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Categorical equivalence of some algebras

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Abstract. We show that two lattices are categorically equivalent if and only if they are isomorphic or dually isomorphic. Two normal bands are categorically equivalent if and only if they are isomorphic or anti-isomorphic. The same result holds for finite bands as well. We also obtain corollaries for rectangular bands and semilattices.

1. Introduction

A variety of algebras is considered as a category: the objects are the algebras in the variety and the morphisms are the homomorphisms between them.

Two algebras A and B are called *categorically equivalent*, if there is a categorical equivalence between the varieties they generate that sends A to B.

All algebraic notions and properties that can be expressed by means of categorical language are preserved under categorical equivalence. For example, for categorically equivalent algebras A and B,

- the endomorphism semigroups of A and B are isomorphic;
- the subalgebra lattices of A and B are isomorphic;
- the congruence lattices of A and B are isomorphic.

(For the proofs, see [1].) Also, since categorical equivalence preserves limits, all corresponding direct powers of categorically equivalent algebras are also categorically equivalent.

Two algebras are called *term-equivalent* if their base sets and term operations coincide. Two algebras A and B are called *weakly isomorphic* if there exists an algebra C isomorphic to A such that C is term-equivalent to B.

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According to R. McKenzie [4], weakly isomorphic algebras are categorically equivalent.

An algebra A is *directly irreducible* if $|A| > 1$, and $A \simeq B \times C$ implies $|B| = 1$ or $|C| = 1$. An algebra A has the *unique factorization property* if

- **(i)** A is isomorphic to a product of directly irreducible algebras;
- **(ii)** if

$$A \simeq \prod_{i\in I} B_i \simeq \prod_{j\in J} C_j$$

for some index sets I and J and for directly irreducible algebras B_i and C_j, then there is a bijection $\phi : I \to J$ such that $B_i \simeq C_{\phi(i)}$ for all $i \in I$.

An algebra A has the *refinement property* (for direct factorizations) if

$$A \simeq \prod_{i\in I} B_i \simeq \prod_{j\in J} C_j$$

implies the existence of algebras D_{ij} ($i \in I$, $j \in J$) such that, for all $i \in I$ and $j \in J$,

$$B_i \simeq \prod_{j\in J} D_{ij} \quad \text{and} \quad C_j \simeq \prod_{i\in I} D_{ij}.$$

It is easy to see ([5]) that the refinement property implies condition **(ii)** of the unique factorization property (but does not imply condition **(i)**).

An ordered set (A, R) is called *connected* if the conditions

$$B \cup C = A, \quad B \cap C = \emptyset, \quad R \subseteq B^2 \cup C^2$$

entail that $B = A$ or $C = A$.

Notions of the direct irreducibility, the unique factorization property, and the refinement property for ordered sets are defined in the same way as for algebras.

In [5], Section 5.6, the following facts, that we will use later, are proved.

Proposition 1.1. *Every connected ordered set has the refinement property.*

Proposition 1.2. *For an algebra A, if* $\operatorname{Con} A$ *is distributive, then A has the refinement property.*

2. Finite groups and semigroups

In 1997, L. Zádori [8] proved the following result.

Theorem 2.1. *Finite groups are categorically equivalent if and only if they are weakly isomorphic.*

In his work Zádori also stated a question of whether two weakly isomorphic finite groups are always isomorphic. However, K. A. Kearnes and Á. Szendrei showed in 2004 (see [2]) that there exist term-equivalent finite groups that are not isomorphic. Thus categorical equivalence of general finite groups implies only weak isomorphism.

But if at least one of the groups is abelian, the situation is slightly different. It is well known that weakly isomorphic groups are isomorphic, provided one of them is abelian, thus categorically equivalent finite groups are isomorphic as soon as one of them is abelian.

Recently, M. Behrisch and T. Waldhauser managed to generalize the result of Zádori. Their result was presented in June 2012 at the Conference on Universal Algebra and Lattice Theory in Szeged.

Theorem 2.2. *Finite semigroups are categorically equivalent if and only if they are weakly isomorphic.*

3. Bands

We say that two groupoids $(A, \cdot)$ and $(B, \cdot)$ are *anti-isomorphic*, if there is a bijection $\phi : A \to B$ such that $\phi(xy) = \phi(y)\phi(x)$ for any $x, y \in A$. We start with the following simple observations.

Lemma 3.1. *Composition of two anti-isomorphisms is an isomorphism.*

Lemma 3.2. *For every groupoid A there exists a unique (up to isomorphism) groupoid A^* anti-isomorphic to A.*

Proof. We define A^* to be groupoid with the same base set as A but with the multiplication $x * y := yx$. Then $\phi : A \to A^*$ defined by $\phi(x) = x$ is an anti-isomorphism.

The uniqueness follows directly from Lemma 3.1. □

Lemma 3.3. *For any groupoids A and B, $(A \times B)^* \simeq A^* \times B^*$.*

Proof. Let $\phi_1 : A \to A^*$ and $\phi_2 : B \to B^*$ be anti-isomorphisms. Then $\phi : A \times B \to A^* \times B^*$ defined by $\phi((x, y)) = (\phi_1(x), \phi_2(y))$ is an anti-isomorphism. □

Proposition 3.4. *Anti-isomorphic groupoids are weakly isomorphic.*

Proof. Let $(A, \cdot)$ and $(B, \cdot)$ be two anti-isomorphic groupoids. Consider the groupoid $(B, *)$ with $x * y := yx$. Groupoids $(B, \cdot)$ and $(B, *)$ are anti-isomorphic, hence $(A, \cdot)$ and $(B, *)$ are isomorphic. On the other hand, $(B, \cdot)$ and $(B, *)$ are term-equivalent. Altogether we get that $(A, \cdot)$ and $(B, \cdot)$ are weakly isomorphic. □

Corollary 3.5. *Anti-isomorphic groupoids are categorically equivalent.*

A *band* is a semigroup consisting of idempotents. A *rectangular band* is a band satisfying the identity $xyx = x$. Every rectangular band is isomorphic to a direct product of a left zero band (satisfies the identity $xy = x$) and a right zero band (satisfies the identity $xy = y$). A band is called *normal* if it satisfies the identity $xyzx = xzyx$.

It is well known (see, for example, [3], p. 262) that every band S is a semilattice Y of rectangular subbands, $S = \cup\{S_e : e \in Y\}$, and $S_e S_f \subset S_{ef}$ for $e, f \in Y$. The semilattice Y is called the *structural semilattice* of S. The rectangular bands S_e, $e \in Y$, are called the *rectangular components* of S. Both the structural semilattice and the rectangular components of a band are uniquely determined (up to isomorphism).

Lemma 3.6. *If a band S is isomorphic to the direct product of a semilattice Y and a rectangular band C, then Y and C are uniquely determined.*

Proof. If $S \simeq Y \times C$, where Y is a semilattice and C a rectangular band, then Y is the structural semilattice of S and the rectangular components are all isomorphic to C. □

Lemma 3.7. *If Y is a semilattice and C a rectangular band, then $(Y \times C)^* \simeq Y \times C^*$.*

Proof. Since semilattices are commutative, $Y^* \simeq Y$. Thus we have

$$(Y \times C)^* \simeq Y^* \times C^* \simeq Y \times C^*.$$

□

Suppose Y is a chain semilattice, i.e., Y is linearly ordered, and $st = \min\{s, t\}$ for all $s, t \in Y$. Then Y^d denotes the dual chain semilattice, i.e., the semigroup $(Y, *)$, where $s * t = \max\{s, t\}$ for any $s, t \in Y$. If C is a rectangular band, then both $Y \times C$ and $Y^d \times C$ are normal bands. Every band isomorphic to $Y \times C$ is called a *chain normal band.* If S and T are bands isomorphic to $Y \times C$ and $Y^d \times C$, respectively, S and T are called *dual chain normal bands.*

In 1985, B. M. Schein [6] characterized normal bands with isomorphic endomorphism semigroups.

Theorem 3.8 (see [6], Theorem 3)**.** *Let S and T be normal bands with isomorphic endomorphism semigroups. Then one of the following holds:*

(1) *S and T are isomorphic;*
(2) *S and T are anti-isomorphic;*
(3) *S and T are dual chain normal bands;*
(4) *S and T are chain normal bands and T is anti-isomorphic to the dual of S.*

Now we are ready to prove the following result.

Theorem 3.9. *Normal bands are categorically equivalent if and only if they are isomorphic or anti-isomorphic.*

Proof. We need to show that categorically equivalent normal bands are isomorphic or anti-isomorphic. By Corollary 3.5, the converse is always true.

Take two categorically equivalent normal bands S and T. Their endomorphism semigroups are isomorphic, thus one of the statements (1)–(4) of Theorem 3.8 is fulfilled. In case of (1) or (2) we are done. So assume that (3) or (4) holds.

In case of (3), $S \simeq Y \times C$ and $T \simeq Y^d \times C$ for some chain semilattice Y and rectangular band C. We may assume that Y has at least two elements, otherwise case (3) reduces to (1).

Consider the normal bands $S^2 \simeq (Y \times C)^2 \simeq Y^2 \times C^2$ and $T^2 \simeq (Y^d \times C)^2 \simeq (Y^d)^2 \times C^2$. They are also categorically equivalent and therefore have isomorphic endomorphism semigroups. Thus one of the cases (1)–(4) is fulfilled also for S^2 and T^2. We analyze each possibility.

Case 3.1. $S^2 \simeq Y^2 \times C^2$ and $T^2 \simeq (Y^d)^2 \times C^2$ are isomorphic.

By Lemma 3.6 we get that the semilattices Y^2 and $(Y^d)^2$ are isomorphic. Semilattices as ordered sets are connected, hence by Proposition 1.1 they have the refinement property. Both Y^2 and $(Y^d)^2$ have factorization into directly irreducible ordered sets (chains Y and Y^d are directly irreducible), thus this factorization is unique. Therefore Y and Y^d are isomorphic as ordered sets and hence also as semilattices. Thus normal bands S and T are also isomorphic.

Case 3.2. $Y^2 \times C^2$ and $(Y^d)^2 \times C^2$ are anti-isomorphic.

Lemma 3.7 gives us that $Y^2 \times C^2 \simeq (Y^d)^2 \times (C^2)^*$, hence by Lemma 3.6, Y^2 and $(Y^d)^2$ are isomorphic. Like in Case 3.1, this implies that semilattices Y and Y^d are isomorphic and therefore $S \simeq Y \times C$ and $T \simeq Y^d \times C$ are also isomorphic.

Case 3.3 and 3.4. $S^2 \simeq Y^2 \times C^2$ is a chain normal band.

By Lemma 3.6, Y^2 must be a chain semilattice. Since the square of at least two-element chain is not a chain, this is not possible. This completes the study of the case (3).

Finally, in case of (4), $S \simeq Y \times C$ and T is anti-isomorphic to $Y^d \times C$. By Lemma 3.7, $T \simeq Y^d \times C^*$. Like in case (3), we consider the normal bands $S^2 \simeq Y^2 \times C^2$ and $T^2 \simeq (Y^d)^2 \times (C^*)^2$, which are also categorically equivalent and therefore have isomorphic endomorphism semigroups. One of the cases (1)–(4) of Theorem 3.8 is fulfilled for S^2 and T^2, and the analysis of them is exactly the same as for the cases (3.1)–(3.4).

First two cases lead to $Y \simeq Y^d$, and thus $S \simeq Y \times C$, $T \simeq Y \times C^*$. From Lemma 3.7 we see that S and T are anti-isomorphic. The last two cases lead to contradiction.

The proof of the theorem is now complete. □

Corollary 3.10. *Categorically equivalent rectangular bands are isomorphic or anti-isomorphic.*

Corollary 3.11. *Categorically equivalent semilattices are isomorphic.*

Proof. Since the semilattice multiplication is commutative, anti-isomorphic semilattices are isomorphic. □

The assertion of Theorem 3.9 holds also for arbitrary finite bands if we apply Theorem 2.2.

Proposition 3.12. *Finite bands are categorically equivalent if and only if they are isomorphic or anti-isomorphic.*

Proof. By Theorem 2.2 it suffices to show that weakly isomorphic finite bands are isomorphic or anti-isomorphic. Consider the binary part of the clone of term operations generated by a band operation xy. Since the band operation is idempotent, it is clear that this binary part consists of the interpretations of the binary terms x, y, xy, yx, xyx, yxy. Hence any binary term operation that generates the clone must equal xy or yx, which yields what we need. Indeed, the projections themselves do not generate any additional term operations; it is straightforward to check that the clone generated by xyx (yxy) does not contain xy. □

Remark 3.13. Proposition 3.12 and the idea of its proof were suggested by the referee. We thank the referee for this observation.

4. Lattices

For arbitrary lattice L we denote by L^d the dual lattice of L. We say that the lattices L and L' are *dually isomorphic*, if $L' \simeq L^d$. We shall need the following easy observation.

Lemma 4.1. *For any positive integer n and lattice L, $(L^n)^d = (L^d)^n$.*

We say that $\operatorname{Sub} L$ *determines* L if for an arbitrary lattice K, $\operatorname{Sub} L \simeq \operatorname{Sub} K$ implies $L \simeq K$ or $L \simeq K^d$.

In [7], the following statement was proved.

Theorem 4.2. *If L is directly reducible, then* $\operatorname{Sub} L$ *determines L.*

Note that in this theorem, L is determined not only in the class of directly reducible lattices, but in the class of all lattices.

Theorem 4.3. *Lattices are categorically equivalent if and only if they are isomorphic or dually isomorphic.*

Proof. Take two categorically equivalent lattices L and K. They have isomorphic sublattice-lattices. If at least one of them, say L, is directly reducible, then $\operatorname{Sub} L$ determines L, thus $L \simeq K$ or $L \simeq K^d$.

Now assume that none of L and K is directly reducible. The lattices L^2 and K^2 are also categorically equivalent and $\operatorname{Sub} L^2 \simeq \operatorname{Sub} K^2$. Since L^2 is reducible, we obtain that $L^2 \simeq K^2$ or $L^2 \simeq (K^2)^d = (K^d)^2$. Lattices have distributive congruence-lattices, thus by Proposition 1.2 they have the refinement property. Since, by assumption, L and K are directly irreducible, both L^2 and K^2 have factorization into directly irreducible lattices and therefore have unique factorization. Thus $L \simeq K$ or $L \simeq K^d$.

Conversely, assume that L and K are either isomorphic or dually isomorphic. The first case is trivial, in the second case L and K are weakly isomorphic, hence categorically equivalent. □

Remark 4.4. The referee pointed out that Theorem 4.3 can be proved also by using Proposition 1.1, in which case Proposition 1.2 may be avoided.

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