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Generalized parallel p_i-equidistant ruled surfaces

Nuri Kuruoğlu and Melek Masal

Abstract. In this paper, parallel p_i-equidistant ruled surfaces in 3-dimensional Euclidean space E^3, [6], were generalized to n-dimensional Euclidean space E^n. Then mean curvatures, Lipschitz–Killing curvatures, Gauss curvatures, scalar normal curvatures, Riemannian curvatures, Ricci curvatures, scalar curvatures of $(m+1)$-dimensional parallel p_i-equidistant ruled surfaces were calculated and some relations between these curvatures were found. Also, examples related to the parallel p_3-equidistant ruled surfaces in the E^3 are given.

1. Introduction

We shall assume throughout that all curves, vector fields, etc. are differentiable of class C^∞. Consider a general submanifold M of the Euclidean space E^n. Also, let $\chi(M)$ be the vector space of vector fields of a manifold M. Let $\bar{D}$ and D be Riemannian connections of E^n and M, respectively. Then, if X and Y are vector fields of M and if V is the second fundamental form of M, by decomposing $\bar{D}_X Y$ in a tangential and a normal component, [5], we have

$$\bar{D}_X Y = D_X Y + V(X,Y). \tag{1}$$

If ξ is any normal vector field on M, then we find the Weingarten equation by decomposing $\bar{D}_X \xi$ in a tangential component and a normal component as

$$\bar{D}_X \xi = -A_\xi(X) + D^\perp_X \xi. \tag{2}$$

Here A_ξ determines a self-adjoint linear map at each point and $D^\perp$ is a metric connection in the normal bundle $M^\perp$. We use the same notation A_ξ for the linear map and the matrix of the linear map. Suppose that X and Y

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are vector fields on M, then, if the standard metric tensor of E^n is denoted by $\langle , \rangle$, we have

$$< V(X,Y) \, , \, \xi >=< A_\xi(X) \, , \, Y >, \tag{3}$$

where ξ is a normal vector field. If $\xi_1, \xi_2, \ldots, \xi_{n-\dim M}$ constitute an orthonormal base field of the normal bundle $M^\perp$, then we have

$$V(X,Y) = \sum_{i=1}^{n-\dim M} < V(X,Y), \xi_i > \xi_i.$$

If $V(X,Y) = 0$, for all vector fields X, Y of M, then M is called totally geodesic in E^n and X and Y are called conjugate vectors in M. Moreover, X is said to be asymptotic vector if $V(X,X) = 0$, [2]. The mean curvature vector H of M is given by

$$H = \sum_{i=1}^{n-\dim M} \frac{tr A_{\xi_i}}{\dim M} \xi_i \, ,$$

where $\|H\|$ is the mean curvature. If $H = 0$ at each point P of M, then M is said to be minimal. The 4^{th} order covariant tensor field denoted by R as

$$R(X_1, X_2, X_3, X_4) = \langle X_1, R(X_3, X_4)X_2 \rangle \, , \; X_i \in \chi(M),$$

is called the Riemannian curvature tensor and its value at a point $P \in M$ is called Riemannian curvature of M at P and

$$K(P) = \langle X, R(X,Y) \rangle \;\text{ for all }\; X, Y \in \chi(M).$$

Therefore we can write, [5],

$$\langle X, R(X,Y)Y \rangle = \langle V(X,X), V(Y,Y) \rangle - \langle V(X,Y), V(X;Y) \rangle .$$

The sectional curvature function K is defined by

$$K(X_P, Y_P) = \frac{\langle R(X_P, Y_P)X_P, Y_P \rangle}{\langle X_P, X_P \rangle \langle Y_P, Y_P \rangle - \langle X_P, Y_P \rangle^2}.$$

$K(X_P, Y_P)$ is called the sectional curvature of M at P.

For a matrix $A = [a_{ij}]$ we write $M(A) = \sum_{i,j} a_{ij}^2$. Suppose that $\xi_1, ...,$ $\xi_{n-\dim M}$ is an orthonormal base field of $\chi(M^\perp)$, then the scalar normal curvature K_N of M is given by, [4],

$$K_N = \sum_{i,j=1}^{n-\dim M} M(A_{\xi_i} A_{\xi_j} - A_{\xi_j} A_{\xi_i}).$$

If ξ is a normal vector field on M, then the Lipschitz–Killing curvature in the direction ξ and at point P of M is given by, [5],

$$G(P, \xi) = \det A_\xi$$

and for the Gauss curvature, we can write

$$G(P) = \sum_{j=1}^{n-\dim M} G(P, \xi_j).$$

The submanifold M is said to be developable if $G(P) = 0$ at $P \in M$.

If R is the Riemannian curvature tensor of M and $\{e_1, \ldots, e_m\}$ is an orthonormal frame field of $\chi(M)$, then we can write Ricci and the scalar curvatures of M for all vector fields $X, Y \in \chi(M)$ as

$$S(X,Y) = \sum_{i=1}^{m} < R(e_i, X)Y, e_i >$$

and

$$r_{sk} = \sum_{i=1}^{m} S(e_i, e_i),$$

respectively, [2].

Let α be curve with arc-parameter and $\{e_1(t), e_2(t), ..., e_m(t)\}$ be an orthonormal set of vectors spanning the m-dimensional subspace $E_m(t)$ of $T_{\alpha(t)}E^n$. We get an $(m+1)$-dimensional surface in E^n if the subspace $E_m(t)$ moves along the curve α. This surface is called an $(m+1)$-dimensional generalized ruled surface and denoted by M. The curve α is called the base curve and the subspace $E_m(t)$ is called the generating space at point $\alpha(t)$. A parametrization of this ruled surface is given by

$$\Phi(t, u_1, u_2, ..., u_m) = \alpha(t) + \sum_{i=1}^{m} u_i e_i(t).$$

Taking the derivative of Φ with respect to t and u_i we get

$$\Phi_t = \dot{\alpha} + \sum_{i=1}^{m} u_i \dot{e}_i(t)\,,\ \Phi_{u_i} = e_i(t), \quad 1 \leq i \leq m.$$

We call $Sp\{e_1, e_2, ..., e_m, \dot{e}_1, \dot{e}_2, ..., \dot{e}_m\}$ the asymptotic bundle of M with respect to $E_m(t)$ and denote it by $A\,(t)$. The space

$$Sp\{e_1, e_2, ..., e_m, \dot{e}_1, \dot{e}_2, ..., \dot{e}_m, \dot{\alpha}\}$$

is called the tangential bundle of M with respect to $E_m(t)$ and it is denoted by $T\,(t)$, [3].

2. The curvatures of $(m+1)$-dimensional parallel p_i-equidistant ruled surfaces in E^n

In this section, the $(m+1)$-dimensional parallel p_i-equidistant ruled surfaces in E^n are defined and the curvatures of these surfaces are obtained.

Let r and r^* be two curves with arc-parameter in E^n and let $\{V_1, V_2, \ldots, V_k\}$ and $\{V_1^*, V_2^*, \ldots, V_k^*\}$, $k \leq n$, be Frenet frames of r and r^*, respectively. If k_i and k_i^*, $1 \leq i \leq k-1$, are the curvatures of r and r^*, respectively, then we can write

$$\begin{aligned} V_1' &= k_1 V_2\,, \\ V_i' &= -k_{i-1} V_{i-1} + k_i V_{i+1}, 1 < i < k, \\ V_k' &= -k_{k-1} V_{k-1} \end{aligned} \tag{4}$$

and

$$\begin{aligned} V_1^{*'} &= k_1^* V_2^*\,, \\ V_i^{*'} &= -k_{i-1}^* V_{i-1}^* + k_i^* V_{i+1}^*\,, \quad 1 < i < k. \\ V_k^{*'} &= -k_{k-1}^* V_{k-1}^*\,. \end{aligned} \tag{5}$$

Let M and M^* be $(m+1)$-dimensional generalized ruled surfaces in E^n, and let $E_m(t)$ and $E_m(t^*)$, $1 \leq m \leq k-2$, be generating spaces of M and M^*, respectively. Then M and M^* can be given by the following parametric form:

$$\begin{aligned} M : X(t, u_1, \ldots, u_m) &= r(t) + \sum_{i=1}^{m} u_i V_i(t), \\ rank\,\{X_t, X_{u_1}, \ldots, X_{u_m}\} &= m+1, \end{aligned} \tag{6}$$

$$\begin{aligned} M^* : X^*\left(t^*, u_1^*, \ldots, u_m^*\right) &= r^*(t^*) + \sum_{i=1}^{m} u_i^* V_i^*(t^*), \\ rank\left\{X_{t^*}^*, X_{u_1^*}^*, \ldots, X_{u_m^*}^*\right\} &= m+1, \end{aligned} \tag{7}$$

where $\{V_1, V_2, \ldots, V_m\}$ and $\{V_1^*, V_2^*, \ldots, V_m^*\}$ are the orthonormal bases of the generating spaces of M and M^*, respectively.

Definition 1. Let M and M^* be $(m+1)$-dimensional ruled surfaces as above in E^n . Let $p_1, p_2, \ldots, p_{k-1}, p_k$ be the distances between the $(k-1)$-dimensional osculator planes $Sp\{V_2, V_3, \ldots, V_k\}$ and $Sp\{V_2^*, V_3^*, \ldots, V_k^*\}$, $Sp\{V_1,\ V_3,\ V_4,\ \ldots,\ V_{k-1},\ V_k\}$ and $Sp\left\{V_1^*,\ V_3^*,\ V_4^*,\ \ldots,\ V_{k-1}^*,\ V_k^*\right\}$, $\ldots$, $Sp\{V_1,\ V_2,\ \ldots,\ V_{k-3},\ V_{k-2},\ V_k\}$ and $Sp\left\{V_1^*,\ V_2^*,\ \ldots,\ V_{k-3}^*,\ V_{k-2}^*,\ V_k^*\right\}$, $Sp\{V_1, V_2, \ldots, V_{k-2}, V_{k-1}\}$ and $Sp\left\{V_1^*, V_2^*, \ldots, V_{k-2}^*, V_{k-1}^*\right\}$, respectively. If

1) V_1 and V_1^* are parallel,
2) the distances p_i, $1 \leq i \leq k$, between the $(k-1)$-dimensional osculator planes at the corresponding points of r and r^* are constant,

then the ruled surfaces M and M^* are called the $(m+1)$-dimensional parallel p_i-equidistant ruled surfaces, [1].

Since the Frenet frames $\{V_1, V_2, \dots, V_k\}$ and $\{V_1^*, V_2^*, \dots, V_k^*\}$ are the orthonormal systems, we have the orthogonal systems

$$E_1 = \frac{dr}{dt}, E_i = \frac{d^i r}{dt^i} - \sum_{j=1}^{i-1} \frac{< \frac{d^i r}{dt^i}, E_j >}{\|E_j\|^2} E_j, \quad 1 < i \leq k,$$

and

$$E_1^* = \frac{dr^*}{dt^*}, E_i^* = \frac{d^i r^*}{dt^{*i}} - \sum_{j=1}^{i-1} \frac{< \frac{d^i r^*}{dt^{*i}}, E_j^* >}{\left\|E_j^*\right\|^2} E_j^*, \quad 1 < i \leq k. \tag{8}$$

At this point, we get

$$V_1 = E_1, V_2 = \frac{E_2}{\|E_2\|}, \dots, V_k = \frac{E_k}{\|E_k\|}$$

and

$$V_1^* = E_1^*, V_2^* = \frac{E_2^*}{\|E_2^*\|}, \dots, V_k^* = \frac{E_k^*}{\|E_k^*\|}.$$

Since $V_1 = V_1^*$ from the Definition 1, we get $\frac{dr}{dt} = \frac{dr^*}{dt^*}$.

In addition, if $\frac{dt}{dt^*}$ is constant, then we have

$$\frac{d^i r^*}{dt^{*i}} = \frac{d^i r}{dt^i} \left(\frac{dt}{dt^*} \right)^{i-1}, \quad 1 \leq i \leq k. \tag{9}$$

If we subsitute the relation (9) into the relation (8), we obtain

$$E_i = E_i^*.$$

So we get

$$V_i = V_i^*, \quad 1 \leq i \leq k.$$

That is, we can say that the Frenet frames $\{V_1, V_2, \dots, V_k\}$ and $\{V_1^*, V_2^*, \dots, V_k^*\}$ are equivalent to each other at the corresponding points of r and r^*.

Hence we can give the following theorem.

Theorem 2. *Let M and M^* be $(m+1)$-dimensional parallel p_i-equidistant ruled surfaces in E^n. The Frenet frames $\{V_1, V_2, \dots, V_k\}$ and $\{V_1^*, V_2^*, \dots, V_k^*\}$ are equivalent at the corresponding points of r and r^*.*

If rr^* is the vector determined by the points $r(t)$ and $r^*(t^*)$ of the base curves r and r^* of $(m+1)$-dimensional parallel p_i-equidistant ruled surfaces, then the vector rr^* can be expressed in terms of $\{V_1, V_2, \dots, V_k\}$ as follows:

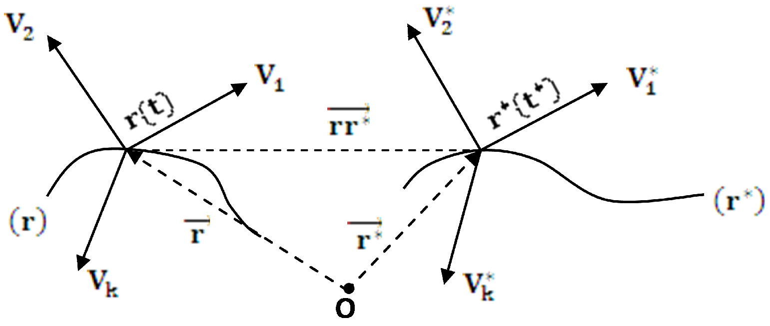

Figure 1

$$rr^* = a_1V_1 + a_2V_2 + \cdots + a_kV_k, a_i \in IR, \quad 1 \le i \le k.$$

Since $< rr^*, V_i >= p_i$ is the distance between the osculator planes, we get $a_i = p_i$ and thus

$$r^* = r + p_1V_1 + p_2V_2 + \cdots + p_kV_k.$$

Hence, we can give the following theorem.

Theorem 3. *Let M and M^* be $(m+1)$-dimensional parallel p_i-equidistant ruled surfaces with the base curves r and r^*, respectively. If p_i, $1 \le i \le k$, is the distance between $(k-1)$-dimensional osculator planes, then we have the relation*

$$r^* = r + p_1V_1 + p_2V_2 + \cdots + p_kV_k.$$

If k_i and k_i^* are the curvatures of r and r^* of $(m+1)$-dimensional parallel p_i-equidistant ruled surfaces, then from the equations (4) and (5) we have

$$k_i =< V_i', V_{i+1} >=< \frac{dV_i}{dt}, V_{i+1} >, \quad 1 \le i < k, \tag{10}$$

and

$$k_i^* =< V_i^{*\prime}, V_{i+1}^* >=< \frac{dV_i^*}{dt^*}, V_{i+1}^* >, \quad 1 \le i < k.$$

From Theorem 2,

$$\frac{dV_i}{dt} = \frac{dV_i^*}{dt^*}\frac{dt^*}{dt}. \tag{11}$$

If we use the equation (11), Theorem 2 and the equation (10), then we get

$$k_i^* = \frac{dt}{dt^*}k_i, \quad 1 \le i < k.$$

Hence, we can give the following theorem.

Theorem 4. *Let M and M^* be $(m+1)$-dimensional parallel p_i-equidistant ruled surfaces with the base curves r and r^*, respectively. If k_i and k_i^* are the curvatures of r and r^*, respectively, then*

$$k_i^* = \frac{dt}{dt^*}k_i, \quad 1 \le i < k.$$

Let $A(t)$ and $A(t^*)$ be asymptotic bundles of M and M^*, respectively. Then from the definition of the asymptotic bundle, we have

$$A(t) = Sp\left\{V_1, V_2, \ldots, V_m, V_1', V_2', \ldots, V_m'\right\} \tag{12}$$

and

$$A(t^*) = Sp\left\{V_1^*, V_2^*, \ldots, V_m^*, V_1^{*'}, V_2^{*'}, \ldots, V_m^{*'}\right\}. \tag{13}$$

If we use the Frenet formulas in the equations (12) and (13), then we get the orthonormal bases $\{V_1, V_2, \ldots, V_{m+1}\}$ and $\left\{V_1^*, V_2^*, \ldots, V_{m+1}^*\right\}$ of $A(t)$ and $A(t^*)$, respectively. Therefore, we have

$$\dim A(t) = m+1, \quad \dim A(t^*) = m+1.$$

Similarly, if $T(t)$ and $T(t^*)$ are the tangential bundles of M and M^*, respectively, then we have

$$T(t) = Sp\left\{V_1, V_2, \ldots, V_m, V_1', V_2', \ldots, V_m', r'\right\}$$

and

$$T(t^*) = Sp\left\{V_1^*, V_2^*, \ldots, V_m^*, V_1^{*'}, V_2^{*'}, \ldots, V_m^{*'}, r^{*'}\right\}.$$

From the Frenet formulas, we get the orthonormal bases $\{V_1, V_2, \ldots, V_{m+1}\}$ and $\left\{V_1^*, V_2^*, \ldots, V_{m+1}^*\right\}$ of $T(t)$ and $T(t^*)$, respectively. Therefore,

$$\dim T(t) = m+1, \quad \dim T(t^*) = m+1,$$

and

$$A(t) = A(t^*) = T(t) = T(t^*).$$

Thus, we can say that the asymptotic bundles of M and M^* are equal and so are the tangential bundles of M and M^*.

Hence, we can give the following theorem.

Theorem 5. *Let M and M^* be $(m+1)$-dimensional parallel p_i-equidistant ruled surfaces. Then the asymptotic and the tangential bundles of M and M^* are equal.*

Let M and M^* be $(m+1)$-dimensional parallel p_i-equidistant ruled surfaces in E^n . From the equations (6) and (7), we have

$$X_t = V_1 + \sum_{i=1}^{m} u_i V_i', X_{u_1} = V_1, \ldots, X_{u_m} = V_m.$$

Therefore, we get the orthonormal bases $\{V_1, \ldots, V_{m+1}\}$ and $\{V_1^*, \ldots, V_{m+1}^*\}$ of M and M^*, respectively. If $\{\xi_1, \ldots, \xi_{k-m-1}, \ldots, \xi_{n-m-1}\}$ and $\left\{\xi_1^*, \ldots, \xi_{k-m-1}^*, \ldots, \xi_{n-m-1}^*\right\}$ are the orthonormal bases of the normal bundles $M^\perp$ and $M^{*\perp}$, respectively, then we get the orthonormal bases $\{V_1, \ldots, V_{m+1}, \xi_1, \ldots, \xi_{k-m-1}, \ldots, \xi_{n-m-1}\}$ and $\left\{V_1^*, \ldots, V_{m+1}^*, \xi_1^*, \ldots, \xi_{k-m-1}^*, \ldots, \xi_{n-m-1}^*\right\}$ of E^n at a point $P \in M$ and at a point $P^* \in M^*$, respectively, where $\xi_i = V_{m+1+i}$ and $\xi_i^* = V_{m+1+i}^*$, $1 \le i \le k-m-1$. Suppose that $\bar{D}$, D and D^* are the Riemannian connections of E^n, M and M^*, respectively. Then we have the following Weingarten equations:

$$\bar{D}_{V_1}\xi_j = \sum_{i=1}^{m+1} a_{1i}^j V_i + \sum_{q=1}^{n-m-1} b_{1q}^j \xi_q, \quad 1 \le j \le n-m-1,$$
$$\vdots$$
$$\bar{D}_{V_{m+1}}\xi_j = \sum_{i=1}^{m+1} a_{(m+1)i}^j V_i + \sum_{q=1}^{n-m-1} b_{(m+1)q}^j \xi_q\,, \quad 1 \le j \le n-m-1.$$

Therefore, we can obtain the matrix $A_{\xi_j}, 1 \le j \le n-m-1$, as

$$A_{\xi_j} = -\begin{bmatrix} a_{11}^j & a_{12}^j & \cdots & a_{1(m+1)}^j \\ \vdots & \vdots & \vdots & \vdots \\ a_{(m+1)1}^j & a_{(m+1)2}^j & \cdots & a_{(m+1)(m+1)}^j \end{bmatrix},$$

where

$$a_{11}^j =< \bar{D}_{V_1}\xi_j, V_1 > \,, \ \ldots \ , \ a_{1(m+1)}^j =< \bar{D}_{V_1}\xi_j, V_{m+1} >,$$
$$\vdots$$
$$a_{(m+1)1}^j =< \bar{D}_{V_{m+1}}\xi_j, V_1 > \,, \ldots, \ a_{(m+1)(m+1)}^j =< \bar{D}_{V_{m+1}}\xi_j, V_{m+1} > . \tag{14}$$

It follows from the equations (2), (3) and (4) that

$$A_{\xi_1} = A_{V_{m+2}} = \begin{bmatrix} 0 & \cdots & 0 & k_{m+1} \\ 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix}_{(m+1)\times(m+1)} \quad \text{and} \quad A_{\xi_j} = 0, \ \ 2 \le j \le n-m-1. \tag{15}$$

Similarly, if ξ^* is a normal vector field on M^*, then the Weingarten equation has the form

$$\bar{D}_{X^*}\xi^* = -A_{\xi^*}(X^*) + D_{X^*}^{*\perp}\xi^*,$$

where A_{ξ^*} is the self-adjoint linear transformation of $\chi(M^*)$ and $D^{*\perp}$ is the metric connection in the normal bundle $M^{*\perp}$. Then we have the following Weingarten equations:

$$\bar{D}_{V_1^*}\xi_j^* = \sum_{i=1}^{m+1} c_{1i}^j V_i^* + \sum_{q=1}^{n-m-1} d_{1q}^j \xi_q^*, \quad 1 \le j \le n-m-1,$$
$$\vdots$$
$$\bar{D}_{V_{m+1}^*}\xi_j^* = \sum_{i=1}^{m+1} c_{(m+1)i}^j V_i^* + \sum_{q=1}^{n-m-1} d_{(m+1)q}^j \xi_q^*, \quad 1 \le j \le n-m-1.$$

Therefore, we can obtain the matrix $A_{\xi_j^*}$, $1 \leq j \leq n-m-1$, as

$$A_{\xi_j^*} = -\begin{bmatrix} c_{11}^j & \cdots & c_{1(m+1)}^j \\ \vdots & & \\ c_{(m+1)1}^j & \cdots & c_{(m+1)(m+1)}^j \end{bmatrix},$$

where

$$\begin{aligned} & c_{11}^j = < \bar{D}_{V_1^*}\xi_j^* \,, V_1^* > \,, \ \ldots\,, \ c_{1(m+1)}^j = < \bar{D}_{V_1^*}\xi_j^* \,, V_{m+1}^* >, \\ & \vdots \\ & c_{(m+1)1}^j = < \bar{D}_{V_{m+1}^*}\xi_j^* \,, V_1^* > \,, \ldots\,, \ c_{(m+1)(m+1)}^j = < \bar{D}_{V_{m+1}^*}\xi_j^* \,, V_{m+1}^* > . \end{aligned} \tag{16}$$

Hence, from the equations (2), (3) and (5), it follows

$$A_{\xi_1^*} = A_{V_{m+2}^*} = \begin{bmatrix} 0 & \cdots 0 & k_{m+1}^* \\ 0 & \cdots 0 & 0 \\ \vdots & & \\ 0 & \cdots 0 & 0 \end{bmatrix}_{(m+1)\times(m+1)} \quad \text{and} \quad A_{\xi_j^*} = 0, \ \ 2 \leq j \leq n-m-1. \tag{17}$$

From Theorem 4 and equations (15) and (17), we have

$$A_{\xi_1^*} = \frac{dt}{dt^*} A_{\xi_1} \ \text{ and } \ A_{\xi_j^*} = A_{\xi_j} = 0\,, \quad 2 \leq j \leq n-m-1.$$

Making use of equations (15) and (17), we can prove the following theorem.

Theorem 6. *Let M and M^* be $(m+1)$-dimensional parallel p_i-equidistant ruled surfaces in E^n. The matrices A_{ξ_j} and $A_{\xi_j^*}$, $1 \leq j \leq n-m-1$ satisfy the relations*

$$A_{\xi_1^*} = \frac{dt}{dt^*} A_{\xi_1}\,, \quad A_{\xi_j^*} = A_{\xi_j} = 0\,, \quad 2 \leq j \leq n-m-1.$$

Corollary 1. *If M and M^* are $(m+1)$-dimensional parallel p_i-equidistant ruled surfaces in E^n, then*

$$\det A_{\xi_j^*} = \det A_{\xi_j} = 0\,, \quad 1 \leq j \leq n-m-1.$$

From Corollary 1 and the definition of Lipschitz–Killing curvature in a direction ξ_j, we can write

$$G(P, \xi_j) = 0 \ \text{ for all } \ P \in M, \quad 1 \leq j \leq n-m-1.$$

Thus, from the definition of Gauss curvature of M, we get

$$G(P) = \sum_{j=1}^{n-m-1} G(P, \xi_j) = 0.$$

Therefore, M is a developable ruled surface. Similarly, from the definition of Lipschitz–Killing and Gauss curvatures of M^*, we have

$$G(P^*, \xi_j^*) = \det A_{\xi_j^*} = 0\,, \quad 1 \le j \le n-m-1\,, \quad \text{for all } \; P^* \in M^*,$$

and

$$G(P^*) = \sum_{j=1}^{n-m-1} G(P^*, \xi_j^*) = 0.$$

Therefore, M^* is also a developable ruled surface.

Hence, we can give the following theorem.

Theorem 7. *Let M and M^* be $(m+1)$-dimensional parallel p_i-equidistant ruled surfaces in E^n. Then for any normal direction the Lipschitz–Killing curvatures of M and M^* vanish and M, M^* are developable surfaces.*

If H (H^*) and K_N (K_{N^*}) are the mean curvature vector and scalar normal curvature of M (M^*), respectively, then from equations (15) *and* (17) *we have*

$$H = H^* = 0 \quad \textit{and} \quad K_N = K_{N^*} = 0.$$

Therefore, M and M^* are the minimal ruled surfaces.

Hence we can give the following theorem.

Theorem 8. *If M and M^* are $(m+1)$-dimensional parallel p_i-equidistant ruled surfaces in E^n, then M and M^* are minimal ruled surfaces and the scalar normal curvatures of M and M^* are zero.*

If X and Y are vector fields and V is the second fundamental form of M, then from equations (1) *and* (2) *we have*

$$< \bar{D}_X Y \,,\, \xi > = < V(X,Y) \,,\, \xi > = < A_\xi(X) \,,\, Y >\,, \quad \xi \in \chi^\perp(M),$$

and

$$V(X,Y) = -\sum_{j=1}^{n-m-1} < Y \,,\, \bar{D}_X \xi_j > \xi_j.$$

In this case, for the Frenet vectors V_i and V_j, $1 \le i, j \le m+1$, we can write

$$V(V_i \,,\, V_j\,) = -\sum_{l=1}^{n-m-1} < V_j \,,\, \bar{D}_{V_i} \xi_l > \xi_l, \quad 1 \le i, j \le m+1.$$

Thus, from equation (14), we get

$$V(V_i \,,\, V_j) = -\sum_{l=1}^{n-m-1} a_{ij}^l \xi_l.$$

Using the equation (15), we obtain

$$\begin{aligned} V(V_1, V_{m+1}) &= -\sum_{l=1}^{n-m-1} a^l_{1(m+1)}\xi_l = k_{m+1}V_{m+2}, \\ V(V_i, V_j) &= -\sum_{l=1}^{n-m-1} a^l_{ij}\xi_l = 0, \quad 1 \leq i, j \leq m+1. \end{aligned} \tag{18}$$

Similarly, for all $X^*, Y^* \in \chi(M^*)$, the Gauss equation is

$$\bar{D}_{X^*}Y^* = D^*_{X^*}Y^* + V^*(X^*, Y^*),$$

where V^* is the second fundamental form of M^*. From (9), we have

$$< \bar{D}_{X^*}Y^*, \xi^* >=< V^*(X^*, Y^*), \xi^* >=< A_{\xi^*}(X^*), Y^* >$$

and

$$V^*(X^*, Y^*) = -\sum_{j=1}^{n-m-1} < Y^*, \bar{D}_{X^*}\xi^*_j > \xi^*_j.$$

Then we have

$$V^*(V^*_i, V^*_j) = -\sum_{l=1}^{n-m-1} < V^*_j, \bar{D}_{V^*_i}\xi^*_l > \xi^*_l, \quad 1 \leq i, j \leq m+1,$$

and from equation (16),

$$V^*(V^*_i, V^*_j) = -\sum_{l=1}^{n-m-1} c^l_{ij}\xi^*_l, \quad 1 \leq i, j \leq m+1.$$

Making use of equation (17), we obtain

$$\begin{aligned} V^*(V^*_1, V^*_{m+1}) &= k^*_{m+1}V^*_{m+2}\,, \\ V^*(V^*_i, V^*_j) &= 0, \quad 1 \leq i, j \leq m+1, \end{aligned} \tag{19}$$

and from Theorems 2 and 4, we have

$$\begin{aligned} V^*(V^*_1, V^*_{m+1}) &= \tfrac{dt}{dt^*}V(V_1, V_{m+1}), \\ V^*(V^*_i, V^*_j) &= V(V_i, V_j) = 0, \quad 1 \leq i, j \leq m+1. \end{aligned} \tag{20}$$

Thus, from equation (20) and from the definition of conjugate vectors, we can give the following result.

Corollary 2. *The vectors V_1 and V_{m+1} are conjugate if and only if V^*_1 and V^*_{m+1} are conjugate vectors.*

The sectional curvatures of M are

$$K(V_i, V_j) =< V(V_i, V_i)\,,\, V(V_j, V_j) > - < V(V_i, V_j)\,,\, V(V_i, V_j) > .$$

Substituting equations (18) and (19) into the last equation, we get

$$\begin{aligned} &K(V_1, V_{m+1}) = -(k_{m+1})^2, K(V_i, V_j) = 0\,, \quad 1 \leq i, j \leq m+1\,, \\ &K(V_1^*, V_{m+1}^*) = -(k_{m+1}^*)^2, K(V_i^*, V_j^*) = 0, \quad 1 \leq i, j \leq m+1\,. \end{aligned} \tag{21}$$

Hence, we can give the following theorem.

Theorem 9. *The sectional curvatures of M and M^* are determined by the equations* (21)*.*

From Theorems 4 and 9, we can give the next corollary.

Corollary 3. *We have the following equations for sectional curvatures of $(m+1)$-dimensional parallel p_i-equidistant ruled surfaces in E^n:*

$$K(V_1^*, V_{m+1}^*) = \left(\frac{dt}{dt^*}\right)^2 K(V_1, V_{m+1}),$$
$$K(V_i^*, V_j^*) = K(V_i, V_j) = 0\,, \quad 1 \leq i, j \leq m+1.$$

For the Ricci curvature in the direction V_i of M, $1 \leq i \leq m+1$, we can write

$$S(V_i, V_i) = \sum_{j=1}^{m+1} \{< V(V_j, V_j)\,, V(V_i, V_i) > - < V(V_i, V_j)\,, V(V_i, V_j) >\}.$$

Using (18), we get

$$S(V_{m+1}, V_{m+1}) = K(V_1, V_{m+1}) = -(k_{m+1})^2,$$
$$S(V_i, V_i) = 0\,,\ 1 \leq i \leq m\ .$$

For the scalar curvature of M we obtain

$$r_{sk} = \sum_{i=1}^{m+1} S(V_i, V_i) = S(V_{m+1}, V_{m+1}) = K(V_1, V_{m+1})\ .$$

Similarly, for the Ricci curvature in the direction V_i^* of M^*, $1 \leq i \leq m+1$, we can write

$$\begin{aligned} (V_i^*, V_i^*) = \sum_{j=1}^{m+1} \{ &< V(V_j^*, V_j^*)\ ,\ V(V_i^*, V_i^*) > \\ &- < V(V_i^*, V_j^*)\ ,\ V(V_i^*, V_j^*) > \}\,. \end{aligned}$$

Using (20), we obtain

$$S(V_{m+1}^*, V_{m+1}^*) = K(V_1^*, V_{m+1}^*) = -(k_{m+1}^*)^2,$$
$$S(V_i^*, V_i^*) = 0\,,\ 1 \leq i \leq m\,.$$

Thus, the scalar curvature of M^* is

$$r^*_{sk} = \sum_{i=1}^{m+1} S(V^*_i, V^*_i) = S(V^*_{m+1}, V^*_{m+1}) = K(V^*_1, V^*_{m+1}).$$

From Theorem 4 we have

$$S(V^*_{m+1}, V^*_{m+1}) = \left(\frac{dt}{dt^*}\right)^2 S(V_{m+1}, V_{m+1}) \text{ and } r^*_{sk} = \left(\frac{dt}{dt^*}\right)^2 r_{sk}.$$

Hence, we can give the following theorem.

Theorem 10. *Let M and M^* be $(m+1)$-dimensional parallel p_i-equidistant ruled surfaces in E^n . If $S(V_i, V_i)$ and $S(V^*_i, V^*_i)$ are the Ricci curvatures of M and M^* and if r_{sk} and r^*_{sk} are the scalar curvatures of M and M^*, respectively, then we have*

$$S(V^*_i, V^*_i) = S(V_i, V_i) = 0\,, \quad 1 \leq i \leq m\,,$$

$$S(V^*_{m+1}, V^*_{m+1}) = \left(\frac{dt}{dt^*}\right)^2 S(V_{m+1}, V_{m+1}) = r^*_{sk} = \left(\frac{dt}{dt^*}\right)^2 r_{sk}.$$

Let $X = \sum\limits_{i=1}^{m+1} a_i V_i$, $Y = \sum\limits_{i=1}^{m+1} b_i V_i \in \chi(M)$. Then we can write

$$V(X,Y) = V(\sum_{i=1}^{m+1} a_i V_i \ , \ \sum_{j=1}^{m+1} b_j V_j),$$

and from (18),

$$V(X,Y) = a_1 b_{m+1} V(V_1, V_{m+1}). \tag{22}$$

If S is totally geodesic, then

$$V(X,Y) = 0\,.$$

Thus, we get

$$a_1 b_{m+1} = 0 \text{ or } V(V_1, V_{m+1}) = 0.$$

Conversely, let $a_1 b_{m+1} = 0$ or $V(V_1, V_{m+1}) = 0$. From equation (22) we get

$$V(X,Y) = 0 \text{ for all } X, Y \in \chi(M).$$

Hence, S is totally geodesic.

Thus, we can give the following theorem.

Theorem 11. *Let $X = \sum\limits_{i=1}^{m+1} a_i V_i$, $Y = \sum\limits_{i=1}^{m+1} b_i V_i \in \chi(M)$. Then M is totally geodesic if and only if $V(V_1, V_{m+1}) = 0$ or $a_1 b_{m+1} = 0$.*

Therefore, from Theorem 9, Theorem 11 and equation (18) we can give the following corollaries.

Corollary 4. *If $K(V_1, V_{m+1})$ is equal to zero, then M is totally geodesic.*

Corollary 5. *If $a_1 b_{m+1} \neq 0$ and M is totally geodesic, then M^* is totally geodesic and the Riemannian curvatures of M and M^* are zero.*

Example 1. The following surfaces (Figure 2) are parallel p_3-equidistant ruled surfaces in E^3 parametrized by

$$\varphi(t,v) = \left(\cos^2 t - v \sin 2t, \frac{1}{2}\sin 2t + v\cos 2t, 0\right),$$

and

$$\varphi^*(t,v) = \left(\cos^2 t - v \sin 2t, \frac{1}{2}\sin 2t + v\cos 2t, 4\right).$$

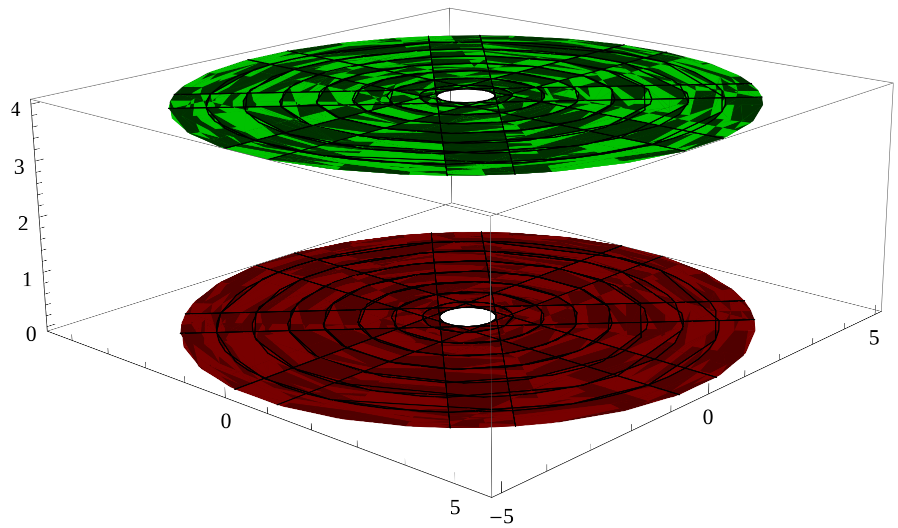

Figure 2

Example 2. The surfaces in E^3 parametrized by

$$\varphi(t,v) = \left(\cos^3 t - v\cos t, \sin^3 t + v \sin t, 0\right),$$

and

$$\varphi^*(t,v) = \left(4\cos^3 t - v\cos t, 4\sin^3 t + v \sin t, 0\right)$$

are parallel p_3-equidistant ruled surfaces (Figure 3).

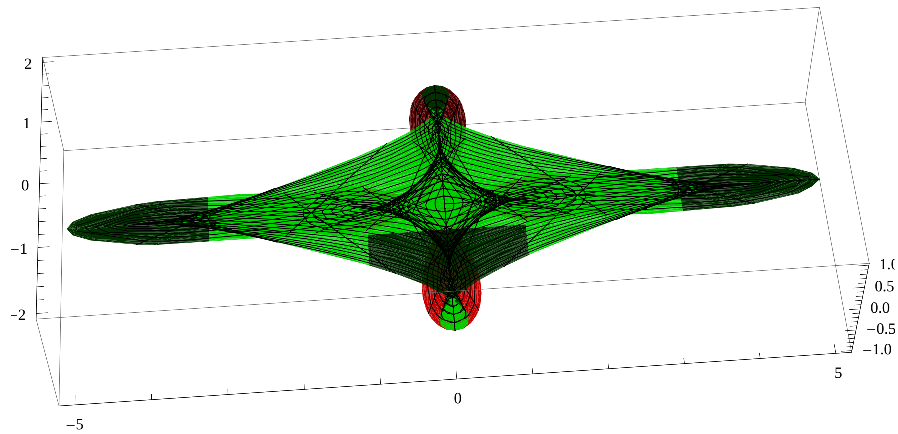

Figure 3

Example 3. The surfaces in E^3 parametrized by

$$\varphi(t, v) = (\cos t - v \sin t, \sin t + v \cos t, 1)$$

and

$$\varphi^*(t, v) = (3 \cos t - v \sin t, 3 \sin t + v \cos t, 4)$$

are parallel p_3-equidistant ruled surfaces (Figure 4).

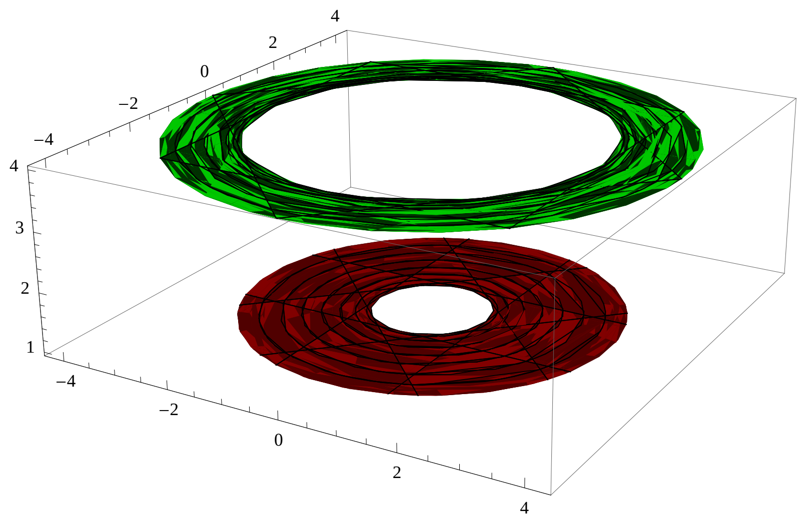

Figure 4

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Bahcesehir University, Faculty of Arts and Sciences, Department of Mathematics and Computer Sciences, Istanbul, Turkey
E-mail address: kuruoglu@bahcesehir.edu.tr

Sakarya University, Faculty of Education, Department of Elementery Education, Hendek-Sakarya, Turkey
E-mail address: mmasal@sakarya.edu.tr