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Fractional difference inequalities of Gronwall–Bellman type

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Abstract. Discrete inequalities, in particular the discrete analogues of the Gronwall–Bellman inequality, have been extensively used in the analysis of finite difference equations. The aim of the present paper is to establish some fractional difference inequalities of Gronwall–Bellman type which provide explicit bounds for the solutions of fractional difference equations.

1. Introduction

Difference equations usually describe the evolution of phenomena over the course of time. The theory of difference equations has been developed as a natural discrete analogue of corresponding theory of differential equations. Many physical problems arising in a wide variety of applications are governed by finite difference equations.

The theory of inequalities is always of great importance for the development of many branches of mathematics. This field is dynamic and experiencing an explosive growth in both theory and applications. As a response to the needs of diverse applications, a large variety of inequalities have been proposed and studied in the literature. Since the integral inequalities with explicit estimates are so important in the study of properties of solutions of differential and integral equations, their finite difference (or discrete) analogues should also be useful in the study of properties of solutions of finite difference equations.

The finite difference version of the well-known Gronwall inequality seems to have appeared first in the work of Mikeladze in 1935. It is well recognized that the discrete version of Gronwall's inequality provides a very

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useful and important tool in proving convergence of the discrete variable methods. In view of wider applications, finite difference inequalities with explicit estimates have been generalized, extended and used considerably in the development of the theory of finite difference equations.

In the year 1973, B. G. Pachpatte [11] established the following remarkable inequality.

Theorem 1.1. *Let $u(n)$, $b(n)$ and $c(n)$ be real valued nonnegative functions defined on $\mathbb{N}_0^+$ and let $C \geq 0$ be a constant. If for all $n \in \mathbb{N}_0^+$*

$$u(n) \leq C + \sum_{j=0}^{n-1} b(j)\Big[u(j) + \sum_{k=0}^{j-1} c(k)u(k)\Big], \tag{1.1}$$

then for all $n \in \mathbb{N}_0^+$

$$u(n) \leq C\Big[1 + \sum_{j=0}^{n-1} b(j) \prod_{k=0}^{j-1} [1 + b(j) + c(j)]\Big]. \tag{1.2}$$

On the other hand, fractional calculus has gained importance during the past three decades due to its applicability in diverse fields of science and engineering. The notions of fractional calculus may be traced back to the works of Euler, but the idea of fractional difference is very recent.

J. B. Diaz and T. J. Osler [7] defined the fractional difference by the rather natural approach of allowing the index of differencing, in the standard expression for the n^{th} difference, to be any real or complex number. Later, R. Hirota [9], defined the fractional order difference operator ∇^α where α is any real number, using Taylor's series. A. Nagai [10] adopted another definition for fractional difference by modifying Hirota's definition. Recently, G. V. S. R. Deekshitulu and J. Jagan Mohan [2] slightly modified the definition of A. Nagai [10] and discussed some basic inequalities, comparison theorems and qualitative properties of the solutions of fractional difference equations [2, 3, 4, 5, 6].

In the present paper, the authors consider an initial value problem of fractional order and obtain some useful fractional difference inequalities of Gronwall–Bellman type.

2. Preliminaries

In this section, we introduce some basic definitions and results concerning nabla discrete fractional calculus. Throughout the article, for notations and terminology we refer to [1]. The extended binomial coefficient $\binom{a}{n}$, where

$a \in \mathbb{R}$, $n \in \mathbb{Z}$, is defined by

$$\binom{a}{n} = \begin{cases} \frac{\Gamma(a+1)}{\Gamma(a-n+1)\Gamma(n+1)} & n > 0, \\ 1 & n = 0, \\ 0 & n < 0. \end{cases} \tag{2.1}$$

H. L. Gray and N. F. Zhang [8] gave the following definition.

Definition 2.1. For any complex numbers α and β, let $(\alpha)_\beta$ be defined as follows.

$$(\alpha)_\beta = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} & \text{when } \alpha,\ \alpha+\beta \text{ are neither zero nor negative integers,} \\ 1 & \text{when } \alpha = \beta = 0, \\ 0 & \text{when } \alpha = 0,\ \beta \text{ is neither zero nor negative integer,} \\ \text{undefined} & \text{otherwise} \end{cases}$$

Remark 1. For any complex numbers α and β, when α, β and $\alpha + \beta$ are neither zero nor negative integers,

$$(\alpha + \beta)_n = \sum_{k=0}^{n} \binom{n}{k} (\alpha)_{n-k} (\beta)_k \tag{2.2}$$

for any positive integer n.

In 2003, A. Nagai [10] gave the following definition for fractional order difference operator.

Definition 2.2. Let $\alpha \in \mathbb{R}$ and let m be an integer such that $m - 1 < \alpha \leq m$. The difference operator ∇ of order α, with step length ε, is defined as

$$\nabla^\alpha u(n) = \begin{cases} \nabla^{\alpha-m}[\nabla^m u(n)] = \varepsilon^{m-\alpha} \sum_{j=0}^{n-1} \binom{\alpha-m}{j} (-1)^j \nabla^m u(n-j) & \alpha > 0, \\ u(n) & \alpha = 0, \\ \varepsilon^{-\alpha} \sum_{j=0}^{n-1} \binom{\alpha}{j} (-1)^j u(n-j) & \alpha < 0. \end{cases}$$

The above definition contains ∇ operator and the term $(-1)^j$ inside the summation index and hence it becomes difficult to study the properties of solutions of fractional difference equations. To avoid this, G. V. S. R. Deekshitulu and J. Jagan Mohan [2] rearranged the terms in the definition of A. Nagai [10] as follows, for $\varepsilon = m = 1$.

Definition 2.3. The fractional sum operator of order α ($\alpha \in \mathbb{R},\ \alpha \geq 0$) is defined as

$$\nabla^{-\alpha} u(n) = \sum_{j=0}^{n-1} \binom{j + \alpha - 1}{j} u(n-j) = \sum_{j=1}^{n} \binom{n - j + \alpha - 1}{n - j} u(j) \tag{2.3}$$

and the fractional order difference operator of order α ($\alpha \in \mathbb{R}$, $0 < \alpha \leq 1$) is defined as

$$\nabla^{\alpha}u(n) = \sum_{j=0}^{n-1}\binom{j-\alpha}{j}\nabla u(n-j) \tag{2.4}$$

$$= \sum_{j=1}^{n}\binom{n-j-\alpha-1}{n-j}u(j) - \binom{n-\alpha-1}{n-1}u(0). \tag{2.5}$$

Theorem 2.1. *Let $u(n)$, $v(n) : \mathbb{N}_0^+ \to \mathbb{R}$, $\alpha, \beta \in \mathbb{R}$ be such that $\alpha, \beta > 0$, $\alpha + \beta \leq 1$ and let c, d be scalars. Then*

(1) $\nabla^{\beta}\nabla^{\gamma}u(n) = \nabla^{\beta+\gamma}u(n)$,
(2) $\nabla^{\alpha}[cu(n) + dv(n)] = c\nabla^{\alpha}u(n) + d\nabla^{\alpha}v(n)$,
(3) $\nabla^{\alpha}u(0) = 0$ *and* $\nabla^{\alpha}u(1) = u(1) - u(0) = \nabla u(1)$.

Proof. Consider

$$\begin{aligned}
\nabla^{\alpha}\nabla^{\beta}u(n) &= \nabla^{\alpha}\Big[\nabla^{\beta}u(n)\Big] \\
&= \sum_{j=1}^{n}\binom{n-j-\alpha-1}{n-j}\nabla^{\beta}u(j) - \binom{n-\alpha-1}{n-1}\Big[\nabla^{\beta}u(n)\Big]_{n=0} \\
&= \sum_{j=1}^{n}\sum_{k=1}^{j}\binom{n-j-\alpha-1}{n-j}\binom{j-k-\beta-1}{j-k}u(k) \\
&\quad -\sum_{j=1}^{n}\binom{n-j-\alpha-1}{n-j}\binom{j-\beta-1}{j-1}u(0) \\
&= S_1 - S_2.
\end{aligned}$$

Now consider

$$\begin{aligned}
S_1 &= \sum_{j=1}^{n}\sum_{k=1}^{j}\binom{n-j-\alpha-1}{n-j}\binom{j-k-\beta-1}{j-k}u(k) \\
&= \sum_{k=1}^{n}\sum_{j=k}^{n}\frac{\Gamma(n-j-\alpha)}{\Gamma(n-j+1)\Gamma(-\alpha)}\frac{\Gamma(j-k-\beta)}{\Gamma(j-k+1)\Gamma(-\beta)}u(k) \\
&= \sum_{k=1}^{n}\sum_{j=0}^{n-k}\frac{\Gamma(n-j-k-\alpha)}{\Gamma(n-j-k+1)\Gamma(-\alpha)}\frac{\Gamma(j-\beta)}{\Gamma(j+1)\Gamma(-\beta)}u(k) \\
&= \sum_{k=1}^{n}\frac{u(k)}{\Gamma(n-k+1)}\sum_{j=0}^{n-k}\binom{n-k}{j}(-\alpha)_{n-k-j}(-\beta)_j \\
&= \sum_{k=1}^{n}\frac{u(k)}{\Gamma(n-k+1)}(-\alpha-\beta)_{n-k} \text{ (using Remark 2.2)}
\end{aligned}$$

$$= \sum_{k=1}^{n} \frac{\Gamma(n-k-\alpha-\beta)}{\Gamma(n-k+1)\Gamma(-\alpha-\beta)} u(k) = \sum_{k=1}^{n} \binom{n-k-\alpha-\beta-1}{n-k} u(k)$$

and

$$\begin{aligned}
S_2 &= \sum_{j=1}^{n} \binom{n-j-\alpha-1}{n-j}\binom{j-\beta-1}{j-1} u(0) \\
&= \sum_{j=1}^{n} \frac{\Gamma(n-j-\alpha)}{\Gamma(n-j+1)\Gamma(-\alpha)} \frac{\Gamma(j-\beta)}{\Gamma(j)\Gamma(-\beta+1)} u(0) \\
&= u(0)\sum_{j=0}^{n-1} \frac{\Gamma(n-j-\alpha-1)}{\Gamma(n-j)\Gamma(-\alpha)} \frac{\Gamma(j-\beta+1)}{\Gamma(j+1)\Gamma(-\beta+1)} \\
&= \frac{u(0)}{\Gamma(n)} \sum_{j=0}^{n-1} \frac{\Gamma(n)}{\Gamma(n-j)\Gamma(j+1)} \frac{\Gamma(n-1-j-\alpha)}{\Gamma(-\alpha)} \frac{\Gamma(j-\beta+1)}{\Gamma(-\beta+1)} \\
&= \frac{u(0)}{\Gamma(n)} \sum_{j=0}^{n-1} \binom{n-1}{j} (-\alpha)_{n-1-j}(-\beta+1)_j \text{ (using Definition 2.1)} \\
&= \frac{u(0)}{\Gamma(n)} (-\alpha-\beta+1)_{n-1} \text{ (using Remark 2.2)} \\
&= \frac{\Gamma(n-\alpha-\beta)}{\Gamma(n)\Gamma(-\alpha-\beta+1)} u(0) = \binom{n-\alpha-\beta-1}{n-1} u(0).
\end{aligned}$$

Therefore

$$\begin{aligned}
\nabla^{\alpha}\nabla^{\beta}u(n) &= \sum_{k=1}^{n} \binom{n-k-\alpha-\beta-1}{n-k} u(k) - \binom{n-\alpha-\beta-1}{n-1} u(0) \\
&= \nabla^{\alpha+\beta}u(n).
\end{aligned}$$

□

Definition 2.4. Let $f(n,r) : \mathbb{N}_0^+ \times \mathbb{R} \to \mathbb{R}$. Then a nonlinear difference equation of order α, $0 < \alpha \le 1$, together with an initial condition is of the form

$$\nabla^{\alpha}u(n+1) = f(n, u(n)),\ u(0) = u_0. \tag{2.6}$$

Recently, the authors established the following fractional order discrete Gronwall–Bellman inequality (see [5]).

Theorem 2.2. *Let $u(n)$, $a(n)$ and $b(n)$ be real valued nonnegative functions defined on $\mathbb{N}_0^+$ and let $\alpha \in \mathbb{R}$ be such that $0 < \alpha \leq 1$. If for all $n \in \mathbb{N}_0^+$*

$$\nabla^{\alpha} u(n+1) \leq a(n)u(n) + b(n), \tag{2.7}$$

then for all $n \in \mathbb{N}_0^+$

$$\begin{aligned} u(n) \quad \leq \quad & u(0) \prod_{j=0}^{n-1} \Big[1 + B(n-1,\alpha;j)a(j)\Big] + \sum_{j=0}^{n-1} B(n-1,\alpha;j)b(j) \\ & \prod_{k=j+1}^{n-1} \Big[1 + B(n-1,\alpha;j)a(k)\Big]. \end{aligned}$$

Corollary 1. *Let $u(n)$, $a(n)$ and $b(n)$ be real valued nonnegative functions defined on $\mathbb{N}_0^+$ and let $\alpha \in \mathbb{R}$ be such that $0 < \alpha \leq 1$. If for all $n \in \mathbb{N}_0^+$*

$$u(n) \leq u(0) + \sum_{j=0}^{n-1} B(n-1,\alpha;j)a(j)[a(j)u(j) + b(j)], \tag{2.8}$$

then for all $n \in \mathbb{N}_0^+$

$$\begin{aligned} u(n) \quad \leq \quad & u(0) \prod_{j=0}^{n-1} \Big[1 + B(n-1,\alpha;j)a(j)\Big] + \sum_{j=0}^{n-1} B(n-1,\alpha;j)b(j) \\ & \prod_{k=j+1}^{n-1} \Big[1 + B(n-1,\alpha;k)a(k)\Big] \\ \leq \quad & u(0) \ \exp\Big[\sum_{j=0}^{n-1} B(n-1,\alpha;j)a(j)\Big] + \sum_{j=0}^{n-1} B(n-1,\alpha;j)b(j) \\ & \exp\Big[\sum_{k=j+1}^{n-1} B(n-1,\alpha;k)a(k)\Big]. \end{aligned}$$

3. Gronwall–Bellman type inequalities

In this section, we shall establish some fractional difference inequalities of Gronwall–Bellman type. Throughout the section we assume that $\alpha \in (0,1]$. Let $u(n)$, $a(n)$, $b(n)$, $c(n)$ and $p(n)$ be real valued nonnegative functions defined on $\mathbb{N}_0^+$.

Theorem 3.1. *If for all $n \in \mathbb{N}_0^+$*

$$u(n) \leq a(n) + p(n) \sum_{j=0}^{n-1} B(n-1,\alpha;j)[b(j)u(j) + c(j)], \tag{3.1}$$

then for all $n \in \mathbb{N}_0^+$

$$\begin{aligned} u(n) \quad \leq \quad & a(n) + p(n)\sum_{j=0}^{n-1} B(n-1,\alpha;j)[a(j)b(j)+c(j)] \\ & \prod_{k=j+1}^{n-1}\Big[1+B(n-1,\alpha;k)b(k)p(k)\Big]. \end{aligned} \tag{3.2}$$

Proof. Define a function $z(n)$ by

$$z(n) = \sum_{j=0}^{n-1} B(n-1,\alpha;j)[b(j)u(j)+c(j)].$$

Then $z(0)=0$, $u(n) \leq a(n)+p(n)z(n)$ and

$$\nabla^{\alpha} z(n+1) \leq b(n)u(n)+c(n) \leq b(n)p(n)z(n)+a(n)b(n)+c(n).$$

Now an application of Theorem 2.2 with $z(0)=0$ yields

$$z(n) \leq \sum_{j=0}^{n-1} B(n-1,\alpha;j)[a(j)b(j)+c(j)] \prod_{k=j+1}^{n-1}\Big[1+B(n-1,\alpha;k)b(k)p(k)\Big]. \tag{3.3}$$

Using (3.3) in $u(n) \leq a(n)+p(n)z(n)$, we get the required inequality in (3.2). □

Theorem 3.2. *If, for all* $n \in \mathbb{N}_0^+$, $a(n)$ *is nondecreasing and*

$$u(n) \leq a(n) + p(n)\sum_{j=0}^{n-1} B(n-1,\alpha;j)[b(j)u(j)], \tag{3.4}$$

then for all $n \in \mathbb{N}_0^+$

$$u(n) \leq a(n)\Big[1+p(n)\sum_{j=0}^{n-1} B(n-1,\alpha;j)b(j) \prod_{k=j+1}^{n-1}\Big[1+B(n-1,\alpha;k)b(k)p(k)\Big]\Big]. \tag{3.5}$$

Proof. First we assume that $a(n)>0$ for $n \in \mathbb{N}_0^+$. From (3.4) we observe that

$$\begin{aligned} \frac{u(n)}{a(n)} \quad & \leq \quad 1 + \frac{p(n)}{a(n)}\sum_{j=0}^{n-1} B(n-1,\alpha;j)[b(j)u(j)] \\ & \leq \quad 1 + p(n)\sum_{j=0}^{n-1} B(n-1,\alpha;j)\Big[b(j)\frac{u(j)}{a(j)}\Big]. \end{aligned}$$

Now an application of Theorem 2.2 yields the desired bound in (3.5). If $a(n) = 0$, then from (3.4) we observe that

$$u(n) \leq \epsilon + p(n)\sum_{j=0}^{n-1} B(n-1,\alpha;j)[b(j)u(j)] \tag{3.6}$$

where $\epsilon > 0$ is an arbitrarily small constant. A suitable application of Theorem 3.2 to (3.6) yields

$$u(n) \leq \epsilon\Big[1 + p(n)\sum_{j=0}^{n-1} B(n-1,\alpha;j)b(j)\prod_{k=j+1}^{n-1}\Big[1 + B(n-1,\alpha;k)b(k)p(k)\Big]\Big]. \tag{3.7}$$

Now by lettting $\epsilon \to 0$ in (3.7), we have $u(n) = 0$ and hence (3.5) holds. □

Theorem 3.3. *If, for all $n \in \mathbb{N}_0^+$, $\nabla^\alpha c(n+1) \geq 0$ and*

$$u(n) \leq a(n) + b(n)\Big[c(n) + \sum_{j=0}^{n-1} B(n-1,\alpha;j)[p(j)u(j)]\Big], \tag{3.8}$$

then for all $n \in \mathbb{N}_0^+$

$$\begin{aligned} u(n) \quad \leq \quad & a(n) + b(n)\Big[c(0)\prod_{j=0}^{n-1}\Big[1 + B(n-1,\alpha;j)[b(j)p(j)]\Big] \\ & + \sum_{j=0}^{n-1} B(n-1,\alpha;j)[\nabla^\alpha c(j+1) + a(j)p(j)] \\ & \prod_{k=j+1}^{n-1}\Big[1 + B(n-1,\alpha;k)b(k)p(k)\Big]\Big]. \end{aligned} \tag{3.9}$$

Proof. Define a function $z(n)$ by

$$z(n) = c(n) + \sum_{j=0}^{n-1} B(n-1,\alpha;j)[p(j)u(j)].$$

Then $z(0) = c(0)$, $u(n) \leq a(n) + b(n)z(n)$ and

$$\begin{aligned} \nabla^\alpha z(n+1) \quad &= \quad \nabla^\alpha c(n+1) + p(n)u(n) \\ &\leq \quad \nabla^\alpha c(n+1) + a(n)p(n) + p(n)b(n)z(n). \end{aligned} \tag{3.10}$$

Now by applying Theorem 2.2 to (3.10), we get

$$\begin{aligned} z(n) \quad \leq \quad & z(0)\prod_{j=0}^{n-1}\Big[1+B(n-1,\alpha;j)[b(j)p(j)]\Big] \\ & +\sum_{j=0}^{n-1}B(n-1,\alpha;j)[\nabla^{\alpha}c(j+1)+a(j)p(j)] \\ & \prod_{k=j+1}^{n-1}\Big[1+B(n-1,\alpha;k)b(k)p(k)\Big]. \end{aligned} \tag{3.11}$$

Using (3.11) in $u(n) \leq a(n) + b(n)z(n)$, we get the required inequality in (3.9). □

Theorem 3.4. *Let $a(n)$ be a real valued positive function defined on $\mathbb{N}_0^+$ and*

$$u(n) \leq a(n)\Big[c+\sum_{j=0}^{n-1}B(n-1,\alpha;j)[b(j)u(j)]\Big] \tag{3.12}$$

for all $n \in \mathbb{N}_0^+$, where $c \geq 0$ is a constant. Then the following inequalities hold.

(1) *If $0 < a(n) \leq 1$, then*

$$u(n) \leq c\ a(n)\prod_{j=0}^{n-1}\Big[1+B(n-1,\alpha;j)[b(j)]\Big]. \tag{3.13}$$

(2) *If $a(n) \geq 1$, then*

$$u(n) \leq c\Big[\prod_{j=0}^{n}a(j)\Big]\prod_{j=0}^{n-1}\Big[1+B(n-1,\alpha;j)[b(j)]\Big] \tag{3.14}$$

for $n \in \mathbb{N}_0^+$.

Proof. (1) If $0 < a(n) \leq 1$, then from (3.12) we have

$$\begin{aligned} \frac{u(n)}{a(n)} \quad & \leq \quad c+\sum_{j=0}^{n-1}B(n-1,\alpha;j)b(j)u(j) \\ & \leq \quad c+\sum_{j=0}^{n-1}B(n-1,\alpha;j)\Big[b(j)\frac{u(j)}{a(j)}\Big] \end{aligned} \tag{3.15}$$

for all $n \in \mathbb{N}_0^+$. Now an application of Corollary 1 yields the desired inequality in (3.13).

(2) If $a(n) \geq 1$, then from (3.12) we have

$$\begin{aligned}\frac{u(n)}{a(n)} &\leq c + \sum_{j=0}^{n-1} B(n-1,\alpha;j)b(j)u(j) \\ &= c + \sum_{j=0}^{n-1} B(n-1,\alpha;j)\Big[b(j)a(j)\frac{u(j)}{a(j)}\Big] \qquad (3.16)\end{aligned}$$

for $n \in \mathbb{N}_0^+$. Now an application of Corollary 1 yields

$$\frac{u(n)}{a(n)} \leq c \prod_{j=0}^{n-1} \Big[1 + B(n-1,\alpha;j)a(j)b(j)\Big]. \qquad (3.17)$$

From (3.17), we observe that

$$\frac{u(n)}{a(n)} \leq c\Big[\prod_{j=0}^{n-1} a(j)\Big] \prod_{j=0}^{n-1} \Big[1 + B(n-1,\alpha;j)b(j)\Big] \qquad (3.18)$$

for $n \in \mathbb{N}_0^+$. The required inequality in (3.14) follows from (3.18). □

4. Applications

In this section we apply the fractional difference inequality established in Theorem 2.2 to obtain a bound for the solution of a fractional difference equation together with an initial condition of the form

$$\nabla^\alpha u(n+1) = f(n,u(n)), u(0) = u_0, \qquad (4.1)$$

where $f(n,r)$ is a function defined for $n \in \mathbb{N}_0^+$, $0 \leq r < \infty$, and $u(n) : \mathbb{N}_0^+ \longrightarrow \mathbb{R}$ is such that

$$|f(n,u(n))| \leq a(n)|u(n)| + b(n) \qquad (4.2)$$

for $n \in \mathbb{N}_0^+$, where $a(n)$, $b(n)$ are as defined in Theorem 2.2. Let $u(n)$ be the solution of (4.1) for $n \in \mathbb{N}_0^+$. Using Theorem 2.2, we get

$$\begin{aligned}|u(n)| &\leq u(0)\prod_{j=0}^{n-1}\Big[1 + B(n-1,\alpha;j)a(j)\Big] + \sum_{j=0}^{n-1} B(n-1,\alpha;j)b(j) \\ &\quad \prod_{k=j+1}^{n-1}\Big[1 + B(n-1,\alpha;k)a(k)\Big] \\ &\leq u(0)\ \exp\Big[\sum_{j=0}^{n-1} B(n-1,\alpha;j)a(j)\Big] + \sum_{j=0}^{n-1} B(n-1,\alpha;j)b(j) \\ &\quad \exp\Big[\sum_{k=j+1}^{n-1} B(n-1,\alpha;k)a(k)\Big]\end{aligned}$$

for $n \in \mathbb{N}_0^+$. The right hand side of the above inequality gives the bound on the solution of (4.1) in terms of the known functions.

Example 1. Now we use Theorem 2.2 to find a bound for the solution of the fractional difference equation together with an initial condition

$$\nabla^\alpha u(n+1) = u(n),\ u(0) = 1, \tag{4.3}$$

where $u(n) : \mathbb{N}_0^+ \longrightarrow \mathbb{R}$.

Solution. Let $u(n)$ be the solution of (4.3) for $n \in \mathbb{N}_0^+$. Using Theorem 2.2, we get

$$|u(n)| \leq \exp\Big[\sum_{j=0}^{n-1} B(n-1,\alpha;j)\Big] = \exp\Big[\binom{n-1+\alpha}{n-1}\Big] \tag{4.4}$$

for $n \in \mathbb{N}_0^+$. The right hand side of the above inequality gives the bound on the solution of (4.3) in terms of the known functions. Clearly, the solution of (4.3) is

$$u(n) = \prod_{j=0}^{n-1} \Big[1 + B(n-1,\alpha;j)\Big]. \tag{4.5}$$

The solution of the corresponding ordinary difference equation

$$\nabla u(n+1) = u(n),\ u(0) = 1 \tag{4.6}$$

is

$$u(n) = 2^n. \tag{4.7}$$

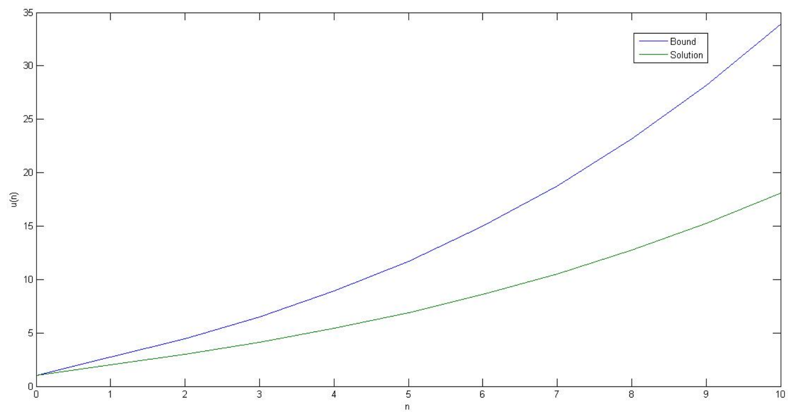

FIGURE 1. Comparison of the solution of (4.3) and its bound

5. Conclusion

To conclude, we note that the bounds established in Theorems 3.1 to 3.4 are independent of the unknown functions and will have many applications to boundedness, uniqueness and other properties of the solutions of initial value problems of fractional order.

The comparison of the solution of (4.3) and its bound obtained in (4.4) for $\alpha = 0.5$ is given in Figure 1.

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