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Slight extensions of some theorems on the rate of pointwise approximation of functions from some subclasses of L^p

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Abstract. In this paper we prove some results on the rate of pointwise approximation of functions by means of some matrix transformations related to the partial sums of a Fourier series, removing the assumptions that entries of the considered matrix belong to the classes $RBVS$ or $HBVS$. In fact, with weaker assumptions, our results give better degrees than those obtained previously. Moreover, some results that have been obtained earlier follow from our results as special cases. Finally, we present some theorems of such type involving the so-called $\gamma RBVS$ or $\gamma HBVS$ classes of numerical sequences.

1. Introduction and preliminaries

Let $L^p\,(1<p<+\infty)$ be the class of all 2π-periodic real-valued functions integrable in the Lebesgue sense with p-th power over $T:=[-\pi,\pi]$ with the norm

$$\|f\|=\|f\|_{L^p}=\left(\int_T |f(t)|^p dt\right)^{1/p}.$$

Consider its trigonometric Fourier series

$$Sf(x):=\frac{a_0}{2}+\sum_{\nu=1}^{\infty}(a_\nu\cos\nu x+b_\nu\sin\nu x)$$

and the conjugate one

$$\widetilde{S}f(x):=\sum_{\nu=1}^{\infty}(b_\nu\cos\nu x-a_\nu\sin\nu x)$$

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with their partial sums $S_k f$ and $\widetilde{S}_k f$, respectively. It is a well-known fact that if $f \in L$, then

$$\tilde{f}(x) := -\frac{1}{2\pi}\int_0^\pi \psi_x(t)\cot\frac{t}{2}dt = \lim_{\varepsilon\to 0}\tilde{f}(x,\varepsilon),$$

where

$$\tilde{f}(x,\varepsilon) := -\frac{1}{2\pi}\int_\varepsilon^\pi \psi_x(t)\cot\frac{t}{2}dt$$

with

$$\psi_x(t) := f(x+t) - f(x-t),$$

exists for almost all x (see, e.g., [10], Theorem (3.1), Chapter IV).

Let $A := (a_{n,k})$ be a lower triangular infinite matrix of real numbers such that

$$a_{n,k} \geq 0, \quad \sum_{k=0}^{n} a_{n,k} = 1, \quad (k, n = 0, 1, \dots),$$

and let the A-transformations of $\{S_k f\}$ and $\{\widetilde{S}_k f\}$ be given by

$$T_{n,A}(f;x) := \sum_{k=0}^{n} a_{n,k} S_k(f;x) \quad (n = 0, 1, \dots)$$

and

$$\widetilde{T}_{n,A}(f;x) := \sum_{k=0}^{n} a_{n,k} \widetilde{S}_k(f;x) \quad (n = 0, 1, \dots),$$

respectively.

The estimates of the deviation $\widetilde{T}_{n,A}(f) - \tilde{f}$ were obtained by K. Qureshi [7, 8] for monotonic sequences $\{a_{n,k}\}$. This deviation was estimated in the norm of L^p by S. Lal and H. Nigam [2], while later on their result was generalized by M. L. Mittal, B. E. Rhoades, and V. N. Mishra [6]. Recently W. Łenski and B. Szal [5] considered the same deviation and additionally the deviations $\widetilde{T}_{n,A} f(\cdot) - \tilde{f}\left(\cdot, \frac{\pi}{n+1}\right)$ and $T_{n,A}(f) - f$ in the case when the sequence $\{a_{n,k}\}$ is of Rest Bounded Variation or of Head Bounded Variation. Also some results of this type are obtained very recently in [1].

A sequence $\mathbf{c} := \{c_n\}$ of nonnegative numbers tending to zero is called of Rest Bounded Variation, or briefly $\mathbf{c} \in RBVS$, if it has the property

$$\sum_{n=m}^{\infty} |c_n - c_{n+1}| \leq K(\mathbf{c}) c_m$$

for all natural numbers m, where $K(\mathbf{c})$ is a constant depending only on $\mathbf{c}$.

A sequence $\mathbf{c} := \{c_n\}$ of nonnegative numbers will be called of Head Bounded Variation, or briefly $\mathbf{c} \in HBVS$, if it has the property

$$\sum_{n=0}^{m-1} |c_n - c_{n+1}| \leq K(\mathbf{c})c_m$$

for all natural numbers m, or only for all $m \leq N$ if the sequence $\mathbf{c}$ has only finitely many nonzero terms, and the last nonzero term is c_N.

As a measure of approximation W. Łenski and B. Szal used the generalized moduli of continuity of f in the space L^p defined for $\beta \geq 0$ by

$$\widetilde{\omega}_\beta f(\delta)_{L^p} := \sup_{0\leq|t|\leq\delta} \left\{ \left|\sin\frac{t}{2}\right|^{\beta p} \int_0^\pi |\psi_x(t)|^p dx \right\}^{1/p},$$

$$\omega_\beta f(\delta)_{L^p} := \sup_{0\leq|t|\leq\delta} \left\{ \left|\sin\frac{t}{2}\right|^{\beta p} \int_0^\pi |\varphi_x(t)|^p dx \right\}^{1/p},$$

where

$$\varphi_x(t) := f(x+t) + f(x-t) - 2f(x).$$

Also they defined two subclasses of L^p class as follows.

Let ω be a function of modulus of continuity type on the interval $[0, 2\pi]$, i.e. a nondecreasing continuous function having the following properties: $\omega(0) = 0$, $\omega(\delta_1 + \delta_2) \leq \omega(\delta_1) + \omega(\delta_2)$ for any $0 \leq \delta_1 \leq \delta_2 \leq \delta_1 + \delta_2 \leq 2\pi$.

Then the above-mentioned classes are

$$L^p(\widetilde{\omega})_\beta = \left\{ f \in L^p : \widetilde{\omega}_\beta f(\delta)_{L^p} \leq \widetilde{\omega}(\delta) \right\},$$

$$L^p(\omega)_\beta = \left\{ f \in L^p : \omega_\beta f(\delta)_{L^p} \leq \omega(\delta) \right\},$$

where ω and $\widetilde{\omega}$ are some functions of modulus of continuity type.

Using the notation

$$a_n = \begin{cases} a_{n,0} & \text{when} \quad \{a_{n,k}\} \in RBVS, \\ a_{n,n} & \text{when} \quad \{a_{n,k}\} \in HBVS, \end{cases}$$

W. Łenski and B. Szal [5] have proved the following three theorems.

Theorem 1. *Let $f \in L^p(\widetilde{\omega})_\beta$ with $\beta < 1 - \frac{1}{p}$, $\{a_{n,k}\} \in HBVS$ (or $\{a_{n,k}\} \in RBVS$) and let $\widetilde{\omega}$ be such that*

$$\left(\int_0^{\pi/(n+1)} \left(\frac{t|\psi_x(t)|}{\widetilde{\omega}(t)} \right)^p \sin^{\beta p}\frac{t}{2} dt \right)^{1/p} = O_x\left((n+1)^{-1}\right) \tag{1}$$

and

$$\left(\int_{\pi/(n+1)}^{\pi} \left(\frac{t^{-\gamma}|\psi_x(t)|}{\widetilde{\omega}(t)} \right)^p \sin^{\beta p}\frac{t}{2} dt \right)^{1/p} = O_x\left((n+1)^{\gamma}\right) \tag{2}$$

hold with $0 < \gamma < \beta + \frac{1}{p}$*. Then*

$$\left|\widetilde{T}_{n,A}f(x) - \widetilde{f}\left(x, \frac{\pi}{n+1}\right)\right| = O_x\left((n+1)^{\beta+\frac{1}{p}+1} a_n \widetilde{\omega}\left(\frac{\pi}{n+1}\right)\right)$$

for considered x.

Theorem 2. *Let* $f \in L^p(\widetilde{\omega})_\beta$ *with* $\beta < 1 - \frac{1}{p}$, $\{a_{n,k}\} \in HBVS$ *(or* $\{a_{n,k}\} \in RBVS$*) and let* $\widetilde{\omega}$ *satisfy* (2) *with* $0 < \gamma < \beta + \frac{1}{p}$,

$$\left(\int_0^{\pi/(n+1)} \left(\frac{|\psi_x(t)|}{\widetilde{\omega}(t)}\right)^p \sin^{\beta p} \frac{t}{2} dt\right)^{1/p} = O_x\left((n+1)^{-1/p}\right) \tag{3}$$

and

$$\left(\int_0^{\pi/(n+1)} \left(\frac{\widetilde{\omega}(t)}{t \sin^\beta \frac{t}{2}}\right)^q dt\right)^{1/q} = O_x\left((n+1)^{\beta+1/p}\, \widetilde{\omega}\left(\frac{\pi}{n+1}\right)\right), \tag{4}$$

where $q = p/(p-1)$*. Then*

$$\left|\widetilde{T}_{n,A}f(x) - \widetilde{f}(x)\right| = O_x\left((n+1)^{\beta+\frac{1}{p}+1} a_n \widetilde{\omega}\left(\frac{\pi}{n+1}\right)\right)$$

for considered x *such that* $\widetilde{f}(x)$ *exists.*

Theorem 3. *Let* $f \in L^p(\omega)_\beta$ *with* $\beta < 1 - \frac{1}{p}$, $\{a_{n,k}\} \in HBVS$ *(or* $\{a_{n,k}\} \in RBVS$*) and let* $\widetilde{\omega}$ *satisfy*

$$\left(\int_{\pi/(n+1)}^{\pi} \left(\frac{t^{-\gamma}|\varphi_x(t)|}{\omega(t)}\right)^p \sin^{\beta p} \frac{t}{2} dt\right)^{1/p} = O_x\left((n+1)^\gamma\right) \tag{5}$$

with $0 < \gamma < \beta + \frac{1}{p}$,

$$\left(\int_0^{\pi/(n+1)} \left(\frac{|\varphi_x(t)|}{\omega(t)}\right)^p \sin^{\beta p} \frac{t}{2} dt\right)^{1/p} = O_x\left((n+1)^{-1/p}\right), \tag{6}$$

and

$$\left(\int_0^{\pi/(n+1)} \left(\frac{\omega(t)}{t \sin^\beta \frac{t}{2}}\right)^q dt\right)^{1/q} = O_x\left((n+1)^{\beta+1/p}\, \omega\left(\frac{\pi}{n+1}\right)\right), \tag{7}$$

where $q = p/(p-1)$*. Then*

$$|T_{n,A}f(x) - f(x)| = O_x\left((n+1)^{\beta+\frac{1}{p}+1} a_n \omega\left(\frac{\pi}{n+1}\right)\right)$$

for considered x.

The aim of the present paper is to prove the counterparts of the above results (Theorems 1–3) without assuming that $\{a_{n,k}\} \in HBVS$ or $\{a_{n,k}\} \in RBVS$.

Throughout this paper we write $u = O(v)$ if there exists a positive constant C such that $u \leq Cv$.

2. Helpful lemmas

To prove the main results we need some auxiliary statements. Also we shall use the following equalities from [5]:

$$T_{n,A}f(x) - f(x) = \frac{1}{\pi}\int_0^{\pi} \varphi_x(t) \sum_{k=0}^{n} a_{n,k} D_k(t)dt,$$

$$\widetilde{T}_{n,A}f(x) - \widetilde{f}(x) = \frac{1}{\pi}\int_0^{\pi} \psi_x(t) \sum_{k=0}^{n} a_{n,k} \widetilde{D}_k^{\circ}(t)dt,$$

and

$$\begin{aligned}\widetilde{T}_{n,A}f(x) - \widetilde{f}\left(x, \frac{\pi}{n+1}\right) &= -\frac{1}{\pi}\int_0^{\pi/(n+1)} \psi_x(t) \sum_{k=0}^{n} a_{n,k} \widetilde{D}_k(t)dt \\ &\quad + \frac{1}{\pi}\int_{\pi/(n+1)}^{\pi} \psi_x(t) \sum_{k=0}^{n} a_{n,k} \widetilde{D}_k^{\circ}(t)dt,\end{aligned}$$

where

$$\widetilde{D}_k^{\circ}(t) = \frac{\cos\frac{(2k+1)t}{2}}{2\sin\frac{t}{2}},$$

$$D_k(t) = \frac{1}{2} + \sum_{\nu=1}^{k} \cos \nu x = \frac{\sin\frac{(2k+1)t}{2}}{2\sin\frac{t}{2}},$$

and

$$\widetilde{D}_k(t) = \sum_{\nu=1}^{k} \sin \nu x = \frac{\cos\frac{t}{2} - \cos\frac{(2k+1)t}{2}}{2\sin\frac{t}{2}}.$$

Lemma 4 ([10])**.** *If $0 < |t| \leq \pi/2$, then*

$$\left|\widetilde{D}_k^{\circ}(t)\right| \leq \frac{\pi}{2|t|} \quad \textit{and} \quad \left|\widetilde{D}_k(t)\right| \leq \frac{\pi}{|t|},$$

and for any real number t we have

$$\left|\widetilde{D}_k(t)\right| \leq \frac{1}{2}k(k+1)|t| \quad \textit{and} \quad \left|\widetilde{D}_k(t)\right| \leq k+1.$$

Lemma 5 ([10]). *If* $0 < |t| \le \pi/2$*, then*

$$\left|D_k(t)\right| \le \frac{\pi}{|t|},$$

and for any real number t *we have*

$$\left|D_k(t)\right| \le k+1.$$

Lemma 6. *For any lower triangular infinite matrix* $(a_{n,k})$, $k,n=0,1,2,\dots,$ *of nonnegative numbers, it holds uniformly in* $0 < t \le \pi$*, that*

$$\left|\sum_{k=0}^{n} a_{n,k}\widetilde{D}_k^{\circ}(t)\right| \le O\left(\frac{1}{t^2}\left(\sum_{k=0}^{n}|\triangle a_{n,k}|\right)\right).$$

Proof. Using the following equality (see [5], page 19)

$$\begin{aligned}
&\sum_{k=m}^{n} a_{n,k}\cos\frac{(2k+1)t}{2}\sin\frac{t}{2} \\
&\quad = a_{n,m}\cos\frac{(2m+1)t}{2}\sin\frac{t}{2} \\
&\quad + \sum_{k=m+1}^{n-1}(a_{n,k}-a_{n,k+1})\sin\frac{(k-m-1)t}{2}\cos\frac{(k+m+1)t}{2} \\
&\quad + a_{n,n}\sin\frac{(n-m-1)t}{2}\cos\frac{(n+m+1)t}{2},
\end{aligned}$$

we obviously obtain

$$\left|\sum_{k=m}^{n} a_{n,k}\cos\frac{(2k+1)t}{2}\sin\frac{t}{2}\right| \le a_{n,m} + \sum_{k=m+1}^{n-1}|a_{n,k}-a_{n,k+1}| + a_{n,n}.$$

But since (a_{nk}) is a lower triangular matrix, that is, $a_{n,k} = 0$ for $k > n$, then

$$a_{nm} \le \sum_{k=m}^{n}|\triangle a_{nk}|$$

holds for $m = 0, 1, 2, \dots, n$.

Therefore from this and above we have

$$\begin{aligned}
\left|\sum_{k=0}^{n} a_{n,k}\widetilde{D}_k^{\circ}(t)\right| &= O\left(\frac{1}{t^2}\left(a_{n,0} + \sum_{k=1}^{n-1}|a_{n,k}-a_{n,k+1}| + a_{n,n}\right)\right) \\
&= O\left(\frac{1}{t^2}\left(\sum_{k=0}^{n}|\triangle a_{n,k}|\right)\right),
\end{aligned}$$

which completes the proof of the lemma. □

Lemma 7. *For any lower triangular infinite matrix* $(a_{n,k})$, $k,n=0,1,2,\dots$, *of nonnegative numbers, it holds uniformly in* $0 < t \le \pi$, *that*

$$\left|\sum_{k=0}^{n} a_{n,k} D_k(t)\right| \le O\left(\frac{1}{t^2}\left(\sum_{k=0}^{n} |\triangle a_{n,k}|\right)\right).$$

Proof. Analogously, we obtain

$$\left|\sum_{k=m}^{n} a_{n,k} \sin\frac{(2k+1)t}{2}\sin\frac{t}{2}\right| \le a_{n,m} + \sum_{k=m+1}^{n-1} |a_{n,k} - a_{n,k+1}| + a_{n,n}.$$

Using again the fact that (a_{nk}) is a lower triangular matrix, that is, $a_{n,k} = 0$ for $k > n$, we obtain that

$$a_{nm} \le \sum_{k=m}^{n} |\triangle a_{nk}|$$

holds for $m = 0, 1, 2, \dots, n$, and it follows that

$$\begin{aligned}\left|\sum_{k=0}^{n} a_{n,k} D_k(t)\right| &= O\left(\frac{1}{t^2}\left(a_{n,0} + \sum_{k=1}^{n-1} |a_{n,k} - a_{n,k+1}| + a_{n,n}\right)\right)\\ &= O\left(\frac{1}{t^2}\left(\sum_{k=0}^{n} |\triangle a_{n,k}|\right)\right).\end{aligned}$$

The proof of the lemma is finished. □

3. Main results

We establish the following.

Theorem 8. *Let* $f \in L^p(\widetilde{\omega})_\beta$ *with* $\beta < 1 - \frac{1}{p}$, *and let* $\widetilde{\omega}$ *satisfy* (1) *and* (2) *with* $0 < \gamma < \beta + \frac{1}{p}$. *Then*

$$\left|\widetilde{T}_{n,A} f(x) - \widetilde{f}\left(x, \frac{\pi}{n+1}\right)\right| = O_x\left((n+1)^{\beta+\frac{1}{p}+1} \sum_{k=0}^{n} |\triangle a_{n,k}| \widetilde{\omega}\left(\frac{\pi}{n+1}\right)\right)$$

for considered x.

Proof. We shall follow the idea of Łenski and Szal's [5]. Starting from the equality

$$\begin{aligned}\widetilde{T}_{n,A} f(x) - \widetilde{f}\left(x, \frac{\pi}{n+1}\right) &= -\frac{1}{\pi}\int_0^{\pi/(n+1)} \psi_x(t) \sum_{k=0}^{n} a_{n,k} \widetilde{D}_k(t) dt\\ &\quad + \frac{1}{\pi}\int_{\pi/(n+1)}^{\pi} \psi_x(t) \sum_{k=0}^{n} a_{n,k} \widetilde{D}_k^{\circ}(t) dt := R_1 + R_2,\end{aligned}$$

we clearly have

$$\left|\widetilde{T}_{n,A}f(x)-\widetilde{f}\left(x,\frac{\pi}{n+1}\right)\right|\le |R_1|+|R_2|.$$

To estimate $|R_1|$ we use Hölder's inequality with $p+q=pq$, Lemma 4, and (1):

$$\begin{aligned}
|R_1| &\le \frac{(n+1)^2}{2\pi}\int_0^{\frac{\pi}{n+1}} t|\psi_x(t)|dt\\
&\le \frac{(n+1)^2}{2\pi}\left(\int_0^{\frac{\pi}{n+1}}\left(\frac{t|\psi_x(t)|}{\widetilde{\omega}(t)}\right)^p \sin^{\beta p}\frac{t}{2}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{\pi}{n+1}}\left(\frac{\widetilde{\omega}(t)}{\sin^{\beta}\frac{t}{2}}\right)^q dt\right)^{\frac{1}{q}}\\
&= O\left((n+1)\left(\int_0^{\frac{\pi}{n+1}}\left(\frac{\widetilde{\omega}(t)}{t^{\beta}}\right)^q dt\right)^{\frac{1}{q}}\right)\\
&= O_x\left((n+1)^{\beta+\frac{1}{p}}\widetilde{\omega}\left(\frac{\pi}{n+1}\right)\right),
\end{aligned}$$

for $\beta<1-\frac{1}{p}$.

Now by Hölder's inequality with $p+q=pq$, Lemma 6 and (2), for $|R_2|$ we have

$$\begin{aligned}
|R_2| &\le \frac{1}{\pi}\int_{\frac{\pi}{n+1}}^{\pi}|\psi_x(t)|\left|\sum_{k=0}^{n}a_{n,k}\widetilde{D}_k^{\circ}(t)\right|dt\\
&= O\left(\sum_{k=0}^{n}|\triangle a_{n,k}|\int_{\frac{\pi}{n+1}}^{\pi}\frac{|\psi_x(t)|}{t^2}dt\right)\\
&= O\Bigg(\sum_{k=0}^{n}|\triangle a_{n,k}|\left(\int_{\frac{\pi}{n+1}}^{\pi}\left(\frac{t^{-\gamma}|\psi_x(t)|}{\widetilde{\omega}(t)}\right)^p\sin^{\beta p}\frac{t}{2}dt\right)^{\frac{1}{p}}\\
&\quad\times\left(\int_{\frac{\pi}{n+1}}^{\pi}\left(\frac{\widetilde{\omega}(t)}{t^{2-\gamma}\sin^{\beta}\frac{t}{2}}\right)^q dt\right)^{\frac{1}{q}}\Bigg)\\
&= O_x\left((n+1)^{\gamma}\sum_{k=0}^{n}|\triangle a_{n,k}|\left(\int_{\frac{\pi}{n+1}}^{\pi}\left(\frac{\widetilde{\omega}(t)}{t^{2-\gamma+\beta}}\right)^q dt\right)^{\frac{1}{q}}\right)\\
&= O_x\left((n+1)^{\gamma+1}\sum_{k=0}^{n}|\triangle a_{n,k}|\widetilde{\omega}\left(\frac{\pi}{n+1}\right)\left(\int_{\frac{\pi}{n+1}}^{\pi}\left(t^{\gamma-\beta-1}\right)^q dt\right)^{\frac{1}{q}}\right)\\
&= O_x\left((n+1)^{\beta+\frac{1}{p}+1}\sum_{k=0}^{n}|\triangle a_{n,k}|\widetilde{\omega}\left(\frac{\pi}{n+1}\right)\right),
\end{aligned}$$

for $0 < \gamma < \beta + \frac{1}{p}$.

We complete the proof by combining the above estimates. □

Theorem 9. *Let $f \in L^p(\widetilde{\omega})_\beta$ with $\beta < 1 - \frac{1}{p}$, and let $\widetilde{\omega}$ satisfy* (2), (3), *and* (4), *with $0 < \gamma < \beta + \frac{1}{p}$ and $q = p/(p-1)$. Then*

$$\left|\widetilde{T}_{n,A}f(x) - \widetilde{f}(x)\right| = O_x\left((n+1)^{\beta+\frac{1}{p}+1}\sum_{k=0}^{n}|\triangle a_{n,k}|\widetilde{\omega}\left(\frac{\pi}{n+1}\right)\right)$$

for considered x such that $\widetilde{f}(x)$ exists.

Proof. From the equality

$$\begin{aligned}\widetilde{T}_{n,A}f(x) - \widetilde{f}\,(x) &= \frac{1}{\pi}\int_0^{\frac{\pi}{n+1}}\psi_x(t)\sum_{k=0}^{n}a_{n,k}\widetilde{D}_k^{\circ}(t)dt\\ &+\frac{1}{\pi}\int_{\frac{\pi}{n+1}}^{\pi}\psi_x(t)\sum_{k=0}^{n}a_{n,k}\widetilde{D}_k^{\circ}(t)dt := R_1^{\circ} + R_2,\end{aligned}$$

we have

$$|\widetilde{T}_{n,A}f(x) - \widetilde{f}\,(x)\,| \le |R_1^{\circ}| + |R_2|.$$

For $|R_1^{\circ}|$ we use the estimation (see [5, p. 23])

$$|R_1^{\circ}| = O_x\left((n+1)^{\beta}\widetilde{\omega}\left(\frac{\pi}{n+1}\right)\right),$$

while for $|R_2|$ we use the one from Theorem 8

$$|R_2| = O_x\left((n+1)^{\beta+\frac{1}{p}+1}\sum_{k=0}^{n}|\triangle a_{n,k}|\widetilde{\omega}\left(\frac{\pi}{n+1}\right)\right).$$

Estimates made for $|R_1^{\circ}|$ and $|R_2|$ completely verify the theorem. □

Theorem 10. *Let $f \in L^p(\omega)_\beta$ with $\beta < 1 - \frac{1}{p}$, and let $\widetilde{\omega}$ satisfy* (5), (6), *and* (7), *where $q = p/(p-1)$. Then*

$$|T_{n,A}f(x) - f(x)| = O_x\left((n+1)^{\beta+\frac{1}{p}+1}\sum_{k=0}^{n}|\triangle a_{n,k}|\omega\left(\frac{\pi}{n+1}\right)\right)$$

for considered x.

Proof. Since

$$\begin{aligned}T_{n,A}f(x) - f\,(x) &= \frac{1}{\pi}\int_0^{\frac{\pi}{n+1}}\varphi_x(t)\sum_{k=0}^{n}a_{n,k}D_k(t)dt\\ &+\frac{1}{\pi}\int_{\frac{\pi}{n+1}}^{\pi}\varphi_x(t)\sum_{k=0}^{n}a_{n,k}D_k(t)dt := R_1^{\circ} + R_2^{\circ},\end{aligned}$$

we have

$$|T_{n,A}f(x) - f(x)| \leq |R_1^\circ| + |R_2^\circ|.$$

On one hand, in the same way as in the proof of Theorem 3 using Lemmas 5, (6), and (7), we obtain

$$|R_1^\circ| = O_x\left((n+1)^\beta \omega\left(\frac{\pi}{n+1}\right)\right).$$

On the other hand, by Hölder's inequality with $p+q=pq$, Lemma 7, and (5) we have

$$\begin{aligned}
|R_2^\circ| &\leq \frac{1}{\pi}\int_{\frac{\pi}{n+1}}^{\pi} |\varphi_x(t)| \left|\sum_{k=0}^{n} a_{n,k}D_k(t)\right| dt \\
&= O\left(\sum_{k=0}^{n} |\triangle a_{n,k}| \int_{\frac{\pi}{n+1}}^{\pi} \frac{|\varphi_x(t)|}{t^2} dt\right) \\
&= O\Bigg(\sum_{k=0}^{n} |\triangle a_{n,k}| \left(\int_{\frac{\pi}{n+1}}^{\pi} \left(\frac{t^{-\gamma}|\varphi_x(t)|}{\omega(t)}\right)^p \sin^{\beta p}\frac{t}{2} dt\right)^{\frac{1}{p}} \\
&\quad \times \left(\int_{\frac{\pi}{n+1}}^{\pi} \left(\frac{\omega(t)}{t^{2-\gamma}\sin^\beta \frac{t}{2}}\right)^q dt\right)^{\frac{1}{q}}\Bigg) \\
&= O_x\left((n+1)^\gamma \sum_{k=0}^{n} |\triangle a_{n,k}| \left(\int_{\frac{\pi}{n+1}}^{\pi} \left(\frac{\omega(t)}{t^{2-\gamma+\beta}}\right)^q dt\right)^{\frac{1}{q}}\right) \\
&= O_x\left((n+1)^{\gamma+1} \sum_{k=0}^{n} |\triangle a_{n,k}| \omega\left(\frac{\pi}{n+1}\right) \left(\int_{\frac{\pi}{n+1}}^{\pi} \left(t^{\gamma-\beta-1}\right)^q dt\right)^{\frac{1}{q}}\right) \\
&= O_x\left((n+1)^{\beta+\frac{1}{p}+1} \sum_{k=0}^{n} |\triangle a_{n,k}| \omega\left(\frac{\pi}{n+1}\right)\right),
\end{aligned}$$

for $0 < \gamma < \beta + \frac{1}{p}$.

This completes the proof. □

4. Concluding remarks

Remark 1. Note that if $\mathbf{c} := \{a_{n,k}\} \in RBVS$, then the inequality

$$a_{n,n} - a_{n,0} \leq |a_{n,0} - a_{n,n}| \leq \sum_{k=0}^{n-1} |\triangle a_{n,k}|$$

implies

$$\begin{aligned}\sum_{k=0}^{n} |\triangle a_{n,k}| &= \sum_{k=0}^{n-1} |\triangle a_{n,k}| + a_{n,n} \\ &\leq 2\sum_{k=0}^{n-1} |\triangle a_{n,k}| + a_{n,0} \leq 2\left(K\left(\mathbf{c}\right)+1\right) a_{n,0}.\end{aligned}$$

Also if $\mathbf{c} : \{a_{n,k}\} \in HBVS$, then

$$\sum_{k=0}^{n} |\triangle a_{n,k}| = \sum_{k=0}^{n-1} |\triangle a_{n,k}| + a_{n,n} \leq 2\left(K\left(\mathbf{c}\right)+1\right) a_{n,n}.$$

Therefore, Theorems $1-3$ immediately follow from Theorems $8-10$.

Remark 2. L. Leindler [4] has extended the definition of $RBVS$ to the so called $\gamma RBVS$. Indeed: For a fixed n, let $\gamma_n := \{\gamma_{n,k}\}$ be a nonnegative sequence. If a null-sequence $\theta_n := \{a_{n,k}\}$ of real numbers has the property

$$\sum_{k=m}^{\infty} |\triangle a_{n,k}| \leq K(\theta_n)\gamma_{n,m}$$

for every positive integer m, then we call the sequence $\theta_n := \{a_{n,k}\}$ a $\gamma RBVS$ and denote $\theta_n \in \gamma RBVS$.

Similarly, the authors of [9] introduced a new kind of sequences as follows.

For a fixed n, let $\gamma_n := \{\gamma_{n,k}\}$ be a nonnegative sequence. If a null-sequence $\theta_n := \{a_{n,k}\}$ of real numbers has the property

$$\sum_{k=0}^{m-1} |\triangle a_{n,k}| \leq K(\theta_n)\gamma_{n,m}$$

for every positive integer m, then we call the sequence $\theta_n := \{a_{n,k}\}$ a $\gamma HBVS$ and denote $\theta_n \in \gamma HBVS$.

By an argument similar to Theorems $8-10$ and using the notation

$$\gamma_n = \begin{cases} \gamma_{n,0} & \text{when} \quad \{a_{n,k}\} \in \gamma RBVS, \\ \gamma_{n,n} & \text{when} \quad \{a_{n,k}\} \in \gamma HBVS, \end{cases}$$

we have the following generalizations of Theorems $1-3$.

Theorem 11. *Let $f \in L^p(\widetilde{\omega})_\beta$ with $\beta < 1 - \frac{1}{p}$, $\{a_{n,k}\} \in \gamma HBVS$ (or $\{a_{n,k}\} \in \gamma RBVS$) and let $\widetilde{\omega}$ satisfy* (1) *and* (2) *with* $0 < \gamma < \beta + \frac{1}{p}$. *Then*

$$\left|\widetilde{T}_{n,A}f(x) - \widetilde{f}\left(x, \frac{\pi}{n+1}\right)\right| = O_x\left((n+1)^{\beta+\frac{1}{p}+1}\gamma_n\widetilde{\omega}\left(\frac{\pi}{n+1}\right)\right)$$

for considered x.

Theorem 12. *Let $f \in L^p(\widetilde{\omega})_\beta$ with $\beta < 1 - \frac{1}{p}$, $\{a_{n,k}\} \in \gamma HBVS$ (or $\{a_{n,k}\} \in \gamma RBVS$) and let $\widetilde{\omega}$ satisfy* (2), (3), *and* (4), *with* $0 < \gamma < \beta + \frac{1}{p}$ *and $q = p/(p-1)$. Then*

$$\left|\widetilde{T}_{n,A}f(x) - \widetilde{f}(x)\right| = O_x\left((n+1)^{\beta+\frac{1}{p}+1}\gamma_n\widetilde{\omega}\left(\frac{\pi}{n+1}\right)\right)$$

for considered x such that $\widetilde{f}(x)$ exists.

Theorem 13. *Let $f \in L^p(\omega)_\beta$ with $\beta < 1 - \frac{1}{p}$, $\{a_{n,k}\} \in \gamma HBVS$ (or $\{a_{n,k}\} \in \gamma RBVS$) and let $\widetilde{\omega}$ satisfy* (5), (6), *and* (7), *where $q = p/(p-1)$. Then*

$$|T_{n,A}f(x) - f(x)| = O_x\left((n+1)^{\beta+\frac{1}{p}+1}\gamma_n\omega\left(\frac{\pi}{n+1}\right)\right)$$

for considered x.

Remark 3. If $\gamma_n = \theta_n$, then clearly $\gamma HBVS \equiv HBVS$ and $\gamma RBVS \equiv RBVS$. Therefore Theorems 1 – 3 are also special cases of Theorems 11 – 13.

Remark 4. If we consider the L^p norms of the above-discussed deviations, we can obtain the same estimations without any difficulty.

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