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Double sequences of interval numbers defined by Orlicz functions

Ayhan Esi

Abstract. We define and study λ_2-convergence of double sequences of interval numbers defined by Orlicz function and λ_2-statistical convergence of double sequences of interval numbers. We also establish some inclusion relations between them.

1. Introduction

The idea of statistical convergence for ordinary sequences was introduced by Fast [7] in 1951. Schoenberg [17] studied statistical convergence as a summability method and listed some elementary properties of statistical convergence. Both of these authors noted that if a bounded sequence is statistically convergent, then it is Cesàro summable. Recently Mursaleen [15] defined and studied λ-statistical convergence for sequences as follows. Let $\lambda = (\lambda_i)$ be a non-decreasing sequence of positive numbers tending to infinity such that $\lambda_{i+1} \leq \lambda_i + 1$, $\lambda_1 = 1$. Then a sequence $x = (x_k)$ is said to be λ-*statistically convergent* to a number L if for every $\varepsilon > 0$,

$$\lim_{i\to\infty} \frac{1}{\lambda_i} \left|\{k \in I_i : d\left(x_k, L\right) \geq \varepsilon\ \}\right| = 0,$$

where $I_i = [i - \lambda_i + 1, i]$.

Recall (see [10]) that an *Orlicz function* M is a continuous, convex, non-decreasing function, defined for $u \geq 0$, such that $M(0) = 0$ and $M(u) > 0$ if $u > 0$. An Orlicz function M is said to satisfy Δ_2-*condition* for all values of u if there exists a number $K > 0$ such that $M(2u) \leq KM(u), u \geq 0$.

Interval arithmetic was first suggested by Dwyer [2] in 1951. Development of interval arithmetic as a formal system and evidence of its value as a computational device was provided by Moore [12] in 1959 and Moore and

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Yang [13] in 1962. Furthermore, Moore and others (see [3], [8], [12] and [11]) have developed applications to differential equations.

Chiao [1] introduced sequences of interval numbers and defined the usual convergence of sequences of interval numbers. Şengönül and Eryılmaz [18] introduced and studied bounded and convergent sequence spaces of interval numbers and showed that these spaces are complete metric spaces. Recently, Esi in [4] and [5] defined and studied λ-statistical and lacunary statistical convergence of interval numbers, respectively.

We denote the set of all real valued closed intervals by $\mathrm{I}\mathbb{R}$. Any element of $\mathrm{I}\mathbb{R}$ is called an interval number and is denoted by $\overline{A} = [x_l, x_r]$. Let x_l and x_r be the smallest and the greatest points of an interval number $\overline{A}$, respectively. For interval numbers $\overline{A}_1 = [x_{1_l}, x_{1_r}]$, $\overline{A}_2 = [x_{2_l}, x_{2_r}]$ and a number $\alpha \in \mathbb{R}$ we have

$$\overline{A}_1 = \overline{A}_2 \iff x_{1_l} = x_{2_l}, x_{1_r} = x_{2_r},$$
$$\overline{A}_1 + \overline{A}_2 = \{x \in \mathbb{R} : x_{1_l} + x_{2_l} \le x \le x_{1_r} + x_{2_r}\}$$

and

$$\alpha\overline{A} = \begin{cases} \{x \in \mathbb{R} : \ \alpha x_{1_l} \le x \le \alpha x_{1_r}\} & \text{if } \alpha \ge 0, \\ \{x \in \mathbb{R} : \ \alpha x_{1_r} \le x \le \alpha x_{1_l}\} & \text{if } \alpha < 0. \end{cases}$$

The set of all interval numbers $\mathrm{I}\mathbb{R}$ is a complete metric space with the distance

$$d\left(\overline{A}_1, \overline{A}_2\right) = \max\left\{\left|x_{1_l} - x_{2_l}\right|, \left|x_{1_r} - x_{2_r}\right|\right\}$$

(see [14]). In the special case $\overline{A}_1 = [a, a]$ and $\overline{A}_2 = [b, b]$ we obtain the usual metric of $\mathbb{R}$.

Now we give the definition of a convergent sequence of interval numbers (see [1]).

Definition 1.1. A sequence $\left(\overline{A}_k\right)$ of interval numbers is said to be *convergent* to an interval number $\overline{A}_0$ if for each $\varepsilon > 0$ there exists a positive integer k_0 such that $d\left(\overline{A}_k, \overline{A}_0\right) < \varepsilon$ for all $k \ge k_0$. We denote it by $\lim_k \overline{A}_k = \overline{A}_0$.

Thus, $\lim_k \overline{A}_k = \overline{A}_0$ if and only if $\lim_k x_{k_l} = x_{0_l}$ and $\lim_k x_{k_r} = x_{0_r}$.

In this paper, we introduce and study the concepts of λ_2-summable and statistically λ_2-convergent double sequences of interval numbers, and relations between them.

2. Definitions

Recall that a double sequence $(a_{k,i})$ of real numbers is said to be *convergent in the Pringsheim sense* to a number L if for every $\varepsilon > 0$ there exists an index n such that $|a_{k,i} - L| < \varepsilon$ whenever $k, i > n$ (see [16]). We transfer this definition to the double sequences of interval numbers in the following way.

Definition 2.1. An interval valued double sequence $\left(\overline{A}_{k,l}\right)$ is said to be *convergent in the Pringsheim sense* to an interval number $\overline{A}_0$ if for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that

$$d\left(\overline{A}_{k,l}, \overline{A}_0\right) < \varepsilon \text{ for } k, l > n.$$

In this case we write $P\text{-}\lim \overline{A}_{k,l} = \overline{A}_0$.

We denote by $\overline{c}^2$ the set of all convergent in the Pringsheim sense double sequences of interval numbers.

Definition 2.2. An interval valued double sequence $\left(\overline{A}_{k,l}\right)$ is said to be *bounded* if there exist an interval number $\overline{A}_0$ and a positive number B such that $d\left(\overline{A}_{k,l}, \overline{0}\right) \leq B$ for all $k, l \in \mathbb{N}$.

We will denote the set of all bounded double sequences of interval number by the symbol $\overline{\ell}_{\infty}^2$. It should be noted that, similarly to the case of double number sequences, $\overline{c}^2$ is not the subset of $\overline{\ell}_{\infty}^2$.

Let a double sequence $\lambda_2 = (\lambda_{i,j})$ of positive real numbers tend to infinity and satisfy

$$\lambda_{i+1,j} \leq \lambda_{i,j} + 1, \ \lambda_{i,j+1} \leq \lambda_{i,j} + 1,$$

$$\lambda_{i,j} - \lambda_{i+1,j} \leq \lambda_{i,j+1} - \lambda_{i+1,j+1}, \lambda_{1,1} = 1.$$

Put

$$I_{i,j} = \left\{(k,l) : \ i - \lambda_{i,j} + 1 \leq k \leq i, \ j - \lambda_{i,j} + 1 \leq l \leq j\right\}.$$

Definition 2.3. An interval valued double sequence $\left(\overline{A}_{k,l}\right)$ is said to be λ_2-*summable* if there exists an interval number $\overline{A}_0$ such that

$$P\text{-}\lim_{i,j} \frac{1}{\lambda_{i,j}} \sum_{(k,l) \in I_{i,j}} d\left(\overline{A}_{k,l}, \overline{A}_0\right) = 0.$$

Definition 2.4. Let M be an Orlicz function, let $\left(\overline{A}_{k,l}\right)$ be an interval valued double sequence, and let $p = (p_{k,l})$ be a double sequence of positive real numbers. Let $\lambda_2 = (\lambda_{i,j})$ be the double sequence defined above. We define

$$\left[\overline{V}^2_{\lambda_2}, M, p\right] = \left\{ \left(\overline{A}_{k,l}\right) : P\text{-}\lim_{i,j} \frac{1}{\lambda_{i,j}} \sum_{(k,l)\in I_{i,j}} \left[M\left(\frac{d\left(\overline{A}_{k,l}, \overline{A}_0\right)}{r} \right) \right]^{p_{k,l}} = 0, \right.$$
$$\left. \text{for some } r > 0 \text{ and } \overline{A}_0 \in \mathbb{IR} \right\},$$

$$\left[\overline{V}^2_{\lambda_2}, M, p\right]_0 = \left\{ \left(\overline{A}_{k,l}\right) : P\text{-}\lim_{i,j} \frac{1}{\lambda_{i,j}} \sum_{(k,l)\in I_{i,j}} \left[M\left(\frac{d\left(\overline{A}_{k,l}, \overline{0}\right)}{r} \right) \right]^{p_{k,l}} = 0, \right.$$
$$\left. \text{for some } r > 0 \right\}$$

and

$$\left[\overline{V}^2_{\lambda_2}, M, p\right]_\infty = \left\{ \left(\overline{A}_{k,l}\right) : \sup_{i,j} \frac{1}{\lambda_{i,j}} \sum_{(k,l)\in I_{i,j}} \left[M\left(\frac{d\left(\overline{A}_{k,l}, \overline{0}\right)}{r} \right) \right]^{p_{k,l}} < \infty, \right.$$
$$\left. \text{for some } r > 0 \right\},$$

where $\overline{0} = [0, 0]$.

If we consider various assignments of M, λ_2 and p in Definition 2.4, then we obtain different special sets of sequences. For example, if $p_{k,l} = 1$ for all $k, l \in \mathbb{N}$, then these sets reduce to the sets denoted, respectively, by $\left[\overline{V}^2_{\lambda_2}, M\right]$, $\left[\overline{V}^2_{\lambda_2}, M\right]_0$ and $\left[\overline{V}^2_{\lambda_2}, M\right]_\infty$. In the special case $\lambda_{i,j} = ij \ \ (i, j \in \mathbb{N})$, we write $\left[\overline{c}^2, M\right]$ instead of $\left[\overline{V}^2_{\lambda_2}, M\right]$.

For $M(x) = x$ we obtain

$$\left[\overline{V}^2_{\lambda_2}, p\right] = \left\{ \left(\overline{A}_{k,l}\right) : P\text{-}\lim_{i,j} \frac{1}{\lambda_{i,j}} \sum_{(k,l)\in I_{i,j}} \left[d\left(\overline{A}_{k,l}, \overline{A}_0\right)\right]^{p_{k,l}} = 0 \right.$$
$$\left. \text{for some } \overline{A}_0 \in \mathbb{IR} \right\},$$

and similarly, $\left[\overline{V}^2_{\lambda_2}, p\right]_0$ and $\left[\overline{V}^2_{\lambda_2}, p\right]_\infty$.

3. Main theorems

Theorem 3.1. *If* $0 < p_{k,l} < q_{k,l}$ *and* $\left(\frac{q_{k,l}}{p_{k,l}}\right)$ *is bounded, then* $\left[\overline{V}^2_{\lambda_2}, M, p\right] \subset \left[\overline{V}^2_{\lambda_2}, M, q\right]$.

Proof. If we take $\left[M\left(\frac{d(\overline{A}_{k,l},\overline{A}_0)}{r}\right)\right]^{p_{k,l}} = w_{k,l}$ for all $k,l \in \mathbb{N}$, then using the same technique employed in the proof of Theorem 2.9 from [9] we get the result. □

Corollary 3.2. *The following statements are valid.*

(i) *If* $0 < \inf_{k,l} p_{k,l} \leq 1$ *for all* $k,l \in \mathbb{N}$, *then* $\left[\overline{V}^2_{\lambda_2}, M, p\right] \subset \left[\overline{V}^2_{\lambda_2}, M\right]$.

(ii) *If* $1 \leq p_{k,l} \leq \sup_{k,l} p_{k,l} = H < \infty$ *for all* $k,l \in \mathbb{N}$, *then* $\left[\overline{V}^2_{\lambda_2}, M\right] \subset \left[\overline{V}^2_{\lambda_2}, M, p\right]$.

Proof. (i) follows from Theorem 3.1 with $q_{k,l} = 1$ for all $k,l \in \mathbb{N}$ and (ii) follows from Theorem 3.1 with $p_{k,l} = 1$ for all $k,l \in \mathbb{N}$. □

The proof of the following result is a routine work, so we omit it.

Proposition 3.3. *Let M be an Orlicz function such that Δ_2-condition is satisfied. Then we have* $\left[\overline{V}^2_{\lambda_2}, p\right]_0 \subset \left[\overline{V}^2_{\lambda_2}, M, p\right]_0$, $\left[\overline{V}^2_{\lambda_2}, p\right] \subset \left[\overline{V}^2_{\lambda_2}, M, p\right]$ *and* $\left[\overline{V}^2_{\lambda_2}, p\right]_\infty \subset \left[\overline{V}^2_{\lambda_2}, M, p\right]_\infty$.

The following definition was presented by Esi [6] for a single sequence of interval numbers. A sequence of interval numbers $\left(\overline{A}_k\right)$ is said to be *statistically λ-convergent* to an interval number $\overline{A}_0$ if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \left|\left\{k \in I_n : d\left(\overline{A}_k, \overline{A}_0\right) \geq \varepsilon \right\}\right| = 0,$$

where the vertical bars indicate the number of elements in the enclosed set.

Now, we will give definitions of statistical convergence and statistical λ_2-convergence for double sequences of interval numbers.

Definition 3.1. A double sequence $\left(\overline{A}_{k,l}\right)$ of interval numbers is said to be *statistically convergent* to an interval number $\overline{A}_0$ provided that for each $\varepsilon > 0$,

$$P\text{-}\lim_{i,j\to\infty} \frac{1}{ij} \left|\left\{(k,l) \in \mathbb{N}\times\mathbb{N};\ k \leq i, l \leq j : d\left(\overline{A}_{k,l}, \overline{A}_0\right) \geq \varepsilon \right\}\right| = 0.$$

We denote the set of all statistically convergent double sequences of interval numbers by $\overline{s}^2$.

Definition 3.2. A double sequence $\left(\overline{A}_{k,l}\right)$ of interval numbers is said to be *statistically λ_2-convergent* to an interval number $\overline{A}_0$ if for each $\varepsilon > 0$,

$$P\text{-}\lim_{i,j\to\infty} \frac{1}{\lambda_{i,j}} \left|\left\{(k,l) \in I_{i,j} : d\left(\overline{A}_{k,l}, \overline{A}_0\right) \geq \varepsilon \right\}\right| = 0.$$

We denote the set of all statistically λ_2-convergent double sequences of interval numbers by $\overline{s}^2_{\lambda_2}$.

Theorem 3.4. *Let M be an Orlicz function. If* $0 < h \leq \inf_{k,l} p_{k,l} \leq \sup_{k,l} p_{k,l} = H < \infty$*, then* $\left[\overline{V}^2_{\lambda_2}, M, p\right] \subset \overline{s}^2_{\lambda_2}$.

Proof. Let $\left(\overline{A}_{k,l}\right) \in \left[\overline{V}^2_{\lambda_2}, M, p\right]$. Then there exists $r > 0$ such that

$$\frac{1}{\lambda_{i,j}} \sum_{(k,l) \in I_{i,j}} \left[M\left(\frac{d\left(\overline{A}_{k,l}, \overline{A}_0\right)}{r}\right)\right]^{p_{k,l}} \to 0$$

in the Pringsheim sense if $i, j \to \infty$. If $\varepsilon > 0$, then we obtain

$$\begin{aligned}
&\frac{1}{\lambda_{i,j}} \sum_{(k,l) \in I_{i,j}} \left[M\left(\frac{d\left(\overline{A}_{k,l}, \overline{A}_0\right)}{r}\right)\right]^{p_{k,l}} \\
&\geq \frac{1}{\lambda_{i,j}} \sum_{\substack{(k,l) \in I_{i,j} \\ d\left(\overline{A}_{k,l}, \overline{A}_0\right) \geq \varepsilon}} \left[M\left(\frac{d\left(\overline{A}_{k,l}, \overline{A}_0\right)}{r}\right)\right]^{p_{k,l}} \\
&\geq \frac{1}{\lambda_{i,j}} \sum_{\substack{(k,l) \in I_{i,j} \\ d\left(\overline{A}_{k,l}, \overline{A}_0\right) \geq \varepsilon}} \left[M\left(\frac{\varepsilon}{r}\right)\right]^{p_{k,l}} \\
&\geq \frac{1}{\lambda_{i,j}} \sum_{\substack{(k,l) \in I_{i,j} \\ d\left(\overline{A}_{k,l}, \overline{A}_0\right) \geq \varepsilon}} \min\left\{M\left(\frac{\varepsilon}{r}\right)^h, M\left(\frac{\varepsilon}{r}\right)^H\right\} \\
&\geq \frac{1}{\lambda_{i,j}} \left|\left\{(k,l) \in I_{i,j} : d\left(\overline{A}_{k,l}, \overline{A}_0\right) \geq \varepsilon\ \right\}\right| \min\left\{M\left(\frac{\varepsilon}{r}\right)^h, M\left(\frac{\varepsilon}{r}\right)^H\right\}.
\end{aligned}$$

Hence $\left(\overline{A}_{k,l}\right) \in \overline{s}^2_{\lambda_2}$, which completes the proof. □

Theorem 3.5. *Let M be an Orlicz function and let* $0 < h \leq \inf_{k,l} p_{k,l} \leq \sup_{k,l} p_{k,l} = H < \infty$*. Then* $\overline{s}^2_{\lambda_2} \cap \overline{\ell}^2_{\infty} \subset \left[\overline{V}^2_{\lambda_2}, M, p\right]$.

Proof. Let $\left(\overline{A}_{k,l}\right) \in \overline{s}^2_{\lambda_2} \cap \overline{\ell}^2_{\infty}$. Then there is a constant $N > 0$ such that $d\left(\overline{A}_{k,l}, \overline{A}_0\right) \leq N$ for all $k, l \in \mathbb{N}$. Given $\varepsilon > 0$, for an arbitrarily fixed $r > 0$

we have

$$
\begin{aligned}
&\frac{1}{\lambda_{i,j}} \sum_{(k,l)\in I_{i,j}} \left[M\left(\frac{d\left(\overline{A}_{k,l}, \overline{A}_0\right)}{r} \right) \right]^{p_{k,l}} \\
&= \frac{1}{\lambda_{i,j}} \sum_{\substack{(k,l)\in I_{i,j} \\ d\left(\overline{A}_{k,l}, \overline{A}_0\right)<\varepsilon}} \left[M\left(\frac{d\left(\overline{A}_{k,l}, \overline{A}_0\right)}{r} \right) \right]^{p_{k,l}} \\
&\quad + \frac{1}{\lambda_{i,j}} \sum_{\substack{(k,l)\in I_{i,j} \\ d\left(\overline{A}_{k,l}, \overline{A}_0\right)\geq\varepsilon}} \left[M\left(\frac{d\left(\overline{A}_{k,l}, \overline{A}_0\right)}{r} \right) \right]^{p_{k,l}} \\
&\leq \frac{1}{\lambda_{i,j}} \sum_{\substack{(k,l)\in I_{i,j} \\ d\left(\overline{A}_{k,l}, \overline{A}_0\right)<\varepsilon}} \left[M\left(\frac{\varepsilon}{r} \right) \right]^{p_{k,l}} + \frac{1}{\lambda_{i,j}} \sum_{\substack{(k,l)\in I_{i,j} \\ d\left(\overline{A}_{k,l}, \overline{A}_0\right)\geq\varepsilon}} \max\left\{ M\left(\frac{N}{r}\right)^h, M\left(\frac{N}{r}\right)^H \right\} \\
&\leq \max\left\{ M\left(\frac{\varepsilon}{r}\right)^h, M\left(\frac{\varepsilon}{r}\right)^H \right\} \\
&\quad + \frac{1}{\lambda_{i,j}} \left| \left\{ (k,l) \in I_{i,j} : d\left(\overline{A}_{k,l}, \overline{A}_0\right) \geq \varepsilon \ \right\} \right| \max\left\{ M\left(\frac{N}{r}\right)^h, M\left(\frac{N}{r}\right)^H \right\}.
\end{aligned}
$$

Hence $\left(\overline{A}_{k,l}\right) \in \left[\overline{V}^2_{\lambda_2}, M, p\right]$. This completes the proof. ☐

The following corollary follows directly from Theorems 3.4 and 3.5.

Corollary 3.6. *If* $0 < h \leq \inf_{k,l} p_{k,l} \leq \sup_{k,l} p_{k,l} = H < \infty$*, then* $\overline{s}^2_{\lambda_2} \cap \overline{\ell}^2_\infty = \left[\overline{V}^2_{\lambda_2}, M, p\right] \cap \overline{\ell}^2_\infty$.

If we take $\lambda_{i,j} = ij$ and $p_{k,l} = 1$ for all $k, l \in \mathbb{N}$ in Theorems 3.4, 3.5 and Corollary 3.6, then we get

Corollary 3.7. *Let M be an Orlicz function. Then the following statements hold.*

(i) $\left[\overline{c}^2, M\right] \subset \overline{s}^2$.

(ii) $\overline{s}^2 \cap \overline{\ell}^2_\infty \subset \left[\overline{c}^2, M\right]$.

(iii) $\left[\overline{c}^2, M\right] \cap \overline{\ell}^2_\infty = \overline{s}^2 \cap \overline{\ell}^2_\infty$.

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Adiyaman University, Science and Art Faculty, Department of Mathematics, 02040, Adiyaman, Turkey

E-mail address: aesi23@hotmail.com.