ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 17, Number 1, June 2013
Available online at www.math.ut.ee/acta/

A study on hypersurface of complex space form

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Abstract. We show that quasi-umbilical, generalized quasi-umbilical, super quasi-umbilical hypersurfaces of a complex space form are quasi-Einstein, mixed generalized quasi-Einstein and mixed super quasi-Einstein manifolds, respectively. We also prove that a Bochner flat space of generalized complex space form is an Einstein manifold.

1. Introduction

The notion of a quasi-Einstein manifold was studied in [3, 4] by M. C. Chaki and R. K. Maity. In [1] A. Bhattacharyya and T. De studied the notion of a mixed generalized quasi-Einstein manifold. A. Bhattacharyya, M. Tarafdar and D. Debnath [2] have obtained results on mixed super quasi-Einstein manifolds, they proved that a super quasi-umbilical hypersurface of Euclidean space is a mixed super quasi-Einstein manifold. S. Sular and C. Özgür [6] have proved that a quasi-umbilical hypersurface of Kenmotsu space forms is a generalized quasi-Einstein hypersurface. In this paper we study quasi-umbilical, generalized quasi-umbilical, super quasi-umbilical hypersurfaces of a complex space form. It is proved that these hypersurfaces are quasi-Einstein, mixed generalized quasi-Einstein and mixed super quasi-Einstein manifolds. respectively.

Definition 1.1 ([7])**.** A Kähler manifold is an even-dimensional manifold M^{2n} with a complex structure J and a positive definite metric g which satisfy the conditions

$$J^2 = -I, \quad g(JX, JY) = g(X, Y), \quad \nabla J = 0,$$

Received February 29, 2012.
2010 *Mathematics Subject Classification.* 53C40, 53B25, 53C15.
Key words and phrases. Quasi-Einstein, generalized quasi-Einstein, mixed generalized quasi-Einstein, mixed super quasi-Einstein, quasi-umbilical, generalized quasi-umbilical and super quasi-umbilical hypersurfaces, Kähler manifold, complex space form.
http://dx.doi.org/10.12697/ACUTM.2013.17.05

where ∇ means the covariant derivation according to the Levi-Civita connection.

2. Preliminaries

Definition 2.1. A complex manifold with constant sectional curvature c is known as a complex space form. The curvature tensor of a complex space form is given by

$$\bar{R}(X,Y)Z = \frac{c}{4}\left[g(Y,Z)X - g(X,Z)Y + g(Z,JY)JX \right. \\ \left. -g(Z,JX)JY + g(X,JY)JZ\right]. \tag{2.1}$$

Definition 2.2 ([5])**.** An almost Hermitian manifold M is called a generalized complex space form $M(f_1, f_2)$ if its Riemannian curvature tensor R satisfies

$$R(X,Y)Z = f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,JZ)JY \\ -g(Y,JZ)JX + 2g(X,JY)JZ\} \tag{2.2}$$

for all $X, Y, Z \in TM$, where f_1 and f_2 are smooth functions on M.

Definition 2.3. A non-flat Kähler manifold M^{2n} ($n \geq 2$) is said to be

(i) a quasi-Einstein manifold if its Ricci tensor S is non zero and satisfies the condition

$$S(X,Y) = ag(X,Y) + bA(X)A(Y),$$

(ii) a generalized quasi-Einstein manifold if

$$S(X,Y) = ag(X,Y) + bA(X)A(Y) + cB(X)B(Y),$$

(iii) a mixed generalized quasi-Einstein manifold if

$$\begin{aligned} S(X,Y) &= ag(X,Y) + bA(X)A(Y) + cB(X)B(Y) \\ &+ d[A(X)B(Y) + B(X)A(Y)], \end{aligned}$$

(iv) a mixed super quasi-Einstein manifold if

$$\begin{aligned} S(X,Y) &= ag(X,Y) + bA(X)A(Y) + cB(X)B(Y) \\ &+ d[A(X)B(Y) + B(X)A(Y)] + eD(X,Y), \end{aligned}$$

where a, b, c, d, e are non zero scalars , A and B are two non zero 1-forms such that

$$g(X,U) = A(X) \ and \ g(X,V) = B(X) \quad \forall X \in TM,$$

U and V are unit vectors which are orthogonal, i.e.,

$$g(U,V) = 0.$$

D is a symmetric (0,2) tensor of zero trace which satisfies the condition

$$D(X,U) = 0, \ \forall X \in TM.$$

3. Hypersurfaces of the complex space form

Let M be a hypersurface of a Kähler manifold M^{2n}.

If TM^{2n} and TM denote the Lie algebra of vector fields on M^{2n} and M, respectively, and $T^{\perp}M$ is the set of all vector fields normal to M, then Gauss and Weingarten formulae are, respectively, given by

$$\bar{\nabla}_X Y = \nabla_X Y + \sigma(X, Y),$$
$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N,$$

for all $X, Y \in TM$ and $N \in T^{\perp}M$, where $\nabla^{\perp}$ denotes the connection on the normal bundle $T^{\perp}M$. σ and A_N are the second fundamental forms and shape operator of immersion of M into M^{2n} corresponding to a normal vector field N and they are related as

$$g(A_N X, Y) = g(\sigma(X, Y), N).$$

The Gauss equation is given by

$$\begin{aligned} R(X, Y, Z, W) = R(X, Y, Z, W) &- g(\sigma(X, W), \sigma(Y, Z)) \\ &+ g(\sigma(Y, W), \sigma(X, Z)), \end{aligned} \tag{3.1}$$

where Z, W are vector fields tangents to M.

We need the following definitions.

Definition 3.1 ([2])**.** A hypersurface of a Kähler manifold M^{2n} is said to be

(i) quasi-umbilical if its second fundamental tensor has the form

$$H(X, Y) = \alpha g(X, Y) + \beta\omega(X)\omega(Y), \tag{3.2}$$

(ii) generalised quasi-umbilical if its second fundamental tensor has the form

$$H(X, Y) = \alpha g(X, Y) + \beta\omega(X)\omega(Y) + \gamma\delta(X)\delta(Y), \tag{3.3}$$

(iii) super quasi-umbilical if its second fundamental tensor has the form

$$H(X, Y) = \alpha g(X, Y) + \beta\omega(X)\omega(Y) + \gamma\delta(X)\delta(Y) + \rho D(X, Y), \tag{3.4}$$

where $\alpha, \beta, \gamma, \rho$ are scalars and the vector fields corresponding to 1-forms ω and δ are unit vector fields.

Theorem 3.1. *Let $M^{2n}(c)$ be a complex space form.*

(i) *A quasi-umbilical hypersurface M of $M^{2n}(c)$ is a quasi-Einstein hypersurface.*

(ii) *A generalized quasi-umbilical hypersurface M of a $M^{2n}(c)$ is a mixed generalized quasi-Einstein manifold.*

(iii) *A super quasi-umbilical hypersurface M of $M^{2n}(c)$ is a mixed super quasi-Einstein manifold.*

Proof. (i) From equation (2.1), we have

$$\begin{aligned}\bar{R}(X,Y,Z,W) = \frac{c}{4}[&g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\\ &+ g(Z,JY)g(JX,W) - g(Z,JX)g(JY,W)\\ &+ g(X,JY)g(JZ,W)]\end{aligned} \tag{3.5}$$

for all tangents X, Y, Z, W to M.

Let N be the unit normal vector field of M in $M^{2n}(c)$. Using $\sigma(X,Z) = H(X,Z)N$ in equation (3.1), we have

$$\begin{aligned}\bar{R}(X,Y,Z,W) = R(X,Y,Z,W) &- H(X,W)H(Y,Z)\\ &+ H(Y,W)H(X,Z).\end{aligned} \tag{3.6}$$

For a quasi-umbilical hypersurface we know that

$$H(X,Z) = ag(X,Z) + b\omega(X)\omega(Z). \tag{3.7}$$

Putting (3.7) in (3.6) and using (3.5), we have

$$\begin{aligned}&\frac{c}{4}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W) + g(Z,JY)g(JX,W)\\ &\qquad - g(Z,JX)g(JY,W) + g(X,JY)g(JZ,W)]\\ &= R(X,Y,Z,W) + a^2[g(X,Z)g(Y,W) - g(Y,Z)g(X,W)]\\ &\quad + ab[g(X,Z)\omega(Y)\omega(W) + g(Y,W)\omega(X)\omega(Z)\\ &\quad - g(Y,Z)\omega(X)\omega(W) - g(X,W)\omega(Y)\omega(Z)].\end{aligned} \tag{3.8}$$

Putting $X = W = e_i$ and taking the sum over i, $(1 \le i \le 2n)$ in equation (3.8), where $\{e_i\}$ is an orthonormal basis for the given space form, we get

$$S(Y,Z) = [\frac{c}{4}(2n+1) + a^2(2n-1) + ab(2n-2)]g(Y,Z) + [2nab]\omega(Y)\omega(Z).$$

Hence M is a quasi-Einstein hypersurface.

Similar proofs of statements (ii) and (iii) are obtained by using equations (3.3), (3.4) and (3.5) in equation (3.6) and putting $X = W = e_i$. □

4. Generalized complex space form with Bochner curvature tensor

Taking $X = e_i$ $(1 \le i \le 2n)$, $Y = X$ and $Z = Y$ in the equation (2.2), we obtain for the curvature

$$\begin{aligned}R(e_i,X)Y = F_1\{g(X,Y)e_i - g(e_i,Y)X\} + F_2\{&g(e_i,JY)JX\\ &- g(X,JY)Je_i + 2g(e_i,JX)JY\}.\end{aligned} \tag{4.1}$$

Hence the Ricci tensor can be written in the form

$$S(X,Y) = \sum_{i=1}^{2n} R(e_i,X,Y,e_i) = \sum_{i=1}^{2n} g(R(e_i,X)Y,e_i),$$

and making use of (4.1) we can express it as

$$\begin{aligned} S(X,Y) &= F_1\{g(X,Y)2n - g(X,Y)\} + F_2\{g(X,Y) + 2g(X,Y)\} \\ &= \{(2n-1)F_1 + 3F_2\}g(X,Y). \end{aligned} \tag{4.2}$$

For a generalized complex space form the Bochner curvature tensor is given by

$$\begin{aligned} B(X,Y)Z = R(X,Y)Z &+ \frac{1}{2n+4}\,[g(X,Z)QY - S(Y,Z)X - g(Y,Z)QX \\ &+S(X,Z)Y + g(JX,Z)QJY - S(JY,Z)JX - g(JY,Z)QJX \\ &+S(JX,Z)JY + 2S(JX,Y)JZ + 2g(JX,Y)QJZ] \\ &- \frac{D+2n}{2n+4}\,[g(JX,Z)JY - g(JY,Z)JX + 2g(JX,Y)JZ] \\ &- \frac{D-4}{2n+4}[g(X,Z)Y - g(Y,Z)X] \end{aligned}$$

with $D = \frac{r+2n}{2n+2}$. R, S, Q and r are the Riemannian curvature tensor, the Ricci tensor, the Ricci operator and the scalar curvature of the manifold, respectively.

Suppose $B(X,Y)Z = 0$. Then the curvature (1,3)-tensor is

$$\begin{aligned} R(X,Y)Z = &\frac{-1}{2n+4}\,[g(X,Z)QY - S(Y,Z)X - g(Y,Z)QX \\ &+S(X,Z)Y + g(JX,Z)QJY - S(JY,Z)JX - g(JY,Z)QJX \\ &+S(JX,Z)JY + 2S(JX,Y)JZ + 2g(JX,Y)QJZ] \\ &+ \frac{D+2n}{2n+4}[g(JX,Z)JY - g(JY,Z)JX + 2g(JX,Y)JZ] \\ &+ \frac{D-4}{2n+4}[g(X,Z)Y - g(Y,Z)X]. \end{aligned}$$

It follows from the previous formula that the curvature (0,4)-tensor can be written as

$$\begin{aligned} R(X,Y,Z,W) &= g(R(X,Y)Z,W) \\ &= -1/(2n+4)\,[g(X,Z)g(QY,W) - S(Y,Z)g(X,W) \\ &\quad -g(Y,Z)g(QX,W) + S(X,Z)g(Y,W) \\ &\quad +g(JX,Z)g(QJY,W) - S(JY,Z)g(JX,W) \\ &\quad -g(JY,Z)g(QJX,W) + S(JX,Z)g(JY,W) \\ &\quad +2S(JX,Y)g(JZ,W) + 2g(JX,Y)g(QJZ,W)] \\ &\quad + \frac{(\frac{r+2n}{2n+2}) + 2n}{2n+4}\,[g(JX,Z)g(JY,W) \\ &\quad -g(JY,Z)g(JX,W) + 2g(JX,Y)g(JZ,W)] \end{aligned}$$

$$+\frac{(\frac{r+2n}{2n+2})-4}{2n+4}[g(X,Z)g(Y,W)-g(Y,Z)g(X,W)].$$

Putting $X=W=e_i$, where $\{e_i\}$, $i=1,2,\dots,2n$, is a local orthonormal basis for vector fields in a generalized complex space form $N(F_1,F_2)$, by virtue of $S(X,Y)=\sum_{i=1}^{2n} g(R(e_i,X)Y,e_i)$, we get

$$S(Y,Z)=\{(2n-1)F_1+3F_2-2\}g(Y,Z). \tag{4.3}$$

Hence $N(F_1,F_2)$ is the Einstein manifold.

From (4.3), we have

$$QY=\{(2n-1)F_1+3F_2-2\}Y \tag{4.4}$$

and

$$r=2n\{(2n-1)F_1+3F_2-2\}. \tag{4.5}$$

Thus we can state the following theorem.

Theorem 4.1. *A Bochner flat space of generalized complex space form $N(F_1,F_2)$ is an Einstein manifold. The Ricci tensor S, Ricci operator Q and scalar curvature r are given in equations* (4.3)–(4.5).

Acknowledgement. The authors are thankful to the referee for his/her valuable suggestions in the improvement of the paper.

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