = s$ and $\phi^{k,l}_{i,j}(\phi^{i,j}_{k,l}(t)) = t$ for all $s\in S_{i,j}$

and $t \in S_{k,l}$, we conclude that the restriction $\phi_{i,j}^{k,l}|_{S_{i,j}}$ is the inverse map of the restriction $\phi_{k,l}^{i,j}|_{S_{k,l}}$. Then the continuity of the map $\phi_{i,j}^{k,l}$ implies that the restriction $\phi_{i,j}^{k,l}|_{S_{i,j}}$ is a homeomorphism which maps elements of the subspace $S_{i,j}$ onto elements of the subspace $S_{k,l}$ in $\mathscr{B}(S,\mathbb{Z},\theta)$. Now the definition of the map $\phi_{i,j}^{k,l}$ implies that the restriction $\phi_{i,i}^{k,k}|_{S_{i,i}}\colon S_{i,i}\to S_{k,k}$ is a topological isomorphism of semitopological subsemigroups $S_{i,i}$ and $S_{k,k}$.

(ii) Since left and right translations in a semitopological semigroup are continuous maps and left and right translations by an idempotent are retractions, [18, Exercise 1.5.C] implies that $(i+1,1_S,i+1)\mathscr{B}(S,\mathbb{Z},\theta)$ and $\mathscr{B}(S,\mathbb{Z},\theta)(j+1,1_S,j+1)$ are closed subsets in $(\mathscr{B}(S,\mathbb{Z},\theta),\tau_{\mathbf{BR}})$. Hence there exists an open neighbourhood $W_{(i,s,j)}$ of the point (i,s,j) in $(\mathscr{B}(S,\mathbb{Z},\theta),\tau_{\mathbf{BR}})$ such that

$$W_{(i,s,j)}\subseteq\mathscr{B}(S,\mathbb{Z},\theta)\setminus\left((i+1,1_S,i+1)\mathscr{B}(S,\mathbb{Z},\theta)\cup\mathscr{B}(S,\mathbb{Z},\theta)(j+1,1_S,j+1)\right).$$

Since the semigroup operation in $(\mathscr{B}(S,\mathbb{Z},\theta),\tau_{\mathbf{BR}})$ is separately continuous, we conclude that there exists an open neighbourhood $U_{(i,s,j)}$ of the point (i,s,j) in $(\mathscr{B}(S,\mathbb{Z},\theta),\tau_{\mathbf{BR}})$ such that

$$U_{(i,s,j)}\subseteq W_{(i,s,j)},\ (i,1_S,i)\cdot U_{(i,s,j)}\subseteq W_{(i,s,j)} \text{ and } U_{(i,s,j)}\cdot(j,1_S,j)\subseteq W_{(i,s,j)}.$$

Next we shall show that $U_{(i,s,j)}\subseteq\bigcup\{S_{i-k,j-k}\colon k=0,1,2,3,\ldots\}$. Suppose the contrary: there exists $(m,a,n)\in U_{(i,s,j)}$ such that $(m,a,n)\notin\bigcup\{S_{i-k,j-k}\colon k=0,1,2,3,\ldots\}$. Then we have $m\leqslant i$, $n\leqslant j$ and $m-n\neq i-j$. If $m-n>i-j$, then we get

$$(m,a,n)\cdot(j,1_S,j)=(m-n+j,\theta^{j-n}(a),j)\notin\mathscr{B}(S,\mathbb{Z},\theta)\setminus(i+1,1_S,i+1)\mathscr{B}(S,\mathbb{Z},\theta)$$

because $m-n+j>i-j+j=i$, and hence $(m,a,n)\cdot(j,1_S,j)\notin W_{(i,s,j)}$. Similarly, if $m-n<i-j$, then we get

$$(i,1_S,i)\cdot(m,a,n)=(i,\theta^{i-m}(a),n-m+i)\notin\mathscr{B}(S,\mathbb{Z},\theta)\setminus\mathscr{B}(S,\mathbb{Z},\theta)(j+1,1_S,j+1)$$

because $n-m+i>j-i+i=j$, and hence $(i,1_S,i)\cdot(m,a,n)\notin W_{(i,s,j)}$. This completes the proof of our statement. □

Theorem 2.4. *Let $(\mathscr{B}(S,\mathbb{Z},\theta),\tau_{\mathbf{BR}})$ be a topological $\mathbb{Z}$-Bruck–Reilly extension of a semitopological semigroup (S,τ). If S contains a left (right or two-sided) compact ideal, then $\tau_{\mathbf{BR}}$ is the direct sum topology on $\mathscr{B}(S,\mathbb{Z},\theta)$.*

Proof. We consider the case when the semitopological semigroup S has a left compact ideal. In other cases the proof is similar. Let L be a left compact ideal in S. Then by Definition 2.2 there exists an integer n such that the subsemigroup $S_{n,n}$ in $(\mathscr{B}(S,\mathbb{Z},\theta),\tau_{\mathbf{BR}})$ is topologically isomorphic to the semitopological semigroup (S,τ). Hence Proposition 2.3 implies that $L_{i,j}$ is a compact subset of $(\mathscr{B}(S,\mathbb{Z},\theta),\tau_{\mathbf{BR}})$ for all $i,j\in\mathbb{Z}$.

We fix an arbitrary element (i,s,j) of the semigroup $\mathscr{B}(S,\mathbb{Z},\theta)$, $i,j\in\mathbb{Z}$ and $s\in S^1$. We also fix an element $(i-1,t,j-1)$ in $L_{i-1,j-1}$ and define a map $h\colon\mathscr{B}(S,\mathbb{Z},\theta)\to\mathscr{B}(S,\mathbb{Z},\theta)$ by the formula $h(x)=x\cdot(j-1,t,j-1)$. Then by Proposition 2.3(*ii*) there exists an open neighbourhood $U_{(i,s,j)}$ of the point (i,s,j) in $(\mathscr{B}(S,\mathbb{Z},\theta),\tau_{\mathbf{BR}})$ such that $U_{(i,s,j)}\subseteq\bigcup\{S_{i-k,j-k}\colon k=0,1,2,3,\ldots\}$. Since left translations in $(\mathscr{B}(S,\mathbb{Z},\theta),\tau_{\mathbf{BR}})$ are continuous, we conclude that the full pre-image $h^{-1}(L_{i-1,j-1})$ is a closed subset of the topological space $(\mathscr{B}(S,\mathbb{Z},\theta),\tau_{\mathbf{BR}})$, and Remark 1.2 implies that $h^{-1}(L_{i-1,j-1})=\bigcup\{S_{i-k,j-k}\colon k=1,2,3,\ldots\}$. Therefore, an arbitrary element (i,s,j) of the semigroup $\mathscr{B}(S,\mathbb{Z},\theta)$, where $i,j\in\mathbb{Z}$ and $s\in S^1$, has an open neighbourhood $U_{(i,s,j)}$ such that $U_{(i,s,j)}\subseteq S_{i,j}$. □

Theorem 2.4 yields the following corollary.

Corollary 2.5 (see [19])**.** *Let τ be a Hausdorff topology on the extended bicyclic semigroup $\mathscr{C}_{\mathbb{Z}}$. If $(\mathscr{C}_{\mathbb{Z}},\tau)$ is a semitopological semigroup, then $(\mathscr{C}_{\mathbb{Z}},\tau)$ is the discrete space.*

Theorem 2.6. *Let $(\mathscr{B}(S,\mathbb{Z},\theta),\tau_{\mathbf{BR}})$ be a topological $\mathbb{Z}$-Bruck–Reilly extension of a topological inverse semigroup (S,τ) in the class of topological inverse semigroups. If the band $E(S)$ contains a minimal idempotent, then $\tau_{\mathbf{BR}}$ is the direct sum topology on $\mathscr{B}(S,\mathbb{Z},\theta)$.*

Proof. Let e_0 be a minimal element of the band $E(S)$. Then (i,e_0,i) is a minimal idempotent in the band of the subsemigroup $S_{i,i}$ for every integer i.

Since the semigroup operation on $(\mathscr{B}(S,\mathbb{Z},\theta),\tau_{\mathbf{BR}})$ is continuous, we conclude that for every idempotent ι from the semigroup $\mathscr{B}(S,\mathbb{Z},\theta)$ the set ${\uparrow}\iota=\{\varepsilon\in E(\mathscr{B}(S,\mathbb{Z},\theta))\colon\varepsilon\cdot\iota=\iota\cdot\varepsilon=\iota\}$ is a closed subset in $E(\mathscr{B}(S,\mathbb{Z},\theta))$ with the topology induced from $(\mathscr{B}(S,\mathbb{Z},\theta),\tau_{\mathbf{BR}})$. We define the maps $\mathfrak{l}\colon\mathscr{B}(S,\mathbb{Z},\theta)\to E(\mathscr{B}(S,\mathbb{Z},\theta))$ and $\mathfrak{r}\colon\mathscr{B}(S,\mathbb{Z},\theta)\to E(\mathscr{B}(S,\mathbb{Z},\theta))$ by the formulae $\mathfrak{l}(x)=x\cdot x^{-1}$ and $\mathfrak{r}(x)=x^{-1}\cdot x$. We fix any element $(i,s,j)\in\mathscr{B}(S,\mathbb{Z},\theta)$. Since the semigroup operation and inversion are continuous in $(\mathscr{B}(S,\mathbb{Z},\theta),\tau_{\mathbf{BR}})$, we conclude that the sets $\mathfrak{l}^{-1}({\uparrow}(i-1,e_0,i-1))$ and $\mathfrak{r}^{-1}({\uparrow}(j-1,e_0,j-1))$ are closed in $(\mathscr{B}(S,\mathbb{Z},\theta),\tau_{\mathbf{BR}})$. Then by Proposition 2.3(ii) there exists an open neighbourhood $U_{(i,s,j)}$ of the point (i,s,j) in $(\mathscr{B}(S,\mathbb{Z},\theta),\tau_{\mathbf{BR}})$ such that $U_{(i,s,j)}\subseteq\bigcup\{S_{i-k,j-k}\colon k=0,1,2,3,\ldots\}$. Now elementary calculations show that

$$W_{(i,s,j)}=U_{(i,s,j)}\setminus\left(\mathfrak{l}^{-1}({\uparrow}(i-1,e_0,i-1))\cup\mathfrak{r}^{-1}({\uparrow}(j-1,e_0,j-1))\right)\subseteq S_{i,j}.$$

This completes the proof of our theorem. □

The following examples show that the arguments stated in Theorems 2.4 and 2.6 are important.

Example 2.7. Let $N_+=\{0,1,2,3,\ldots\}$ be the discrete topological space with the usual operation of addition of integers. We define a topology $\tau_{\mathbf{BR}}$ on $\mathscr{B}(N_+,\mathbb{Z})$ as follows:

(i) for every point $x \in N_+ \setminus \{0\}$ the base of the topology $\tau_{\mathbf{BR}}$ at (i,x,j) coincides with some base of the direct sum topology $\tau_{\mathbf{BR}}^{\mathbf{ds}}$ at (i,x,j) for all $i,j \in \mathbb{Z}$;

(ii) for any $i,j \in \mathbb{Z}$ the family $\mathscr{B}_{(i,0,j)} = \{U_{i,j}^k \colon k = 1,2,3,\ldots\}$, where

$$U_{i,j}^k = \{(i,0,j)\} \cup \{(i-1,s,j-1) \colon s = k,k+1,k+2,k+3,\ldots\},$$

is the base of the topology $\tau_{\mathbf{BR}}$ at the point $(i,0,j)$.

Simple verifications show that $(\mathscr{B}(N_+,\mathbb{Z}),\tau_{\mathbf{BR}})$ is a Hausdorff topological semigroup.

Example 2.8. Let $N_{\mathbf{m}} = \{0,1,2,3,\ldots\}$ be the discrete topological space with the semigroup operation $x \cdot y = \max\{x,y\}$. We identify the set $\mathscr{B}(N_{\mathbf{m}},\mathbb{Z})$ with $\mathscr{B}(N_+,\mathbb{Z})$. Let $\tau_{\mathbf{BR}}$ be the topology on $\mathscr{B}(N_+,\mathbb{Z})$ defined as in Example 2.7. Then simple verifications show that $(\mathscr{B}(N_{\mathbf{m}},\mathbb{Z}),\tau_{\mathbf{BR}})$ is a Hausdorff topological inverse semigroup.

Definition 2.9. We shall say that a semitopological semigroup S has the *open ideal property* (or shortly, S is an *OIP-semigroup*) if there exists a family $\mathscr{I} = \{I_\alpha\}_{\alpha\in\mathcal{A}}$ of open ideals in S such that for every $x \in S$ there exist an open ideal $I_\alpha \in \mathscr{I}$ and an open neighbourhood $U(x)$ of the point x in S such that $U(x) \cap I_\alpha = \varnothing$.

We observe that in Definition 2.9 the family $\mathscr{I} = \{I_\alpha\}_{\alpha\in\mathcal{A}}$ of open ideals in S satisfies the finite intersection property. Thus every semitopological OIP-semigroup does not contain a minimal ideal.

Theorem 2.10. *Let (S,τ) be a Hausdorff semitopological OIP-semigroup and let $\theta\colon S^1 \to H(1_S)$ be a continuous homomorphism. Then there exists a topological $\mathbb{Z}$-Bruck–Reilly extension $(\mathscr{B}(S,\mathbb{Z},\theta),\tau_{\mathbf{BR}})$ of (S,τ) in the class of semitopological semigroups such that the topology $\tau_{\mathbf{BR}}$ is strictly coarser than the direct sum topology $\tau_{\mathbf{BR}}^{\mathbf{ds}}$ on $\mathscr{B}(S,\mathbb{Z},\theta)$.*

Proof. Let $\mathscr{I} = \{I_\alpha\}_{\alpha\in\mathcal{A}}$ be a family of open ideals in (S,τ) such that for every $x \in S$ there exist $I_\alpha \in \mathscr{I}$ and an open neighbourhood $U(x)$ of the point x in (S,τ) such that $U(x) \cap I_\alpha = \varnothing$.

We shall define a base of the topology $\tau_{\mathbf{BR}}$ on $\mathscr{B}(S,\mathbb{Z},\theta)$ in the following way:

(1) for every $s \in S \setminus H(1_S)$ and $i,j \in \mathbb{Z}$ the base of the topology $\tau_{\mathbf{BR}}$ at the point (i,s,j) coincides with some base of the direct sum topology $\tau_{\mathbf{BR}}^{\mathbf{ds}}$ at (i,s,j);

(2) the family

$$\mathscr{B}_{(i,a,j)} = \left\{(U_a)_{i,j}^\alpha = (U_a)_{i,j} \cup \left(\theta^{-1}(U_a) \cap I_\alpha\right)_{i-1,j-1} \colon U_a \in \mathscr{B}_a, I_\alpha \in \mathscr{I}\right\},$$

where $\mathscr{B}_a$ is a base of the topology τ at the point a in S, is a base of the topology $\tau_{\mathbf{BR}}$ at the point (i,a,j), for every $a \in H(1_S)$ and all $i,j \in \mathbb{Z}$.

Since (S,τ) is a Hausdorff semitopological OIP-semigroup, we conclude that $\tau_{\mathbf{BR}}$ is a Hausdorff topology on $\mathscr{B}(S,\mathbb{Z},\theta)$ and, moreover, $\tau_{\mathbf{BR}}$ is a proper subfamily of $\tau_{\mathbf{BR}}^{\mathbf{ds}}$. Hence $\tau_{\mathbf{BR}}$ is a coarser topology on $\mathscr{B}(S,\mathbb{Z},\theta)$ than $\tau_{\mathbf{BR}}^{\mathbf{ds}}$.

Now we shall show that the semigroup operation on $(\mathscr{B}(S,\mathbb{Z},\theta),\tau_{\mathbf{BR}})$ is separately continuous. Since by Proposition 2.1 the semigroup operation on $(\mathscr{B}(S,\mathbb{Z},\theta),\tau_{\mathbf{BR}}^{\mathbf{ds}})$ is separately continuous, the definition of the topology $\tau_{\mathbf{BR}}$ on $\mathscr{B}(S,\mathbb{Z},\theta)$ implies that it is sufficient to show the separate continuity of the semigroup operation on $(\mathscr{B}(S,\mathbb{Z},\theta),\tau_{\mathbf{BR}})$ in the following three cases:

1) $(i,h,j)\cdot(m,g,n)$; 2) $(i,h,j)\cdot(m,s,n)$; and 3) $(m,s,n)\cdot(i,h,j)$,

where $s\in S\setminus H(1_S)$, $g,h\in H(1_S)$ and $i,j,m,n\in\mathbb{Z}$.

Consider case 1). Then we have

$$(i,h,j)\cdot(m,g,n)=\begin{cases}(i-j+m,\theta^{m-j}(h)\cdot g,n), & \text{if } j<m,\\ (i,h\cdot g,n), & \text{if } j=m,\\ (i,h\cdot\theta^{j-m}(g),n-m+j), & \text{if } j>m.\end{cases}$$

Suppose that $j<m$. Then the separate continuity of the semigroup operation on (S,τ) and the continuity of the homomorphism $\theta\colon S\to H(1_S)$ imply that for every open neighbourhood $U_{\theta^{m-j}(h)\cdot g}$ of the point $\theta^{m-j}(h)\cdot g$ in (S,τ) there exist open neighbourhoods V_h and W_g of the points h and g in (S,τ), respectively, such that

$$\theta^{m-j}(h)\cdot W_g\subseteq U_{\theta^{m-j}(h)\cdot g}\qquad\text{and}\qquad \theta^{m-j}(V_h)\cdot g\subseteq U_{\theta^{m-j}(h)\cdot g}.$$

Hence for every $I_\alpha\in\mathscr{I}$ we get

$$\begin{aligned}
&(i,h,j)\cdot(W_g)^\alpha_{m,n}\subseteq(i,h,j)\cdot\big((W_g)_{m,n}\cup\big(\theta^{-1}(W_g)\cap I_\alpha\big)_{m-1,n-1}\big)\\
&\subseteq\big((i,h,j)\cdot(W_g)_{m,n}\big)\cup\big((i,h,j)\cdot\big(\theta^{-1}(W_g)\cap I_\alpha\big)_{m-1,n-1}\big)\subseteq\\
&\begin{cases}\big(\theta^{m-j}(h)\cdot W_g\big)_{i-j+m,n}\cup\big(\theta^{m-1-j}(h)\cdot\big(\theta^{-1}(W_g)\cap I_\alpha\big)\big)_{i-j+m-1,n-1}, & \text{if } j<m-1,\\ \big(\theta(h)\cdot W_g\big)_{i-j+m,n}\cup\big(h\cdot\big(\theta^{-1}(W_g)\cap I_\alpha\big)\big)_{i,n-1}, & \text{if } j=m-1\end{cases}\\
&\subseteq\big(U_{\theta^{m-j}(h)\cdot g}\big)^\alpha_{i-j+m,n}
\end{aligned}$$

because $\theta\left(\theta^{m-1-j}(h)\cdot\big(\theta^{-1}(W_g)\cap I_\alpha\big)\right)\subseteq\theta^{m-j}(h)\cdot W_g\subseteq U_{\theta^{m-j}(h)\cdot g}$, and

$$\begin{aligned}
&(V_h)^\alpha_{i,j}\cdot(m,g,n)\subseteq\big((V_h)_{i,j}\cup\big(\theta^{-1}(V_h)\cap I_\alpha\big)_{i-1,j-1}\big)\cdot(m,g,n)\\
&\subseteq\big((V_h)_{i,j}\cdot(m,g,n)\big)\cup\big(\big(\theta^{-1}(V_h)\cap I_\alpha\big)_{i-1,j-1}\cdot(m,g,n)\big)\\
&\subseteq\big(\theta^{m-j}(V_h)\cdot g\big)_{i-j+m,n}\cup\big(\theta^{m-j+1}\big(\theta^{-1}(V_h)\cap I_\alpha\big)\cdot g\big)_{i-j+m,n}\\
&\subseteq\big(\theta^{m-j}(V_h)\cdot g\big)_{i-j+m,n}\cup\big(\theta^{m-j}(V_h)\cdot g\big)_{i-j+m,n}\\
&\subseteq\big(\theta^{m-j}(V_h)\cdot g\big)_{i-j+m,n}\subseteq\big(U_{\theta^{m-j}(h)\cdot g}\big)_{i-j+m,n}\subseteq\big(U_{\theta^{m-j}(h)\cdot g}\big)^\alpha_{i-j+m,n}.
\end{aligned}$$

Suppose that $j = m$. Then the separate continuity of the semigroup operation on (S,τ) implies that for every open neighbourhood $U_{h\cdot g}$ of the point $h\cdot g$ in (S,τ) there exist open neighbourhoods V_h and W_g of the points h and g in (S,τ), respectively, such that

$$V_h\cdot g\subseteq U_{h\cdot g} \qquad \text{and} \qquad h\cdot W_g\subseteq U_{h\cdot g}.$$

Then for every $I_\alpha\in\mathscr{I}$ we have

$$\begin{aligned}(V_h)^\alpha_{i,j}\cdot(m,g,n)&\subseteq\big((V_h)_{i,j}\cdot(m,g,n)\big)\cup\big(\big(\theta^{-1}(V_h)\cap I_\alpha\big)_{i-1,j-1}\cdot(m,g,n)\big)\\ &\subseteq (V_h\cdot g)_{i,n}\cup\big(\theta\left(\theta^{-1}(V_h)\cap I_\alpha\right)\cdot g\big)_{i,n}\subseteq (V_h\cdot g)_{i,n}\cup(V_h\cdot g)_{i,n}\\ &=(V_h\cdot g)_{i,n}\subseteq (U_{h\cdot g})^\alpha_{i,n},\end{aligned}$$

and

$$\begin{aligned}(i,h,j)\cdot(W_g)^\alpha_{m,n}&\subseteq\big((i,h,j)\cdot(W_g)_{m,n}\big)\cup\big((i,h,j)\cdot\big(\theta^{-1}(W_g)\cap I_\alpha\big)_{m-1,n-1}\big)\\ &\subseteq (h\cdot W_g)_{i,n}\cup\big(h\cdot\theta\left(\theta^{-1}(W_g)\cap I_\alpha\right)\big)_{i,n}\subseteq (h\cdot W_g)_{i,n}\cup(h\cdot W_g)_{i,n}\\ &=(h\cdot W_g)_{i,n}\subseteq (U_{h\cdot g})^\alpha_{i,n}.\end{aligned}$$

Suppose that $j > m$. Then the separate continuity of the semigroup operation on (S,τ) and the continuity of the homomorphism $\theta\colon S\to H(1_S)$ imply that for every open neighbourhood $U_{h\cdot\theta^{j-m}(g)}$ of the point $h\cdot\theta^{j-m}(g)$ in (S,τ) there exist open neighbourhoods V_h and W_g of the points h and g in (S,τ), respectively, such that

$$h\cdot\theta^{j-m}(W_g)\subseteq U_{h\cdot\theta^{j-m}(g)} \qquad \text{and} \qquad V_h\cdot\theta^{j-m}(g)\subseteq U_{h\cdot\theta^{j-m}(g)}.$$

Hence for every $I_\alpha\in\mathscr{I}$ we get

$$\begin{aligned}(i,h,j)\cdot(W_g)^\alpha_{m,n}&\subseteq\big((i,h,j)\cdot(W_g)_{m,n}\big)\cup\big((i,h,j)\cdot\big(\theta^{-1}(W_g)\cap I_\alpha\big)_{m-1,n-1}\big)\\ &\subseteq\big(h\cdot\theta^{j-m}(W_g)\big)_{i,n-m+j}\cup\big(h\cdot\theta^{j-m+1}\left(\theta^{-1}(W_g)\cap I_\alpha\right)\big)_{i,n-m+j}\\ &\subseteq\big(h\cdot\theta^{j-m}(W_g)\big)_{i,n-m+j}\cup\big(h\cdot\theta^{j-m}(W_g)\big)_{i,n-m+j}\\ &=\big(h\cdot\theta^{j-m}(W_g)\big)_{i,n-m+j}\subseteq\big(U_{h\cdot\theta^{j-m}(g)}\big)^\alpha_{i,n-m+j},\end{aligned}$$

and

$$\begin{aligned}&(V_h)^\alpha_{i,j}\cdot(m,g,n)\subseteq\big((V_h)_{i,j}\cdot(m,g,n)\big)\cup\big(\big(\theta^{-1}(V_h)\cap I_\alpha\big)_{i-1,j-1}\cdot(m,g,n)\big)\subseteq\\ &\begin{cases}\big(V_h\cdot\theta^{j-m}(g)\big)_{i,n-m+j}\cup\big(\left(\theta^{-1}(V_h)\cap I_\alpha\right)\cdot g\big)_{i-1,n}, & \text{if } j-1=m,\\ \big(V_h\cdot\theta^{j-m}(g)\big)_{i,n-m+j}\cup\big(\left(\theta^{-1}(V_h)\cap I_\alpha\right)\cdot\theta^{j-1-m}(g)\big)_{i-1,n-m+j-1}, & \text{if } j-1>m\end{cases}\\ &\subseteq\big(U_{h\cdot\theta^{j-m}(g)}\big)^\alpha_{i,n-m+j}\end{aligned}$$

because $\theta\left(\left(\theta^{-1}(V_h)\cap I_\alpha\right)\cdot\theta^{j-1-m}(g)\right)=V_h\cdot\theta^{j-m}(g)\subseteq U_{h\cdot\theta^{j-m}(g)}$.

We observe that if $g\in H(1_S)$ and $s\in S\setminus H(1_S)$ then $g\cdot s, s\cdot g\in S\setminus H(1_S)$. Otherwise, if $g\cdot s\in H(1_S)$, then we have $g^{-1}\cdot g\cdot s=1_S\cdot s=s\in H(1_S)$, which contradicts the fact that every translation by an element of the group of units of S is a bijective map (see [12, Vol. 1, p. 18]).

Consider case 2). Then we have

$$(i,h,j)\cdot(m,s,n)=\begin{cases}(i-j+m,\theta^{m-j}(h)\cdot s,n), & \text{if } j<m,\\ (i,h\cdot s,n), & \text{if } j=m,\\ (i,h\cdot\theta^{j-m}(s),n-m+j), & \text{if } j>m.\end{cases}$$

Suppose that $j<m$. Then the separate continuity of the semigroup operation on (S,τ) and the continuity of the homomorphism $\theta\colon S\to H(1_S)$ imply that for every open neighbourhood $U_{\theta^{m-j}(h)\cdot s}$ of the point $\theta^{m-j}(h)\cdot s$ in (S,τ) there exist open neighbourhoods V_h and W_s of the points h and g in (S,τ), respectively, such that

$$\theta^{m-j}(h)\cdot W_s\subseteq U_{\theta^{m-j}(h)\cdot s} \qquad \text{and} \qquad \theta^{m-j}(V_h)\cdot s\subseteq U_{\theta^{m-j}(h)\cdot s}.$$

Hence for every $I_\alpha\in\mathscr{I}$ we get that

$$(i,h,j)\cdot(W_s)_{m,n}\subseteq\left(\theta^{m-j}(h)\cdot W_s\right)_{i-j+m,n}\subseteq\left(U_{\theta^{m-j}(h)\cdot s}\right)_{i-j+m,n}$$

and

$$\begin{aligned}(V_h)^{\alpha}_{i,j}\cdot(m,s,n)&\subseteq\big((V_h)_{i,j}\cdot(m,s,n)\big)\cup\big(\left(\theta^{-1}(V_h)\cap I_\alpha\right)_{i-1,j-1}\cdot(m,s,n)\big)\\ &\subseteq\left(\theta^{m-j}(V_h)\cdot s\right)_{i-j+m,n}\cup\left(\theta^{m-j+1}\left(\theta^{-1}(V_h)\cap I_\alpha\right)\cdot s\right)_{i-j+m,n}\\ &\subseteq\left(\theta^{m-j}(V_h)\cdot s\right)_{i-j+m,n}\cup\left(\theta^{m-j}(V_h)\cdot s\right)_{i-j+m,n}\\ &\subseteq\left(\theta^{m-j}(V_h)\cdot s\right)_{i-j+m,n}\subseteq\left(U_{\theta^{m-j}(h)\cdot s}\right)_{i-j+m,n}.\end{aligned}$$

Suppose that $j=m$. Then the separate continuity of the semigroup operation on (S,τ) implies that for every open neighbourhood $U_{h\cdot s}$ of the point $h\cdot s$ in (S,τ) there exist open neighbourhoods V_h and W_s of the points h and s in (S,τ), respectively, such that

$$V_h\cdot s\subseteq U_{h\cdot s} \qquad \text{and} \qquad h\cdot W_s\subseteq U_{h\cdot s}.$$

Then for every $I_\alpha\in\mathscr{I}$ we have $(i,h,j)\cdot(W_s)_{m,n}\subseteq(h\cdot W_s)_{i,n}\subseteq(U_{h\cdot s})_{i,n}$ and

$$\begin{aligned}(V_h)^{\alpha}_{i,j}\cdot(m,s,n)&\subseteq\big((V_h)_{i,j}\cdot(m,s,n)\big)\cup\big(\left(\theta^{-1}(V_h)\cap I_\alpha\right)_{i-1,j-1}\cdot(m,s,n)\big)\\ &\subseteq(V_h\cdot s)_{i,n}\cup\left(\theta\left(\theta^{-1}(V_h)\cap I_\alpha\right)\cdot s\right)_{i,n}\subseteq(V_h\cdot s)_{i,n}\cup(V_h\cdot s)_{i,n}\\ &=(V_h\cdot s)_{i,n}\subseteq(U_{h\cdot s})_{i,n}.\end{aligned}$$

If $j>m$ then the separate continuity of the semigroup operation on (S,τ) and the continuity of the homomorphism $\theta\colon S\to H(1_S)$ imply that for every open neighbourhood $U_{h\cdot\theta^{j-m}(s)}$ of the point $h\cdot\theta^{j-m}(s)$ in (S,τ) there exist open neighbourhoods V_h and W_s of the points h and s in (S,τ), respectively, such that

$$h\cdot\theta^{j-m}(W_s)\subseteq U_{h\cdot\theta^{j-m}(s)} \qquad \text{and} \qquad V_h\cdot\theta^{j-m}(s)\subseteq U_{h\cdot\theta^{j-m}(s)}.$$

Hence for every $I_\alpha \in \mathscr{I}$ we get that

$$(i,h,j)\cdot(W_s)_{m,n} \subseteq \left(h\cdot\theta^{j-m}(W_s)\right)_{i,n-m+j} \subseteq \left(U_{h\cdot\theta^{j-m}(s)}\right)^{\alpha}_{i,n-m+j}$$

and

$$(V_h)^{\alpha}_{i,j}\cdot(m,s,n)\subseteq\left((V_h)_{i,j}\cdot(m,s,n)\right)\cup\left(\left(\theta^{-1}(V_h)\cap I_\alpha\right)_{i-1,j-1}\cdot(m,s,n)\right)\subseteq$$

$$\begin{cases}\left(V_h\cdot\theta^{j-m}(s)\right)_{i,n-m+j}\cup\left(\left(\theta^{-1}(V_h)\cap I_\alpha\right)\cdot s\right)_{i-1,n}, & \text{if } j-1=m,\\ \left(V_h\cdot\theta^{j-m}(s)\right)_{i,n-m+j}\cup\left(\left(\theta^{-1}(V_h)\cap I_\alpha\right)\cdot\theta^{j-1-m}(s)\right)_{i-1,n-m+j-1}, & \text{if } j-1>m\end{cases}$$

$$\subseteq \left(V_h\cdot\theta^{j-m}(s)\right)^{\alpha}_{i,n-m+j} \subseteq \left(U_{h\cdot\theta^{j-m}(s)}\right)^{\alpha}_{i,n-m+j}$$

because $\theta\left(\left(\theta^{-1}(V_h)\cap I_\alpha\right)\cdot\theta^{j-1-m}(s)\right)\subseteq V_h\cdot\theta^{j-m}(s)\subseteq U_{h\cdot\theta^{j-m}(s)}$.

In case 3) we have

$$(m,s,n)\cdot(i,g,j)=\begin{cases}(m-n+i,\theta^{i-n}(s)\cdot g,j), & \text{if } n<i,\\ (m,s\cdot g,j), & \text{if } n=i,\\ (m,s\cdot\theta^{n-i}(g),j-i+n), & \text{if } n>i.\end{cases}$$

In this case the proof of separate continuity of the semigroup operation on $(\mathscr{B}(S,\mathbb{Z},\theta),\tau_{\mathbf{BR}})$ is similar to case 2). □

We observe that in the case when $\theta(s)=1_S$ for all $s\in S^1$ a base of the topology $\tau_{\mathbf{BR}}$ on $\mathscr{B}(S,\mathbb{Z})$ is determined in the following way:

(1) for every $s\in S^1\setminus\{1_S\}$ and $i,j\in\mathbb{Z}$ the base of the topology $\tau_{\mathbf{BR}}$ at the point (i,s,j) coincides with some base of the direct sum topology $\tau^{\mathbf{ds}}_{\mathbf{BR}}$ at (i,s,j); and

(2) the family $\mathscr{B}_{(i,1_S,j)}=\left\{U^{\alpha}_{i,j}=U_{i,j}\cup(I_\alpha)_{i-1,j-1}\colon U\in\mathscr{B}_{1_S}, I_\alpha\in\mathscr{I}\right\}$, where $\mathscr{B}_{1_S}$ is a base of the topology τ at the point 1_S in S, is a base of the topology $\tau_{\mathbf{BR}}$ at the point $(i,1_S,j)$, for all $i,j\in\mathbb{Z}$.

Then Theorem 2.10 yields the following theorem.

Theorem 2.11. *Let (S,τ) be a Hausdorff semitopological OIP-semigroup. Then there exists a topological $\mathbb{Z}$-Bruck extension $(\mathscr{B}(S,\mathbb{Z}),\tau_{\mathbf{BR}})$ of (S,τ) in the class of semitopological semigroups such that the topology $\tau_{\mathbf{BR}}$ is strictly coarser than the direct sum topology $\tau^{\mathbf{ds}}_{\mathbf{BR}}$ on $\mathscr{B}(S,\mathbb{Z})$.*

Now we need the following proposition.

Proposition 2.12. *Let (S,τ) be a topological (inverse) OIP-semigroup. Let $\tau_{\mathbf{BR}}$ be the topology on the semigroup $\mathscr{B}(S,\mathbb{Z},\theta)$ defined in the proof of Theorem 2.10. Then $(\mathscr{B}(S,\mathbb{Z},\theta),\tau_{\mathbf{BR}})$ is a topological (inverse) semigroup.*

Proof. If (S,τ) is a topological semigroup, then Proposition 2.1 implies that the semigroup operation is continuous on $\left(\mathscr{B}(S,\mathbb{Z},\theta),\tau^{\mathbf{ds}}_{\mathbf{BR}}\right)$. Similarly, if the inversion in an inverse topological semigroup (S,τ) is continuous, then Proposition 2.1 implies that the inversion in $\left(\mathscr{B}(S,\mathbb{Z},\theta),\tau^{\mathbf{ds}}_{\mathbf{BR}}\right)$ is continuous too. Therefore it is sufficient to show that the semigroup operation is jointly continuous on $(\mathscr{B}(S,\mathbb{Z},\theta),\tau_{\mathbf{BR}})$ in the following three cases:

$$1)\ (i,h,j)\cdot(m,g,n);\ 2)\ (i,h,j)\cdot(m,s,n);\ \text{ and }\ 3)\ (m,s,n)\cdot(i,g,j).$$

Also in the case when (S,τ) is a topological inverse semigroup, it is sufficient to show that the inversion is continuous at the point (i,h,j) for all $h,g\in H(1_S)$, $s\in S\setminus H(1_S)$ and $i,j,m,n\in\mathbb{Z}$.

Consider case 1). Then we have

$$(i,h,j)\cdot(m,g,n)=\left\{\begin{array}{ll} (i-j+m,\theta^{m-j}(h)\cdot g,n), & \text{if } j<m,\\ (i,h\cdot g,n), & \text{if } j=m,\\ (i,h\cdot\theta^{j-m}(g),n-m+j), & \text{if } j>m.\end{array}\right.$$

If $j<m$ then the continuity of the semigroup operation on (S,τ) and the continuity of the homomorphism $\theta\colon S\to H(1_S)$ yield that for every open neighbourhood $U_{\theta^{m-j}(h)\cdot g}$ of the point $\theta^{m-j}(h)\cdot g$ in (S,τ) there exist open neighbourhoods V_h and W_g of the points h and g in (S,τ), respectively, such that $\theta^{m-j}(V_h)\cdot W_g\subseteq U_{\theta^{m-j}(h)\cdot g}$. Hence for every $I_\alpha\in\mathscr{I}$ we get

$$\begin{aligned}&(V_h)_{i,j}^{\alpha}\cdot(W_g)_{m,n}^{\alpha}\subseteq\big((V_h)_{i,j}\cdot(W_g)_{m,n}\big)\cup\big((V_h)_{i,j}\cdot\big(\theta^{-1}(W_g)\cap I_\alpha\big)_{m-1,n-1}\big)\cup\\ &\big(\big(\theta^{-1}(V_h)\cap I_\alpha\big)_{i-1,j-1}\cdot(W_g)_{m,n}\big)\cup\big(\big(\theta^{-1}(V_h)\cap I_\alpha\big)_{i-1,j-1}\cdot\big(\theta^{-1}(W_g)\cap I_\alpha\big)_{m-1,n-1}\big)\\ &\subseteq\big(\theta^{m-j}(V_h)\cdot W_g\big)_{i-j+m,n}\cup A\cup\big(\theta^{m-j+1}\left(\theta^{-1}(V_h)\cap I_\alpha\right)\cdot W_g\big)_{i-j+m,n}\cup\\ &\big(\theta^{m-j}\left(\theta^{-1}(V_h)\cap I_\alpha\right)\cdot\left(\theta^{-1}(W_g)\cap I_\alpha\right)\big)_{i-j+m-1,n-1}\subseteq\big(U_{\theta^{m-j}(h)\cdot g}\big)_{i-j+m,n}^{\alpha},\end{aligned}$$

where

$$A=\left\{\begin{array}{ll}\big(V_h\cdot\left(\theta^{-1}(W_g)\cap I_\alpha\right)\big)_{i,n-1}, & \text{if } j=m-1,\\ \big(\theta^{m-1-j}(V_h)\cdot\left(\theta^{-1}(W_g)\cap I_\alpha\right)\big)_{i-j+m-1,n-1}, & \text{if } j<m-1,\end{array}\right.$$

because

$$\begin{aligned}&\theta^{m-j+1}\left(\theta^{-1}(V_h)\cap I_\alpha\right)\cdot W_g\subseteq\theta^{m-j}(V_h)\cdot W_g\subseteq U_{\theta^{m-j}(h)\cdot g},\\ &\theta\left(\theta^{m-j}\left(\theta^{-1}(V_h)\cap I_\alpha\right)\cdot\left(\theta^{-1}(W_g)\cap I_\alpha\right)\right)\subseteq\theta^{m-j}(V_h)\cdot W_g\subseteq U_{\theta^{m-j}(h)\cdot g},\end{aligned}$$

and

$$\theta(A)=\left\{\begin{array}{ll}\theta(V_h)\cdot W_g, & \text{if } j=m-1,\\ \theta^{m-j}(V_h)\cdot W_g, & \text{if } j<m-1,\end{array}\right.\subseteq U_{\theta^{m-j}(h)\cdot g}.$$

The proof of the continuity of the semigroup operation on $(\mathscr{B}(S,\mathbb{Z},\theta),\tau_{\mathbf{BR}})$ in the case when $j>m$ is similar to the previous case.

If $j=m$ then the continuity of the semigroup operation on (S,τ) implies that for every open neighbourhood $U_{h\cdot g}$ of the point $h\cdot g$ in (S,τ) there exist open neighbourhoods V_h and W_g of the points h and g in (S,τ), respectively, such that $V_h\cdot W_g\subseteq U_{h\cdot g}$. Then for every $I_\alpha\in\mathscr{I}$ we get that

$$(V_h)_{i,j}^{\alpha}\cdot(W_g)_{m,n}^{\alpha}\subseteq(V_h\cdot W_g)_{i,n}^{\alpha}\subseteq(U_{h\cdot g})_{i,n}^{\alpha}.$$

In case 2) we have

$$(i,h,j)\cdot(m,s,n)=\begin{cases}(i-j+m,\theta^{m-j}(h)\cdot s,n), & \text{if } j<m,\\ (i,h\cdot s,n), & \text{if } j=m,\\ (i,h\cdot\theta^{j-m}(s),n-m+j), & \text{if } j>m,\end{cases}$$

where $\theta^{m-j}(h)\cdot s, h\cdot s\in S\setminus H(1_S)$ and $h\cdot\theta^{j-m}(s)\in H(1_S)$.

If $j<m$ then the continuity of the semigroup operation on (S,τ) and the continuity of the homomorphism $\theta\colon S\to H(1_S)$ imply that for every open neighbourhood $U_{\theta^{m-j}(h)\cdot s}$ of the point $\theta^{m-j}(h)\cdot s$ in (S,τ) there exist open neighbourhoods V_h and W_s of the points h and s in (S,τ), respectively, such that $\theta^{m-j}(V_h)\cdot W_s\subseteq U_{\theta^{m-j}(h)\cdot s}$. Hence for every $I_\alpha\in\mathscr{I}$ we have

$$\begin{aligned}(V_h)_{i,j}^{\alpha}\cdot(W_s)_{m,n} &\subseteq \left((V_h)_{i,j}\cdot(W_s)_{m,n}\right)\cup\left(\left(\theta^{-1}(V_h)\cap I_\alpha\right)_{i-1,j-1}\cdot(W_s)_{m,n}\right)\\ &\subseteq\left(\theta^{m-j}(V_h)\cdot W_s\right)_{i-j+m,n}\cup\left(\theta^{m-j+1}\left(\theta^{-1}(V_h)\cap I_\alpha\right)\cdot W_s\right)_{i-j+m,n}\\ &\subseteq\left(\theta^{m-j}(V_h)\cdot W_s\right)_{i-j+m,n}\subseteq\left(U_{\theta^{m-j}(h)\cdot s}\right)_{i-j+m,n}.\end{aligned}$$

If $j=m$ then the continuity of the semigroup operation on (S,τ) implies that for every open neighbourhood $U_{h\cdot s}$ of the point $h\cdot s$ in (S,τ) there exist open neighbourhoods V_h and W_s of the points h and s in (S,τ), respectively, such that $V_h\cdot W_s\subseteq U_{h\cdot s}$. Then for every $I_\alpha\in\mathscr{I}$ we get that

$$\begin{aligned}(V_h)_{i,j}^{\alpha}\cdot(W_s)_{m,n} &\subseteq\left((V_h)_{i,j}\cdot(W_s)_{m,n}\right)\cup\left(\left(\theta^{-1}(V_h)\cap I_\alpha\right)_{i-1,j-1}\cdot(W_s)_{m,n}\right)\\ &\subseteq(V_h\cdot W_s)_{i,n}\cup\left(\theta\left(\theta^{-1}(V_h)\cap I_\alpha\right)\cdot W_s\right)_{i,n}\subseteq(V_h\cdot W_s)_{i,n}\subseteq(U_{h\cdot s})_{i,n}.\end{aligned}$$

If $j>m$ then the continuity of the semigroup operation on (S,τ) and the continuity of the homomorphism $\theta\colon S\to H(1_S)$ imply that for every open neighbourhood $U_{h\cdot\theta^{j-m}(s)}$ of the point $h\cdot\theta^{j-m}(s)$ in (S,τ) there exist open neighbourhoods V_h and W_s of the points h and s in (S,τ), respectively, such that $V_h\cdot\theta^{j-m}(W_s)\subseteq U_{h\cdot\theta^{j-m}(s)}$. Hence for every $I_\alpha\in\mathscr{I}$ we have

$$\begin{aligned}&(V_h)_{i,j}^{\alpha}\cdot(W_s)_{m,n}\subseteq\left((V_h)_{i,j}\cdot(W_s)_{m,n}\right)\cup\left(\left(\theta^{-1}(V_h)\cap I_\alpha\right)_{i-1,j-1}\cdot(W_s)_{m,n}\right)\subseteq\\ &\begin{cases}\left(V_h\cdot\theta^{j-m}(W_s)\right)_{i,n-m+j}\cup\left(\left(\theta^{-1}(V_h)\cap I_\alpha\right)\cdot W_s\right)_{i-1,n}, & \text{if } j-1=m,\\ \left(V_h\cdot\theta^{j-m}(W_s)\right)_{i,n-m+j}\cup\left(\left(\theta^{-1}(V_h)\cap I_\alpha\right)\cdot\theta^{j-1-m}(W_s)\right)_{i-1,n-m+j-1}, & \text{if } j-1>m\end{cases}\\ &\subseteq\left(V_h\cdot\theta^{j-m}(W_s)\right)_{i,n-m+j}\cup\left(\theta^{-1}\left(U_{h\cdot\theta^{j-m}(s)}\right)\cap I_\alpha\right)_{i-1,n-m+j-1}\\ &\subseteq\left(U_{h\cdot\theta^{j-m}(s)}\right)_{i,n}^{\alpha}\end{aligned}$$

because

$$\theta\left(\left(\theta^{-1}(V_h)\cap I_\alpha\right)\cdot\theta^{j-1-m}(W_s)\right)\subseteq V_h\cdot\theta^{j-m}(W_s)\subseteq U_{h\cdot\theta^{j-m}(s)}.$$

The proof of the continuity of the semigroup operation on $(\mathscr{B}(S,\mathbb{Z},\theta),\tau_{\mathbf{BR}})$ in case 3) is similar to case 2).

If (S,τ) is a topological inverse semigroup, then for every ideal I in S we have $I^{-1}=I$, and for every open neighbourhoods V_s and $U_{s^{-1}}$ of the points s and s^{-1} in (S,τ), respectively, such that $(V_s)^{-1}\subseteq U_{s^{-1}}$ we have

$$\big((V_s)_{i,j}\big)^{-1}\subseteq \big(U_{s^{-1}}\big)_{j,i}, \text{ for } s\in S\setminus H(1_S) \quad \text{and}$$
$$\big((V_s)_{i,j}^{\alpha}\big)^{-1}\subseteq (U_{s^{-1}})_{j,i}^{\alpha}, \text{ for } s\in H(1_S),$$

for all $I_\alpha\in\mathscr{I}$. Hence $(\mathscr{B}(S,\mathbb{Z},\theta),\tau_{\mathbf{BR}})$ is a topological inverse semigroup. This completes the proof of the proposition. □

Theorem 2.10 and Proposition 2.12 imply the following result.

Theorem 2.13. *Let (S,τ) be a topological (inverse) OIP-semigroup. Then there exists a topological $\mathbb{Z}$-Bruck–Reilly extension $(\mathscr{B}(S,\mathbb{Z},\theta),\tau_{\mathbf{BR}})$ of (S,τ) in the class of topological (inverse) semigroups such that the topology $\tau_{\mathbf{BR}}$ is strictly coarser than the direct sum topology $\tau_{\mathbf{BR}}^{\mathbf{ds}}$ on $\mathscr{B}(S,\mathbb{Z},\theta)$.*

Theorem 2.13 yields the following corollary.

Corollary 2.14. *Let (S,τ) be a topological (inverse) OIP-semigroup. Then there exists a topological $\mathbb{Z}$-Bruck extension $(\mathscr{B}(S,\mathbb{Z}),\tau_{\mathbf{BR}})$ of (S,τ) in the class of topological (inverse) semigroups such that the topology $\tau_{\mathbf{BR}}$ is strictly coarser than the direct sum topology $\tau_{\mathbf{BR}}^{\mathbf{ds}}$ on $\mathscr{B}(S,\mathbb{Z})$.*

Recall (see [12]) that a topological semilattice E is said to be a *U-semilattice* if for every $x\in E$ and every open neighbourhood $U={\uparrow}U$ of x in E, there exists $y\in U$ such that $x\in \operatorname{Int}_E({\uparrow}y)$.

Remark 2.15. Let S be a Clifford inverse semigroup. We define a map $\varphi\colon S\to E(S)$ by the formula $\varphi(x)=x\cdot x^{-1}$. From [13, Theorem 4.11] it follows that if I is an ideal of $E(S)$, then $\varphi^{-1}(I)$ is an ideal of S.

The following theorem provides examples of topological OIP-semigroups.

Theorem 2.16. *Let (S,τ) be a topological inverse Clifford semigroup. If the band $E(S)$ of S has no smallest idempotent and satisfies one of the following conditions:*

(1) *for every $x\in E(S)$ there exists $y\in{\downarrow}x$ such that there is an open neighbourhood U_y of y with the compact closure $\operatorname{cl}_{E(S)}(U_y)$;*
(2) *$E(S)$ is locally compact;*
(3) *$E(S)$ is a U-semilattice,*

then (S,τ) is an OIP-semigroup.

Proof. Suppose condition (1) holds. We fix an arbitrary $x\in E(S)$. By [21, Proposition VI-1.14] the partial order on the topological semilattice $E(S)$ is closed, and hence the compact set $K=\operatorname{cl}_{E(S)}(U_y)$ has a minimal element e, which must also be a minimal element of ${\uparrow}K$. If ${\uparrow}K=E(S)$, then e is a minimal element of $E(S)$. Hence e is the least element of $E(S)$, because

$ef \leqslant e$ for any $f \in E(S)$ implies $e = ef$, i.e., $e \leqslant f$. This contradicts the fact that $E(S)$ does not have the least element.

Then the set $I_x = E(S) \setminus \uparrow \left(\mathrm{cl}_{E(S)}(U_y)\right)$ is an open ideal in $E(S)$, and by [21, Proposition VI-1.13(iii)] the set $U_x = \uparrow U_y$ is an open neighbourhood of the point x in $E(S)$ such that $I_x \cap U_x = \varnothing$. Therefore for every $x \in E(S)$ we constructed an open neighbourhood U_x of the point x in $E(S)$ and an open ideal I_x in $E(S)$ such that $I_x \cap U_x = \varnothing$. Hence the topological semilattice $E(S)$ is an OIP-semigroup. Now we apply Remark 2.15 and get that (S, τ) is an OIP-semigroup.

We observe that every locally compact semilattice satisfies condition (1).

Suppose condition (3) holds. We fix an arbitrary $x \in E(S)$. Since the semilattice $E(S)$ does not contain a minimal idempotent, we conclude that there exists an idempotent $e \in {\downarrow} x \setminus \{x\}$. Then by [21, Proposition VI-1.13(i)] the set $U_x = E(S) \setminus {\downarrow} e$ is open in $E(S)$, and it is obvious that $x \in U_x = \uparrow U_x$. Let $y_{[x,e]} \in U_x$ be such that $x \in \mathrm{Int}_{E(S)}(\uparrow y_{[x,e]})$. We put $V_x = \mathrm{Int}_{E(S)}(\uparrow y_{[x,e]})$ and $I_{[x,e]} = E(S) \setminus \uparrow y_{[x,e]}$. Then V_x is an open neighbourhood of x in $E(S)$ and $I_{[x,e]}$ is an open ideal in $E(S)$. Hence similar arguments as in case (1) show that (S, τ) is an OIP-semigroup. □

3. On I-bisimple topological inverse semigroups

A bisimple semigroup S is called an *I-bisimple semigroup* if and only if $E(S)$ is order isomorphic to $\mathbb{Z}$ under the reverse of the usual order.

In [44] Warne proved the following theorem.

Theorem 3.1 ([44, Theorem 1.3]). *A regular semigroup S is I-bisimple if and only if S is isomorphic to $\mathscr{B}_W = \mathbb{Z} \times G \times \mathbb{Z}$, where G is a group, under the multiplication*

$$(a, g, b) \cdot (c, h, d) = \begin{cases} (a, g \cdot f^{-1}_{b-c,c} \cdot \theta^{b-c}(h) \cdot f_{b-c,d}, d - c + b), & \text{if } b \geqslant c, \\ (a - b + c, f^{-1}_{c-b,a} \cdot \theta^{c-b}(g) \cdot f_{c-b,b} \cdot h, d), & \text{if } b \leqslant c, \end{cases} \tag{4}$$

where θ is an endomorphism of G, θ^0 denoting the identity automorphism of G, and for $m \in \mathbb{N}$, $n \in \mathbb{Z}$ one has

(1) *$f_{0,n} = e$ is the identity of G;*
(2) *$f_{m,n} = \theta^{m-1}(u_{n+1}) \cdot \theta^{m-2}(u_{n+2}) \cdot \ldots \cdot \theta(u_{n+(m-1)}) \cdot u_{n+m}$, where $\{u_n \colon n \in \mathbb{Z}\}$ is a collection of elements of G with $u_n = e$ if $n \in \mathbb{N}$.*

For arbitrary $i, j \in \mathbb{Z}$ we denote $G_{i,j} = \{(i, g, j) \in \mathscr{B}_W \colon g \in G\}$.

Theorem 3.2. *Let S be a regular I-bisimple semitopological semigroup. Then there exist a group G with the identity element e, an endomorphism $\theta \colon G \to G$, a collection $\{u_n \colon n \in \mathbb{Z}\}$ of elements of G with the property $u_n = e$ if $n \in \mathbb{N}$ and a topology on the semigroup $\mathscr{B}_W$ such that the following assertions hold:*

(i) *S is topologically isomorphic to a semitopological semigroup $\mathscr{B}_W$ (not necessarily with the product topology);*
(ii) *$G_{i,j}$ and $G_{k,l}$ are homeomorphic subspaces of $\mathscr{B}_W$ for all $i,j,k,l \in \mathbb{Z}$;*
(iii) *$G_{i,i}$ and $G_{k,k}$ are topologically isomorphic semitopological subgroups of $\mathscr{B}_W$ with the topology induced from $\mathscr{B}_W$ for all $i,k \in \mathbb{Z}$;*
(iv) *θ is a continuous endomorphism of the semitopological group $G = G_{i,i}$ with the topology induced from $\mathscr{B}_W$ for an arbitrary integer i;*
(v) *for every element $(i,g,j) \in \mathscr{B}_W$ there exists an open neighbourhood $U_{(i,g,j)}$ of the point (i,g,j) in $\mathscr{B}_W$ such that $U_{(i,g,j)} \subseteq \bigcup\{G_{i-k,j-k} : k = 0,1,2,3,\ldots\}$;*
(vi) *$E(S)$ is a discrete subspace of S.*

Proof. The first part of the theorem and assertion (i) follow from Theorem 3.1.

(ii) We fix arbitrary $i,j,k,l \in \mathbb{Z}$ and define the map $\varphi_{i,j}^{k,l} \colon \mathscr{B}_W \to \mathscr{B}_W$ by the formula $\varphi_{i,j}^{k,l}(x) = (k,e,i) \cdot x \cdot (j,e,l)$. Then formula (4) implies that the restriction $\varphi_{i,j}^{k,l}|_{G_{i,j}} \colon G_{i,j} \to G_{k,l}$ is a bijective map. Now the compositions $\varphi_{i,j}^{k,l}|_{G_{i,j}} \circ \varphi_{k,l}^{i,j}|_{G_{k,l}}$ and $\varphi_{k,l}^{i,j}|_{G_{k,l}} \circ \varphi_{i,j}^{k,l}|_{G_{i,j}}$ are identity maps of the sets $G_{i,j}$ and $G_{k,l}$, respectively, and hence the map $\varphi_{i,j}^{k,l}|_{G_{i,j}} \colon G_{i,j} \to G_{k,l}$ is invertible to $\varphi_{k,l}^{i,j}|_{G_{k,l}} \colon G_{k,l} \to G_{i,j}$. Since $\mathscr{B}_W$ is a semitopological semigroup, we conclude that $\varphi_{i,j}^{k,l}|_{G_{i,j}} \colon G_{i,j} \to G_{k,l}$ and $\varphi_{k,l}^{i,j}|_{G_{k,l}} \colon G_{k,l} \to G_{i,j}$ are continuous maps, and hence the map $\varphi_{i,j}^{k,l}|_{G_{i,j}} \colon G_{i,j} \to G_{k,l}$ is a homeomorphism.

(iii) Formula (4) implies that $G_{i,i}$ and $G_{k,k}$ are semitopological subgroups of $\mathscr{B}_W$ with the topology induced from $\mathscr{B}_W$ for all $i,k \in \mathbb{Z}$. Simple verifications show that the map $\varphi_{i,i}^{k,k}|_{G_{i,i}} \colon G_{i,i} \to G_{k,k}$ is a topological isomorphism.

(iv) Assertion (iii) implies that for arbitrary $i,k \in \mathbb{Z}$ the subspaces $G_{i,i}$ and $G_{k,k}$ with the induced semigroup operation are topologically isomorphic subgroups of $\mathscr{B}_W$, and hence the semitopological group G is correctly defined. Next we consider the map $f \colon G = G_{0,0} \to G = G_{1,1}$ defined by the formula $f(x) = x \cdot (1,e,1)$. Then by formula (4) we have

$$(0,g,0)\cdot(1,e,1) = (1, f_{1,0}^{-1}\cdot\theta(g)\cdot f_{1,0}\cdot e, 1) = (1, e^{-1}\cdot\theta(g)\cdot e\cdot e, 1) = (1,\theta(g),1),$$

and since the translations in $\mathscr{B}_W$ are continuous, we conclude that θ is a continuous endomorphism of the semitopological group G.

(v) Since left and right translations in a semitopological semigroup are continuous maps and left and right translations by an idempotent are retractions, [18, Exercise 1.5.C] implies that $(i+1,e,i+1)\mathscr{B}_W$ and $\mathscr{B}_W(j+1,e,j+1)$ are closed subsets in $\mathscr{B}_W$. Hence there exists an open neighbourhood $W_{(i,g,j)}$ of the point (i,g,j) in $\mathscr{B}_W$ such that

$$W_{(i,g,j)} \subseteq \mathscr{B}_W \setminus \left((i+1,e,i+1)\mathscr{B}_W \cup \mathscr{B}_W(j+1,e,j+1)\right).$$

Since the semigroup operation in $\mathscr{B}_W$ is separately continuous, we conclude that there exists an open neighbourhood $U_{(i,g,j)}$ of the point (i,g,j) in $\mathscr{B}_W$ such that

$$U_{(i,g,j)} \subseteq W_{(i,g,j)}, \quad (i,e,i)\cdot U_{(i,g,j)} \subseteq W_{(i,g,j)} \quad \text{and} \quad U_{(i,g,j)}\cdot(j,e,j) \subseteq W_{(i,g,j)}.$$

Next we shall show that $U_{(i,g,j)} \subseteq \bigcup\{G_{i-k,j-k}\colon k = 0,1,2,3,\ldots\}$. Suppose the contrary: there exists $(m,a,n) \in U_{(i,g,j)}$ such that $(m,a,n) \notin \bigcup\{G_{i-k,j-k}\colon k = 0,1,2,3,\ldots\}$. Then we have $m \leqslant i$, $n \leqslant j$ and $m-n \neq i-j$. If $m-n > i-j$ then formula (4) implies that there exists $u \in G$ such that

$$(m,a,n)\cdot(j,e,j) = (m-n+j,u,j) \notin \mathscr{B}_W \setminus (i+1,e,i+1)\mathscr{B}_W$$

because $m-n+j > i-j+j = i$, and hence $(m,a,n)\cdot(j,e,j) \notin W_{(i,g,j)}$. Similarly, if $m-n < i-j$ then formula (4) implies that there exists $v \in G$ such that

$$(i,e,i)\cdot(m,a,n) = (i,v,n-m+i) \notin \mathscr{B}_W \setminus \mathscr{B}_W(j+1,e,j+1)$$

because $n-m+i > j-i+i = j$, and hence $(i,e,i)\cdot(m,a,n) \notin W_{(i,g,j)}$. This completes the proof of our assertion.

(vi) The definition of an I-bisimple semigroup implies that $E(S)$ is order isomorphic to $\mathbb{Z}$ under the reverse of the usual order, and hence $E(S)$ is a subsemigroup of S. Then $E(S) = \{(n,e,n)\colon n \in \mathbb{Z}\}$ (see [44]). We fix an arbitrary $(i,e,i) \in E(S)$. Since translations by (i,e,i) in S are continuous retractions, [18, Theorem 1.4.1] implies that the set $\{x \in S\colon x\cdot(i-1,e,i-1) = (i-1,e,i-1)\}$ is closed in S, and [18, Exercise 1.5.C] implies that $(i+1,e,i+1)S$ is a closed subset in S too. It now follows that (i,e,i) is an isolated point of $E(S)$ with the topology induced from S. This completes the proof of our assertion. □

Theorem 3.3. *Let S be a regular I-bisimple semitopological semigroup. If S has a maximal compact subgroup then the following statements hold:*

(i) *S is topologically isomorphic to $\mathscr{B}_W = \mathbb{Z}\times G\times\mathbb{Z}$ with the product topology;*
(ii) *S is a locally compact topological inverse semigroup.*

Proof. (i) By Theorem 3.2(i) we know that the semitopological semigroup S is topologically isomorphic to a semitopological semigroup $\mathscr{B}_W = \mathbb{Z}\times G\times\mathbb{Z}$. It is obvious to show that for arbitrary $i,j \in \mathbb{Z}$ the $\mathscr{H}$-class $G_{i,j}$ of $\mathscr{B}_W$ is an open subset in $\mathscr{B}_W$. We fix an arbitrary $(i,g,j) \in G_{i,j}$. Then by Theorem 3.2(v) there exists an open neighbourhood $U_{(i,g,j)}$ of the point (i,g,j) in $\mathscr{B}_W$ such that $U_{(i,g,j)} \subseteq \bigcup\{G_{i-k,j-k}\colon k = 0,1,2,3,\ldots\}$. Since the semitopological semigroup S has a maximal compact subgroup, Theorem 3.2(ii) implies that every $\mathscr{H}$-class $G_{m,n}$ of $\mathscr{B}_W$ is a compact subset in $\mathscr{B}_W$. Then the separate continuity of the semigroup operation on $\mathscr{B}_W$ and [18, Theorem 1.4.1] imply that $\{x \in \mathscr{B}_W\colon x\cdot(i-1,e,i-1) \in G_{i-1,i-1}\}$ is a closed set in $\mathscr{B}_W$. Therefore there exists an open neighbourhood $V_{(i,g,j)} \subseteq U_{(i,g,j)}$ of the point (i,g,j) in $\mathscr{B}_W$ such that $V_{(i,g,j)} \subseteq G_{i,j}$. This completes the proof of the statement.

(ii) Statement (i), Theorem 3.2(ii) and [18, Theorem 3.3.13] imply that S is a locally compact space. Then statement (i), [18, Corollary 3.3.10] and the Ellis theorem (see [17, Theorem 2] or [12, Vol. 1, Theorem 1.18]) imply that every maximal subgroup $G_{n,n}$ of $\mathscr{B}_W$ is a topological group. We put $G = G_{n,n}$ for some $n \in \mathbb{Z}$ with the topology induced from $\mathscr{B}_W$. Theorem 3.2(iii) implies that the topological group G is correctly defined. Let $\mathfrak{B}_G$ be a base of the topology of the topological group G. Then statement (i) and Theorem 3.2(ii) imply that the family

$$\mathfrak{B}_{\mathscr{B}_W} = \{U_{i,j} \colon U \in \mathfrak{B}_G \text{ and } i,j \in \mathbb{Z}\},$$

where $U_{i,j} = \{(i,x,j) \colon x \in U\} \subseteq G_{i,j}$, is a base of the topology of the semitopological semigroup $\mathscr{B}_W$.

Since G is a topological group and $\theta \colon G \to G$ is a continuous homomorphism, we conclude that for arbitrary integers a,b,c,d with $b \geqslant c$, arbitrary $g,h \in G$ and any open neighbourhood W of the point $g \cdot f_{b-c,c}^{-1} \cdot \theta^{b-c}(h) \cdot f_{b-c,d}$ in the topological space G there exist open neighbourhoods W_g and W_h of the points g and h in G, respectively, such that

$$W_g \cdot f_{b-c,c}^{-1} \cdot \theta^{b-c}(W_h) \cdot f_{b-c,d} \subseteq W.$$

Then in the case when $b \geqslant c$ we obtain

$$(a,W_g,b)\cdot(c,W_h,d) \subseteq (a, W_g \cdot f_{b-c,c}^{-1} \cdot \theta^{b-c}(W_h) \cdot f_{b-c,d}, d-c+b) \subseteq (a,W,d-c+b).$$

Similarly, the continuity of the group operation on G and the continuity of the homomorphism θ imply that for arbitrary integers a,b,c,d with $b \leqslant c$, arbitrary $g,h \in G$ and any open neighbourhood U of $f_{c-b,a}^{-1} \cdot \theta^{c-b}(g) \cdot f_{c-b,b} \cdot h$ in the topological space G there exist open neighbourhoods U_g and U_h of the points g and h in G, respectively, such that

$$f_{c-b,a}^{-1} \cdot \theta^{c-b}(U_g) \cdot f_{c-b,b} \cdot U_h \subseteq U.$$

Then in the case when $b \leqslant c$ we obtain

$$(a,U_g,b)\cdot(c,U_h,d) \subseteq (a-b+c, f_{c-b,a}^{-1} \cdot \theta^{c-b}(U_g) \cdot f_{c-b,b} \cdot U_h, d) \subseteq (a-b+c,U,d).$$

Hence the semigroup operation is continuous on $\mathscr{B}_W$.

Also, since the inversion in G is continuous, we know that for every element g of G and any open neighbourhood $W_{g^{-1}}$ of its inverse g^{-1} in G there exists open neighbourhood U_g of g in G such that $(U_g)^{-1} \subseteq W_{g^{-1}}$. Then we get $(a,U_g,b)^{-1} \subseteq (b,W_{g^{-1}},a)$ for arbitrary integers a and b. This completes the proof that $\mathscr{B}_W$ is a topological inverse semigroup. $\square$

If S is a topological inverse semigroup then the maps $\mathfrak{l}\colon S \to E(S)$ and $\mathfrak{r}\colon S \to E(S)$ defined by the formulae $\mathfrak{l}(x) = x \cdot x^{-1}$ and $\mathfrak{r}(x) = x^{-1} \cdot x$ are continuous. Hence Theorem 3.2 implies the following corollary.

Corollary 3.4. *Let S be a regular I-bisimple topological inverse semigroup. Then every $\mathscr{H}$-class of S is a closed-and-open subset of S.*

A topological space X is called *Baire* if for each sequence $A_1, A_2, \dots, A_i, \dots$ of nowhere dense subsets of X the union $\bigcup\limits_{i=1}^{\infty} A_i$ is a co-dense subset of X (see [18]).

Since every Hausdorff Baire topology on a countable topological group is discrete, Corollary 3.4 implies the following

Corollary 3.5. *Every regular I-bisimple countable Hausdorff Baire topological inverse semigroup is discrete.*

A Tychonoff space X is called *Čech complete* if for every compactification cX of X the remainder $cX \setminus c(X)$ is an F_σ-set in cX (see [18]). Since every Čech complete space (and hence every locally compact space) is Baire, Corollary 3.5 implies the following

Corollary 3.6. *Every regular I-bisimple countable Hausdorff Čech complete (locally compact) topological inverse semigroup is discrete.*

The following provides an example of a Hausdorff locally compact zero-dimensional I-bisimple topological semigroup S with locally compact (discrete) maximal subgroup G such that S is not topologically isomorphic to $\mathscr{B}_W = \mathbb{Z} \times G \times \mathbb{Z}$ with the product topology, and hence S is not a topological inverse semigroup.

Example 3.7. Let Z be the additive group of integers and let $\theta\colon Z \to Z$ be an annihilating homomorphism, i.e., $\theta(m) = e$ is the identity of Z for every $m \in Z$. We let $\mathscr{B}(Z, \mathbb{Z})$ to be the $\mathbb{Z}$-Bruck extension of the group Z. Then Theorem 3.1 implies that $\mathscr{B}(Z, \mathbb{Z})$ is an I-bisimple semigroup.

We determine the topology τ on $\mathscr{B}(Z, \mathbb{Z})$ in the following way:

(i) all non-idempotent elements of the semigroup $\mathscr{B}(Z, \mathbb{Z})$ are isolated points in $(\mathscr{B}(Z, \mathbb{Z}), \tau)$; and

(ii) the family $\mathfrak{B}_{(i,e,j)} = \left\{U^n_{i,j} \colon i, j \in \mathbb{Z}, n \in \mathbb{Z}\right\}$, where $U^n_{i,j} = \{(i, e, j)\} \cup \{(i-1, k, j-1) \colon k \geqslant n\}$, is a base of the topology τ at the point $(i, e, j) \in \mathscr{B}(Z, \mathbb{Z})$, $i, j \in \mathbb{Z}$.

Simple verifications show that τ is a Hausdorff locally compact zero-dimensional topology on $\mathscr{B}(Z, \mathbb{Z})$. We shall prove that τ is a semigroup topology on $\mathscr{B}(Z, \mathbb{Z})$.

We remark that the semigroup operation on $\mathscr{B}(Z,\mathbb{Z})$ is defined by the formula

$$(i,g,j)\cdot(m,h,n)=\begin{cases}(i-j+m,h,n), & \text{if } j<m,\\ (i,g\cdot h,n), & \text{if } j=m,\\ (i,g,n-m+j), & \text{if } j>m\end{cases}$$

for arbitrary $i,j,m,n\in\mathbb{Z}$ and $g,h\in Z$. Since all non-idempotent elements of the semigroup $\mathscr{B}(Z,\mathbb{Z})$ are isolated points in $(\mathscr{B}(Z,\mathbb{Z}),\tau)$, it is sufficient to show that the semigroup operation on $(\mathscr{B}(Z,\mathbb{Z}),\tau)$ is continuous in the following cases:

a) $(i,g,j)\cdot(m,e,n)$; **b)** $(i,e,j)\cdot(m,g,n)$; **c)** $(i,e,j)\cdot(m,e,n)$,

where e is the unity of G and $g\in G\setminus\{e\}$.

Then we have in case **a)**:

(1) if $j<m-1$, then $(i,g,j)\cdot(m,e,n)=(i-j+m,e,n)$ and $\{(i,g,j)\}\cdot U^k_{m,n}\subseteq U^k_{i-j+m,n}$;
(2) if $j=m-1$ then $(i,g,j)\cdot(m,e,n)=(i+1,e,n)$ and $\{(i,g,j)\}\cdot U^k_{m,n}\subseteq U^{k+g}_{i+1,n}$;
(3) if $j\geqslant m$, then $(i,g,j)\cdot(m,e,n)=(i,g,n-m+j)$ and $\{(i,g,j)\}\cdot U^k_{m,n}\subseteq\{(i,g,n-m+j)\}$,

in case **b)**:

(1) if $j\leqslant m$, then $(i,e,j)\cdot(m,g,n)=(i-j+m,g,n)$ and $U^k_{i,j}\cdot\{(m,g,n)\}\subseteq\{(i-j+m,g,n)\}$;
(2) if $j=m+1$ then $(i,e,j)\cdot(m,g,n)=(i,e,n+1)$ and $U^k_{i,j}\cdot\{(m,g,n)\}\subseteq U^{k+g}_{i,n+1}$;
(2) if $j>m+1$, then $(i,e,j)\cdot(m,g,n)=(i,e,n-m+j)$ and $U^k_{i,j}\cdot\{(m,g,n)\}\subseteq U^k_{i,n-m+j}$,

and in case **c)**:

(1) if $j<m$, then $(i,e,j)\cdot(m,e,n)=(i-j+m,e,n)$ and $U^k_{i,j}\cdot U^l_{m,n}\subseteq U^l_{i-j+m,n}$;
(2) if $j=m$, then $(i,e,j)\cdot(m,e,n)=(i,e,n)$ and $U^k_{i,j}\cdot U^l_{m,n}\subseteq U^{k+l}_{i,n}$;
(3) if $j>m$, then $(i,e,j)\cdot(m,e,n)=(i,e,n-m+j)$ and $U^k_{i,j}\cdot U^l_{m,n}\subseteq U^k_{i,n-m+j}$,

for arbitrary integers k and l. Hence $(\mathscr{B}(Z,\mathbb{Z}),\tau)$ is a topological semigroup. It is obvious that the inversion in $(\mathscr{B}(Z,\mathbb{Z}),\tau)$ is not continuous.

Remark 3.8. (1) We observe that propositions similar to Theorems 3.2 and 3.3, Corollaries 3.4, 3.5 and 3.6 hold for ω-bisimple (semi)topological semigroups as topological Bruck–Reilly extensions.

(2) Example 3.7 also shows that there exists a Hausdorff locally compact zero-dimensional ω-bisimple topological semigroup S with a locally compact (discrete) maximal subgroup G such that S is not topologically isomorphic to the Bruck–Reilly extension with the product topology, and hence S is not a topological inverse semigroup.

(3) The statement of Theorem 3.3 is true in the case when the subsemigroup $C(S) = \{(i, g, i) \colon i \in \mathbb{Z} \text{ and } g \in G\}$ is weakly uniform (see [43] for the definition of a weakly uniform topological semigroup). In this case the inversion in $C(S)$ is continuous (see [14] and [15]). Hence by Proposition 2.3 we get that every $\mathscr{H}$-class of S is an open-and-closed subset of S. This implies that the inversion in S is continuous, too.

The following provides an example of a Hausdorff locally compact zero-dimensional I-bisimple semitopological semigroup S with continuous inversion and a locally compact (discrete) maximal subgroup G such that S is not topologically isomorphic to $\mathscr{B}_W = \mathbb{Z} \times G \times \mathbb{Z}$ with the product topology, and hence S is not a topological inverse semigroup.

Example 3.9. Let Z be the additive group of integers and let $\theta \colon Z \to Z$ be an annihilating homomorphism.

We determine the topology τ on $\mathscr{B}(Z, \mathbb{Z})$ in the following way:

(i) all non-idempotent elements of the semigroup $\mathscr{B}(Z, \mathbb{Z})$ are isolated points in $(\mathscr{B}(Z, \mathbb{Z}), \tau)$; and

(ii) the family $\mathfrak{B}_{(i,e,j)} = \left\{U^{m,n}_{i,j} \colon i, j \in \mathbb{Z}, m, n \in \mathbb{Z}\right\}$, where

$$U^{m,n}_{i,j} = \{(i, e, j)\} \cup \{(i-1, k, j-1) \colon k \leqslant -n\} \\ \cup \{(i-1, k, j-1) \colon k \geqslant n\},$$

is a base of the topology τ at the point $(i, e, j) \in \mathscr{B}(Z, \mathbb{Z})$, $i, j \in \mathbb{Z}$.

Simple verifications show that τ is a Hausdorff locally compact zero-dimensional topology on $\mathscr{B}(Z, \mathbb{Z})$. The proof of the separate continuity of semigroup operation and the continuity of inversion in $(\mathscr{B}(Z, \mathbb{Z}), \tau)$ is similar to Example 3.7.

Remark 3.10. Example 3.9 shows that there exists a Hausdorff locally compact zero-dimensional ω-bisimple semitopological semigroup S with continuous inversion and a locally compact (discrete) maximal subgroup G such that S is not topologically isomorphic to the Bruck–Reilly extension with the product topology, and hence S is not a topological inverse semigroup.

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