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A description of relatively (p, r)-compact sets

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Abstract. We introduce the notion of (p, r)-null sequences in a Banach space and we prove a Grothendieck-like result: a subset of a Banach space is relatively (p, r)-compact if and only if it is contained in the closed convex hull of a (p, r)-null sequence. This extends a recent description of relatively p-compact sets due to Delgado and Piñeiro, providing to it an alternative straightforward proof.

1. Introduction

Let X be a Banach space and let $c_0(X)$ denote the space of X-valued null sequences. By a classical result due to Grothendieck [6] (see, e.g., [7, p. 30]), a subset K of X is relatively compact if and only if there exists $(x_n) \in c_0(X)$ such that $K \subset \overline{\mathrm{conv}}(x_n)$, the closed convex hull of the sequence (x_n).

There is a strong form of compactness, the p-compactness, where $p \geq 1$ is a real number, that has been studied during the last years in the literature (see, e.g., [3], [4], [5], [12]). Recently, Delgado and Piñeiro [10] introduced and studied an interesting class $c_{0,p}(X)$ of p-null sequences which is a linear subspace of $c_0(X)$ (in [8], it is proved that $c_{0,p}(X)$ coincides with the Chevet–Saphar tensor product $c_0 \hat{\otimes}_{d_p} X$). One of their main results is as follows.

Theorem 1 (Delgado–Piñeiro; see [10, Theorem 2.5])**.** *Let $1 \leq p < \infty$. A subset K of a Banach space X is relatively p-compact if and only if there exists $(x_n) \in c_{0,p}(X)$ such that $K \subset \overline{\mathrm{conv}}(x_n)$.*

The proof of Theorem 1 in [10] is not self-contained. It relies on some theory of p-compact operators developed by Delgado, Piñeiro, and Serrano in [5] (see [5, Corrollary 3.4, Propositions 3.5 and 3.8, and Theorem 3.13]) and uses a characterization of operators having absolutely p-summing adjoints (see [10, Proposition 2.4]).

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The aim of the present note is to give a very easy direct proof of the Delgado–Piñeiro theorem. However, we shall proceed in a more general setting of relatively (p, r)-compact sets, which is a very recent concept from [1]. This will be done in Section 3.

In Section 2 we extend the notion of p-null sequences due to Delgado and Piñeiro [10] to (p, r)-null sequences, and we show that a (p, r)-null sequence converges to 0 and is relatively (p, r)-compact.

Our notation is standard. We consider Banach spaces over the same, either real or complex, field $\mathbb{K}$. The closed unit ball of a Banach space X is denoted by B_X. The Banach space of all p-summable sequences in X is denoted by $\ell_p(X)$ and its norm by $\|(x_n)\|_p$. If $1 \leq p \leq \infty$, then p^* denotes the conjugate index of p (i.e., $1/p + 1/p^* = 1$ with the convention $1/\infty = 0$).

2. Relatively (p, r)-compact sets and (p, r)-null sequences

Let X be a Banach space and let $p \geq 1$ be a real number. The *p-convex hull* of a sequence $(z_k) \in \ell_p(X)$ is defined as

$$p\text{-conv}(z_k) = \left\{ \sum_{k=1}^{\infty} a_k z_k : (a_k) \in B_{\ell_{p^*}} \right\}.$$

Let $1 \leq p \leq \infty$ and $1 \leq r \leq p^*$. We define the *(p, r)-convex hull* of a sequence $(z_k) \in \ell_p(X)$ (where $(z_k) \in c_0(X)$ if $p = \infty$) by

$$(p, r)\text{-conv}(z_k) = \left\{ \sum_{k=1}^{\infty} a_k z_k : (a_k) \in B_{\ell_r} \right\}.$$

As in [1], we say that a subset K of X is *relatively (p, r)-compact* if $K \subset (p, r)\text{-conv}(x_n)$ for some $(x_n) \in \ell_p(X)$, (where $(x_n) \in c_0(X)$ if $p = \infty$). According to Grothendieck's criterion, the $(\infty, 1)$-compactness coincides with the usual compactness (because $(\infty, 1)\text{-conv}(x_n)$ is precisely the closed absolutely convex hull of (x_n)). The $(p, 1)$-compactness was occasionally considered in the 1980s by Reinov [11] and by Bourgain and Reinov [2] in the study of approximation properties of order $s \leq 1$. The (p, p^*)-compactness was introduced in 2002 by Sinha and Karn [12] under the name of *p-compactness*.

In general, for $p \neq q$, $r \neq s$, relatively (p, r)-compact sets and relatively (q, s)-compact sets do not coincide. In [10, Proposition 3.5], it was proved (see also [9, Proposition 20] for a short proof within the framework of operator ideals) that in all infinite-dimensional Banach spaces, relatively p-compact subsets differ from relatively q-compact subsets for $p \neq q$, $2 \leq p, q \leq \infty$. In our setting, this means that relatively (p, p^*)-compact sets differ from relatively (q, q^*)-compact sets.

Let $1 \leq p < \infty$ and $1 \leq r \leq p^*$. Extending the notion of p-null sequences due to Delgado and Piñeiro [10], we call a sequence (x_n) in X *(p, r)-null* if for every $\epsilon > 0$ there exist $N \in \mathbb{N}$ and $(z_k) \in \ell_p(X)$ with $\|(z_k)\|_p \leq \epsilon$ such that $x_n \in (p, r)\text{-conv}(z_k)$ for all $n \geq N$. The *p-null* sequences in [10] are defined using the $p\text{-conv}(z_k)$, and they are precisely the (p, p^*)-null sequences.

Let us point out a couple of easy properties of (p, r)-null sequences.

Proposition 2. *Let $1 \leq p < \infty$ and $1 \leq r \leq p^*$. If a sequence (x_n) in a Banach space X is (p, r)-null, then $x_n \to 0$ and (x_n) is relatively (p, r)-compact.*

Proof. Suppose that $(x_n) \subset X$ is (p, r)-null. Then for every $\epsilon > 0$ there exist $N \in \mathbb{N}$ and $(z_k) \in \ell_p(X)$, $\|(z_k)\|_p \leq \epsilon$, such that $x_n = \sum_{k=1}^{\infty} a_k^n z_k$, where $(a_k^n)_{k=1}^{\infty} \in B_{\ell_r}$, for all $n \geq N$.

We have, for all $n \geq N$,

$$\|x_n\| \leq \sum_{k=1}^{\infty} \|a_k^n z_k\| \leq \|(a_k^n)_k\|_{p^*} \|(z_k)\|_p \leq \|(a_k^n)_k\|_r \|(z_k)\|_p \leq \epsilon,$$

and therefore $x_n \to 0$.

We already have $\{x_N, x_{N+1}, ...\} \subset (p, r)\text{-conv}(z_k)$. We also have $\{x_1, ..., x_{N-1}\} \subset (p, r)\text{-conv}(w_k)$, where

$$w_k = \begin{cases} x_k & \text{if } k < N, \\ 0 & \text{if } k \geq N. \end{cases}$$

Now the sequence

$$y_k = \begin{cases} w_k & \text{if } k \text{ is odd}, \\ z_k & \text{if } k \text{ is even}, \end{cases}$$

is in $\ell_p(X)$ and $x_n \in (p, r)\text{-conv}(y_k)$ for all $n \in \mathbb{N}$. $\square$

Sometimes (see, e.g., the next section), it is convenient to look at (p, r)-convex hulls in the following way.

Let $1 \leq p < \infty$ and $1 \leq r \leq p^*$. It is well known and easy to see that every $(z_k) \in \ell_p(X)$ defines a compact operator $\Phi_{(z_k)} : \ell_r \to X$ through the equality

$$\Phi_{(z_k)}(a_k) = \sum_{k=1}^{\infty} a_k z_k, \ (a_k) \in \ell_r.$$

Clearly,

$$(p, r)\text{-conv}(z_k) = \Phi_{(z_k)}(B_{\ell_r}),$$

so that a subset K of X is relatively (p, r)-compact if and only if $K \subset \Phi_{(z_k)}(B_{\ell_r})$ for some $(z_k) \in \ell_p(X)$; in particular, $\Phi_{(z_k)}(B_{\ell_r})$ itself is relatively (p, r)-compact.

3. Relatively (p, r)-compact sets are contained in closed convex hulls of (p, r)-null sequences

In the case when $r = p^*$, the following theorem reduces to the Delgado–Piñeiro theorem (see Theorem 1).

Theorem 3. *Let $1 \le p < \infty$ and $1 \le r \le p^*$. A subset K of a Banach space X is relatively (p, r)-compact if and only if K is contained in the closed convex hull of a (p, r)-null sequence.*

Proof. For the "if" part, let $(x_n) \subset X$ be a (p, r)-null sequence. By Proposition 2, (x_n) is relatively (p, r)-compact. Thus $(x_n) \subset \Phi_{(z_k)}(B_{\ell_r})$ for some $(z_k) \in \ell_p(X)$. The set $\Phi_{(z_k)}(B_{\ell_r})$ is clearly absolutely convex. It is also weakly compact. Indeed, if $1 < r < \infty$, then B_{ℓ_r} is weakly compact; if $r = 1$ (or $r = \infty$), then $B_{\ell_1} = B_{c_0^*}$ (or $B_{\ell_\infty} = B_{\ell_1^*}$) is weak* compact and $\Phi_{(z_k)} \in \mathcal{L}(c_0^*, X)$ (or $\Phi_{(z_k)} \in \mathcal{L}(\ell_1^*, X)$) is weak* to weakly continuous (because $\Phi^*_{(z_k)}(X^*) \subset c_0$ (or $\Phi^*_{(z_k)}(X^*) \subset \ell_1$)). Hence, $\Phi_{(z_k)}(B_{\ell_r})$ is a closed absolutely convex subset of X containing (x_n). Since $\overline{\mathrm{conv}}(x_n)$ is contained in the relatively (p, r)-compact set $\Phi_{(z_k)}(B_{\ell_r})$, it is also relatively (p, r)-compact.

For the "only if" part, let us assume that $K \subset X$ is relatively (p, r)-compact. We clearly may assume that $K = \Phi_{(z_k)}(B_{\ell_r})$ for some $(z_k) \in \ell_p(X)$. We are going to construct a (p, r)-null sequence (x_n) such that $K \subset \overline{\mathrm{conv}}(x_n)$.

Similarly to the very beginning of the proof of [4, Theorem 2.1, (c)$\Rightarrow$(a)] (or of [10, Theorem 2.5]), we choose $\alpha_k \searrow 0$ such that $(\alpha_k^{-1} z_k) \in \ell_p(X)$, and we consider a compact diagonal operator $D : \ell_r \to \ell_r$, defined by $D(\beta_k) = (\alpha_k \beta_k)$, $(\beta_k) \in \ell_r$, and $\Phi := \Phi_{(\alpha_k^{-1} z_k)} : \ell_r \to X$. Then, clearly, $\Phi_{(z_k)} = \Phi D$.

Since $D(B_{\ell_r})$ is a relatively compact subset of ℓ_r, by Grothendieck's criterion, there exists a sequence $(\Gamma_n) \subset \ell_r$ such that $\Gamma_n \to 0$ and $D(B_{\ell_r}) \subset \overline{\mathrm{conv}}(\Gamma_n)$. Denote $x_n = \Phi \Gamma_n$. Then $K \subset \overline{\mathrm{conv}}(x_n)$, and it remains to show that (x_n) is a (p, r)-null sequence.

Let $\Gamma_n = (\gamma_k^n)_{k=1}^\infty$. Then $\sum_{k=1}^\infty |\gamma_k^n|^r = \|\Gamma_n\|_r^r \underset{n}{\to} 0$ if $r < \infty$ or $\sup_k |\gamma_k^n| \underset{n}{\to} 0$ if $r = \infty$. Let us only consider the former case, the latter case being similar.

Let $\epsilon > 0$ be fixed. Choose $\delta > 0$ satisfying $\delta^r \le 1 - 2^{-r}$ and $\delta^p(2^p \nu^p + 1) \le \epsilon^p$, where $\nu := \|(\alpha_k^{-1} z_k)\|_p$. Then there exists $N \in \mathbb{N}$ such that $\sum_{k=1}^\infty |\gamma_k^n|^r < \delta^r$ if $n \ge N$, and also $\sum_{k>N} \|\alpha_k^{-1} z_k\|^p < \delta^p$. Now

$$
\begin{aligned}
x_n &= \sum_{k=1}^\infty \gamma_k^n \alpha_k^{-1} z_k = \sum_{k=1}^N \gamma_k^n \alpha_k^{-1} z_k + \sum_{k>N} \gamma_k^n \alpha_k^{-1} z_k \\
&= \sum_{k=1}^N \frac{\gamma_k^n}{2\delta} 2\delta \alpha_k^{-1} z_k + \sum_{k>N} \gamma_k^n \alpha_k^{-1} z_k = \sum_{k=1}^\infty \delta_k^n y_k,
\end{aligned}
$$

where

$$(\delta_k^n)_{k=1}^{\infty} := (\frac{\gamma_1^n}{2\delta}, ..., \frac{\gamma_N^n}{2\delta}, \gamma_{N+1}^n, \gamma_{N+2}^n, ...) \in \ell_r$$

and

$$(y_k)_{k=1}^{\infty} := (2\delta\alpha_1^{-1}z_1, ..., 2\delta\alpha_N^{-1}z_N, \alpha_{N+1}^{-1}z_{N+1}, \alpha_{N+2}^{-1}z_{N+2}, ...) \subset X.$$

Observe that

$$\begin{aligned} \|(y_k)\|_p^p &= 2^p\delta^p \sum_{k=1}^{N} \|\alpha_k^{-1}z_k\|^p + \sum_{k>N} \|\alpha_k^{-1}z_k\|^p \\ &< 2^p\delta^p\nu^p + \delta^p = \delta^p(2^p\nu^p + 1) \leq \epsilon^p, \end{aligned}$$

i.e., $(y_k) \in \ell_p(X)$ and $\|(y_k)\|_p \leq \epsilon$. Observe also that, for every $n \geq N$,

$$\|(\delta_k^n)_k\|_r^r = \sum_{k=1}^{N} \frac{|\gamma_k^n|^r}{2^r\delta^r} + \sum_{k>N} |\gamma_k^n|^r < \frac{\delta^r}{2^r\delta^r} + \delta^r = \frac{1}{2^r} + \delta^r \leq 1,$$

i.e., $(\delta_k^n)_k \in B_{\ell_r}$ if $n \geq N$. Hence, for every $n \geq N$, we have

$$x_n = \sum_{k=1}^{\infty} \delta_k^n y_k \in (p,r)\text{-conv}(y_k),$$

as desired. □

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