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A note on almost sure behavior of randomly weighted sums of ϕ-mixing random variables with ϕ-mixing weights

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Abstract. Randomly weighted sums play an important role in various applied and theoretical problems, e.g., in actuarial mathematics or statistics. The almost sure convergence of randomly weighted sums is usually studied under the assumption that sequences are independent and identically distributed. In this note, we assume that both sequences are ϕ-mixing. Under some additional conditions, we prove a strong law of large numbers for sequences of randomly weighted sums.

1. Introduction

Let $\{Y_i,\ i \geq 1\}$ be a sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. Let $\mathcal{F}_j^k$ be a σ-algebra generated by the random variables Y_l, $l = j, \ldots, k$.

Define the ϕ-mixing coefficient (uniform mixing coefficient)

$$\phi(m) = \sup\{|P(B|A) - P(B)|\},$$

where the supremum is taken over $A \in \mathcal{F}_1^k$, $B \in \mathcal{F}_{k+m}^n$, $P(A) \neq 0$, $1 \leq k \leq n - m$.

The uniform mixing coefficient was introduced independently by Rozanov and Volkonski [11] and Ibragimov [4]. Since then many authors have studied sequences of ϕ-mixing random variables, and a number of useful results have been established. In [8], Nagaev proved probability and maximal inequalities for ϕ-mixing random variables. The summability of ϕ-mixing random variables was studied by Kiesel in [5, 6], whereas strong laws of large numbers were obtained, e.g., in [5, 6, 7, 12].

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Randomly weighted partial sums $\sum_{i=1}^{n} A_i X_i$ play an important role in various applied and theoretical problems. In actuarial mathematics, if X_i is regarded as the net loss within the time period i of the company and each A_i as the discount factor from time i to time 0, then $\sum_{i=1}^{n} A_i X_i$ can be interpreted as the total discounted amount of the net loss from time 0 to time n. For example, in the field of queueing theory, the $\sum_{i=1}^{n} A_i X_i$ can be used to represent the total output for customer being served by n machines. In statistics, Arenal-Gutiérez et al. [1] obtained a strong law of large numbers for the bootstrap mean, assuming that $\{X_i,\ i \geq 1\}$ is a sequence of pairwise independent and identically distributed random variables. The latter was further generalized by Rosalsky and Sreehari in [10]. In this note, we study strong limit theorems of randomly weighted partial sums $\sum_{i=1}^{n} A_i X_i$, assuming that both sequences are weakly dependent, which generalize the results obtained in [1, 10]. In contrast to [1], we assume that $\{X_i,\ i \geq 1\}$ is ϕ-mixing. On the sequence of weights $\{A_i,\ i \geq 1\}$, we assume that this sequence is a sequence of positive, identically distributed (i.d.) ϕ-mixing random variables such that A_i and X_i are independent, for each $i \geq 1$. We establish strong laws of large numbers for a non-identically distributed sequence $\{X_i,\ i \geq 1\}$ using the notion of a regular cover, which was introduced by Pruss in [9].

Definition 1.1. Let $X_1, \dots, X_n$ be random variables, and X be a random variable possibly defined on a different probability space. Then $X_1, \dots, X_n$ are said to be a regular cover of X provided

$$E\left(G\left(X\right)\right) = \frac{1}{n}\sum_{i=1}^{n} E\left(G\left(X_i\right)\right), \tag{1}$$

for any measurable function G for which both sides make sense.

2. Technical lemmas

Let $\{Y_i,\ i \geq 1\}$ be a sequence of ϕ-mixing random variables. Set $S_k = \sum_{i=1}^{k} Y_i$ and $M_n = \max_{1\leq k\leq n} |S_k|$.

Define

$$\phi^{+}\left(m\right) = \sup\left\{P\left(B|A\right) - P\left(B\right)\right\},$$

where the supremum is taken over $A \in \mathcal{F}_1^k$, $B \in \mathcal{F}_{k+m}^n$, $P\left(A\right) \neq 0$, $1 \leq k \leq n-m$.

In [8], it was pointed out that $\phi^{+}\left(n\right) < \phi\left(n\right)$. Assume that $\phi^{+}\left(1\right) < 1$ and let $\delta > 0$ satisfy the condition $\delta + \phi^{+}\left(1\right) < 1$. Set $\rho = \delta + \phi^{+}\left(1\right)$. Let α be a number such that the following condition is satisfied:

$$P\left(2M_n > \alpha\right) < \delta.$$

Under the above notation, Nagaev [8] proved the following inequality.

Lemma 2.1. *For any $p > 0$ and $0 < \varepsilon < \frac{1}{6}$ such that $s(\varepsilon) > p$,*

$$EM_n^p < c_1(p)\sum_{i=1}^{n} E|Y_i|^p + c_2(p)\alpha^p,$$

where $s(\varepsilon) = -\log\rho/\log(1+\varepsilon)$,

$$c_1(p) < \frac{2^{p+1}}{\varepsilon^{3p+1}\rho}B(p+1, s(\varepsilon)-p+1),$$
$$c_2(p) < \rho^{-1}\varepsilon^{-p}B(p+1, s(\varepsilon)-p)\,p+1,$$

and $B(\cdot,\cdot)$ is the Euler Beta function.

In the proof of the main result we will need the following lemma.

Lemma 2.2. *Let $\{A,\ A_i, i \ge 1\}$ be a sequence of positive i.d. random variables with $EA^p < \infty$, for some $1 \le p \le 2$. Then*

$$\frac{\max_{1\le i\le n} A_i}{n} \to 0 \ \ a.s.$$

Proof. Note that the condition $EA^p < \infty$ implies $\sum_{n=1}^{\infty} P(A_n > \epsilon n) < \infty$, for every $\epsilon > 0$. Hence, by the Borel–Cantelli lemma we have that

$$\frac{A_n}{n} \to 0 \ \ a.s.$$

Thus, by Lemma 1 in [3] we get the assertion. □

Throughout this paper, C_1 and C_2 always stand for positive constants which may differ from one place to another.

3. Main results

Theorem 3.1. *Let $\{A,\ A_i, i \ge 1\}$ be a sequence of positive i.d. ϕ-mixing random variables with $EA^p < \infty$, for some $1 \le p \le 2$, and let $\{X_i,\ i \ge 1\}$ be a sequence of ϕ-mixing random variables that is independent of $\{A,\ A_i, i \ge 1\}$. Let X be a random variable, possibly defined on a different probability space, satisfying condition (1). Moreover, additionally assume that $EX_n = 0$, for all $n \ge 1$. Let $b_0 = 0$ and b_n be an increasing sequence of positive numbers satisfying*

$$\frac{b_n}{n} \to \infty \quad \text{and} \quad b_n^p \sum_{i=n}^{\infty} \frac{1}{b_i^p} = O(n). \tag{2}$$

If $s(\varepsilon) > p$, for some $0 < \varepsilon < \frac{1}{6}$, and $E|X|^p < \infty$, then

$$\lim_{n\to\infty} \frac{1}{n^{1/p} b_n} \sum_{i=1}^{n} A_i X_i = 0 \ \ a.s. \tag{3}$$

Proof. For $n \geq 1$, set $X_i' = X_i I\left(|X_i| \leq b_i\right)$ and $X_i'' = X_i I\left(|X_i| > b_i\right)$. Then

$$\sum_{i=1}^{n} A_i X_i = \sum_{i=1}^{n} A_i \left(X_i' - EX_i'\right) + \sum_{i=1}^{n} A_i X_i'' + \sum_{i=1}^{n} A_i EX_i'.$$

In order to show that $\left(n^{1/p} b_n\right)^{-1} \sum_{i=1}^{n} A_i X_i \to 0$ a.s., we only need to show that all terms above are $o\left(n^{1/p} b_n\right)$ a.s.

First, we show that

$$\sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n} A_i \left(X_i' - EX_i'\right)\right| > n^{1/p} b_n \epsilon\right) < \infty, \tag{4}$$

for all $\epsilon > 0$.

Because $X_i' = X_i I\left(|X_i| \leq b_i\right)$ is also ϕ-mixing, by Theorem 5.2 in [2], we have that $\{A_i X_i', i \geq 1\}$ is also ϕ-mixing. Hence, by Markov's inequality and Lemma 2.1, we have that

$$\begin{aligned}
\sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n} A_i \left(X_i' - EX_i'\right)\right| > n^{1/p} b_n \epsilon\right) &\leq \sum_{n=1}^{\infty} \frac{E\left|\sum_{i=1}^{n} A_i \left(X_i' - EX_i'\right)\right|^p}{n b_n^p \epsilon^p} \\
&\leq C_1 \sum_{n=1}^{\infty} \frac{EA^p}{n b_n^p} \sum_{i=1}^{n} E\left|X_i' - EX_i'\right|^p \\
&+ C_2 \sum_{n=1}^{\infty} \frac{1}{n b_n^p} \\
&\leq C_1 \sum_{n=1}^{\infty} \frac{1}{n b_n^p} \sum_{i=1}^{n} E\left|X_i'\right|^p + C_2 \sum_{n=1}^{\infty} \frac{1}{b_n^p} \\
&= I_1 + I_2.
\end{aligned}$$

Note that the second part of (2) ensures that $I_2 < \infty$. Hence, it remains to show that $I_1 < \infty$.

Because b_n is increasing, we have that $b_i \leq b_n$, for all $i \leq n$, and

$$\begin{aligned}
I_1 &= C_1 \sum_{n=1}^{\infty} \frac{1}{n b_n^p} \sum_{i=1}^{n} E\left|X_i'\right|^p \\
&\leq C_1 \sum_{n=1}^{\infty} \frac{1}{n b_n^p} \sum_{i=1}^{n} E\left|X_i\right|^p I\left(|X_i| \leq b_n\right).
\end{aligned}$$

Let $G(x) = |x|^p I[|x| \le b_n]$; then by the definition of regular cover

$$I_1 \le C_1 \sum_{n=1}^{\infty} \frac{1}{n b_n^p} \sum_{i=1}^{n} E|X_i|^p I(|X_i| \le b_n)$$
$$\le C_1 \sum_{n=1}^{\infty} \frac{1}{b_n^p} E|X'|^p,$$

where $X' = XI(|X| \le b_n)$.

Further,

$$I_1 \le C_1 \sum_{n=1}^{\infty} \frac{1}{b_n^p} E|X'|^p = C \sum_{n=1}^{\infty} \frac{1}{b_n^p} \sum_{n=1}^{n} E|X|^p I[b_{i-1} < |X| \le b_i]$$
$$\le C_1 \sum_{n=1}^{\infty} \frac{1}{b_n^p} \sum_{i=1}^{n} b_i^p P(b_{i-1} < |X| \le b_i)$$
$$= C_1 \sum_{i=1}^{\infty} b_i^p P(b_{i-1} < |X| \le b_i) \sum_{n=i}^{\infty} \frac{1}{b_n^p}. \tag{5}$$

Then, by (5) and (2),

$$I_1 \le C_1 \sum_{i=1}^{\infty} iP(b_{i-1} < |X| \le b_i)$$
$$= C_1 \sum_{i=1}^{\infty} P(|X| > b_i) \le C_1 E|X|^p \sum_{i=1}^{\infty} \frac{1}{b_i^p} < \infty,$$

and (4) holds. Thus, $\left(n^{1/p} b_n\right)^{-1} \sum_{i=1}^{n} A_i (X_i' - EX_i')$ converges completely to 0, which implies that $\sum_{i=1}^{n} A_i (X_i' - EX_i')$ is $o\left(n^{1/p} b_n\right)$ a.s.

Next, by the definition of regular cover, condition $E|X|^p < \infty$, and (2), we have that

$$\sum_{n=1}^{\infty} P(|X_n| > b_n) = \sum_{n=1}^{\infty} \sum_{i=n}^{\infty} P(b_{i-1} < |X_n| \le b_i)$$
$$= \sum_{i=1}^{\infty} \sum_{n=1}^{i} P(b_{i-1} < |X_n| \le b_i)$$
$$= \sum_{i=1}^{\infty} \sum_{n=1}^{i} EI[b_{i-1} < |X_n| \le b_i]$$
$$= \sum_{i=1}^{\infty} iEI[b_{i-1} < |X| \le b_i]$$

$$= \sum_{i=1}^{\infty} iP\left(b_{i-1} < |X| \le b_i\right)$$

$$\le E\left|X\right|^p \sum_{i=1}^{\infty} \frac{1}{b_i^p} < \infty.$$

Hence, by the Borel–Cantelli lemma, $\sum_{i=1}^{n} |X_i''|$ is bounded a.s. By (2) and Lemma 2.2, it follows that

$$\frac{1}{n^{1/p} b_n} \left| \sum_{i=1}^{n} A_i X_i'' \right| \le \frac{1}{n^{1/p} b_n} \max_{1 \le i \le n} A_i \sum_{i=1}^{n} \left| X_i'' \right|$$

$$= \frac{n^{1-1/p}}{b_n} \frac{\max_{1 \le i \le n} A_i}{n} \sum_{i=1}^{n} \left| X_i'' \right| \to 0 \ \ a.s.$$

Finally, by Markov's inequality, Lemma 2.1, condition $EA^p < \infty$, and the definition of regular cover,

$$\sum_{n=1}^{\infty} P\left(\left| \sum_{i=1}^{n} A_i E X_i' \right| > n^{1/p} b_n \epsilon \right) \le C_1 \sum_{n=1}^{\infty} \frac{1}{n b_n^p} \sum_{i=1}^{n} E A_i^p \left| E X_i' \right|^p + C_2 \sum_{n=1}^{\infty} \frac{1}{n b_n^p}$$

$$\le C_1 \sum_{n=1}^{\infty} \frac{1}{n b_n^p} \sum_{i=1}^{n} E \left| X_i' \right|^p + C_2 \sum_{n=1}^{\infty} \frac{1}{n b_n^p}.$$

Hence, using the same arguments as in the estimation of I_1 and I_2, we obtain that

$$\sum_{n=1}^{\infty} P\left(\left| \sum_{i=1}^{n} A_i E X_i' \right| > n^{1/p} b_n \epsilon \right) < \infty,$$

for every $\epsilon > 0$. Thus, $\left(n^{1/p} b_n\right)^{-1} \sum_{i=1}^{n} A_i E X_i'$ converges completely to 0, which implies that $\sum_{i=1}^{n} A_i E X_i'$ is $o\left(n^{1/p} b_n\right)$ a.s. This completes the proof. $\square$

In [9], it was pointed out that every i.d. sequence of random variables satisfies the regular cover condition (1) with $X = X_1$. Thus, from Theorem 3.1 we have the following corollary.

Corollary 3.2. *Let $\{A,\ A_i, i \ge 1\}$ be a sequence of positive i.d. ϕ-mixing random variables with $EA^p < \infty$, for some $1 \le p \le 2$, and let $\{X,\ X_i,\ i \ge 1\}$ be a sequence of i.d. ϕ-mixing random variables that is independent of $\{A,\ A_i, i \ge 1\}$. Moreover, additionally assume that $EX_n = 0$, for all $n \ge 1$. Let $b_0 = 0$ and b_n be an increasing sequence of positive numbers satisfying (2). If $s(\varepsilon) > p$, for some $0 < \varepsilon < \frac{1}{6}$, and $E|X|^p < \infty$, then (3) holds.*

It is known that every sequence of independent and identically distributed (i.i.d.) random variables is ϕ-mixing with $\phi(n) = 0$, for each $n \ge 1$. Thus, from Theorem 3.1 we obtain the following corollary.

Corollary 3.3. *Let $\{A,\ A_i, i \geq 1\}$ be a sequence of positive i.i.d. random variables with $EA^p < \infty$, for some $1 \leq p \leq 2$, and let $\{X_i,\ i \geq 1\}$ be a sequence of ϕ-mixing random variables that is independent of $\{A,\ A_i, i \geq 1\}$. Let X be a random variable, possibly defined on a different probability space, satisfying condition* (1). *Moreover, additionally assume that $EX_n = 0$, for all $n \geq 1$. Let $b_0 = 0$ and b_n be an increasing sequence of positive numbers satisfying* (2). *If $s(\varepsilon) > p$, for some $0 < \varepsilon < \frac{1}{6}$, and $E|X|^p < \infty$, then* (3) *holds.*

We conclude with some remarks.

Remark 1. Using Theorem 5.2 in [2], under some additional conditions, one can obtain the counterpart of Theorem 3.1 for other mixing coefficients.

Remark 2. It should be stressed that the assumption of independence of $\{A_i, i \geq 1\}$ and $\{X_i,\ i \geq 1\}$ in Theorem 3.1 is very crucial. Assuming only that both sequences are ϕ-mixing does not guarantee that $\{A_i X_i,\ i \geq 1\}$ will be ϕ-mixing (see [2, Theorem 5.2], and discussion below the theorem).

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