ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 17, Number 2, December 2013
Available online at http://acutm.math.ut.ee

Generalized statistical convergence in 2-normed spaces

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Abstract. In this paper we introduce the concept of λ-statistical convergence in 2-normed spaces. Some inclusion relations between the sets of statistically convergent, λ-statistically convergent and statistically λ-convergent sequences in 2-normed spaces are established, where $\lambda = (\lambda_m)$ is a non-decreasing sequence of positive numbers tending to infinity such that $\lambda_{m+1} \leq \lambda_m + 1, \lambda_1 = 1$.

1. Introduction and preliminaries

The concept of statistical convergence plays a vital role not only in pure mathematics but also in other branches of science involving mathematics, especially in information theory, computer science, biological science, dynamical systems, geographic information systems, population modelling, and motion planning in robotics.

The notion of statistical convergence was introduced by Fast [6] and Schoenberg [21] independently. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Later on it was further investigated from the sequence space point of view and linked with the summability theory by Fridy [8], Šalát [20], Cakalli [1], Caserta, et al. [2], Caserta and Kočinac [3], Di Maio and Kočinac [5], Miller [17], Maddox [16] and many others. In the recent years, generalizations of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on Stone–Čech compactification of the natural numbers. Moreover, the statistical convergence is closely related to the

Received August 11, 2012.
2010 *Mathematics Subject Classification.* 40A05, 40A35, 46A19, 46A45.
Key words and phrases. Statistical convergence, λ-statistical convergence, de la Vallée-Poussin means, 2-norm.
http://dx.doi.org/10.12697/ACUTM.2013.17.12

concept of convergence in probability (see [4]). Mursaleen [18] introduced the λ-statistical convergence for real sequences.

The notion of statistical convergence depends on the density of subsets of $\mathbb{N}$. A subset E of $\mathbb{N}$ is said to have *natural density* $\delta(E)$ if

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k) \text{ exists.}$$

Definition 1.1. A sequence $x = (x_k)$ is said to be *statistically convergent* to a number l if for every $\varepsilon > 0$,

$$\delta(\{k \in \mathbb{N} : |x_k - l| \geq \varepsilon\}) = 0.$$

Let $\lambda = (\lambda_m)$ be a non-decreasing sequence of positive numbers tending to infinity such that

$$\lambda_{m+1} \leq \lambda_m + 1, \lambda_1 = 1.$$

The collection of such sequences λ will be denoted by Δ.

The generalized de la Vallée-Poussin means are defined by

$$t_m(x) = \frac{1}{\lambda_m} \sum_{k\in I_m} x_k,$$

where $I_m = [m - \lambda_m + 1, m]$, $m \in \mathbb{N}$.

Definition 1.2 ([14])**.** A sequence $x = (x_k)$ is said to be (V, λ)-*summable* to a number l if

$$\lim_{m\to\infty} t_m(x) = l.$$

In this case we write $\lambda\text{-}\lim x_k = l$.

Definition 1.3 ([7])**.** The sequence x is said to be *strongly* (V, λ)-*summable* to l if

$$\lim_{m\to\infty} \frac{1}{\lambda_m} \sum_{k\in I_m} |x_k - l| = 0.$$

If $\lambda_m = m$, then the (V, λ)-summability and the strong (V, λ)-summability reduce, respectively, to the $(C, 1)$-summability and the strong $(C, 1)$-summability.

Let $K \subseteq \mathbb{N}$. If the limit

$$\delta(E) = \lim_{m\to\infty} \frac{1}{\lambda_m} \sum_{k\in I_m} \chi_K(k)$$

exists, then $\delta_\lambda(K)$ is called λ-*density* of K. If $\lambda_m = m$, then λ-density reduces to the natural density.

Definition 1.4 ([18]). A sequence $x = (x_k)$ is said to be λ-*statistically convergent* to l if for every $\varepsilon > 0$,

$$\lim_{m\to\infty} \frac{1}{\lambda_m} |\{k \in I_m : |x_k - l| \geq \varepsilon\}| = 0$$

or, equivalently, $\delta_\lambda(K_\varepsilon) = 0$, where

$$K_\varepsilon = \{k \in \mathbb{N} : |x_k - l| \geq \varepsilon\}.$$

It is clear that, for $\lambda_m = m$, the λ-statistical convergence reduces to the statistical convergence.

The concept of 2-normed space was initially introduced by Gähler [10]. Since then, many other authors have studied this concept and obtained various results (see, for instance, Gähler [9], Gunawan and Mashadi [11] and Lewandowska [15]).

Let X be a real vector space of dimension $d \geq 2$ (d may be infinite). A real-valued function $\|\cdot,\cdot\|$ from X^2 into $\mathbb{R}$ satisfying the conditions

(1) $\|x_1, x_2\| = 0$ if and only if x_1, x_2 are linearly dependent,
(2) $\|x_1, x_2\|$ is invariant under permutation,
(3) $\|\alpha x_1, x_2\| = |\alpha| \|x_1, x_2\|$, for any $\alpha \in \mathbb{R}$,
(4) $\|x + \overline{x}, x_2\| \leq \|x, x_2\| + \|\overline{x}, x_2\|$

is called a 2-*norm* on X and the pair $(X, \|\cdot,\cdot\|)$ is called a 2-*normed space.*

A trivial example of a 2-normed space is $X = \mathbb{R}^2$ equipped with the Euclidean 2-norm

$$\|x, y\|_E = |x_1 y_2 - x_2 y_1|,$$

where $x = (x_1, x_2), y = (y_1, y_2)$ are points of $\mathbb{R}^2$.

Definition 1.5 ([12]). A sequence (x_k) in a 2-normed space $(X, \|\cdot,\cdot\|)$ is said to be *statistically convergent* to $l \in X$ with respect to the 2-norm if for each $\varepsilon > 0$ the set $K_\varepsilon(z) = \{k \in \mathbb{N} : \|x_k - l, z\| \geq \varepsilon\}$ has natural density zero for every nonzero $z \in X$.

In other words, the sequence (x_k) converges statistically to l in a 2-normed space X if for each $\varepsilon > 0$ and nonzero $z \in X$,

$$\lim_{m\to\infty} \frac{1}{m} |\{k \leq m : \|x_k - l, z\| \geq \varepsilon\}| = 0.$$

Let S^{2N} denote the set of all statistically convergent sequences in a 2-normed space X.

Gürdal and Pehlivan [12, 13] studied statistical convergence in 2-normed spaces.

In the present paper we study λ-statistical convergence and statistical λ-convergence in 2-normed spaces. We show that some properties of λ-statistical convergence and statistical λ-convergence of sequences of real numbers also hold for sequences in 2-normed spaces.

2. λ-Statistically convergent sequences in 2-normed spaces

In this section we define λ-statistically convergent sequences in 2-normed linear space X. We also obtain some basic properties of this notion.

Definition 2.1. A sequence $x = (x_k)$ in a 2-normed space $(X, \|\cdot,\cdot\|)$ is said to be *strongly* (V,λ)*-summable* to $l \in X$ if, with respect to the 2-norm,

$$t_m(x) \to l \text{ as } m \to \infty.$$

If $\lambda_m = m$, then the strong (V,λ)-summability reduces to the strong $(C,1)$-summability with respect to the 2-norm. If a sequence $x = (x_k)$ is strongly (V,λ)-summable to l in the 2-normed space X, then we write $x_k \to l([V,\lambda]^{2N})$. The set of all strongly (V,λ)-summable sequences in X is denoted by $[V,\lambda]^{2N}$.

Definition 2.2. A sequence $x = (x_k)$ in a 2-normed space $(X, \|\cdot,\cdot\|)$ is said to be λ*-satistically convergent* (briefly, S_λ*-convergent*) to $l \in X$ with respect to the 2-norm if for every $\varepsilon > 0$,

$$\lim_{m\to\infty} \frac{1}{\lambda_m} |\{k \in I_m : \|x_k - l, z\| \geq \varepsilon\}| = 0 \text{ for each nonzero } z \in X,$$

i.e., the set $K_\varepsilon(z) = \{k \in \mathbb{N} : \|x_k - l, z\| \geq \varepsilon\}$ has λ-density zero for each $\varepsilon > 0$ and nonzero $z \in X$.

In this case we write $S_\lambda^{2N}\text{-}\lim x_k = l$ or $x_k \to l(S_\lambda^{2N})$, and denote by S_λ^{2N} the set of all λ-statistically convergent sequences in the 2-normed space X.

Mursaleen and Alotaibi [19] introduced the notion of statistical λ-convergence for real sequences. We define the statistical λ-convergence in 2-normed spaces as follows.

Definition 2.3. A sequence $x = (x_k)$ in a 2-normed space $(X, \|\cdot,\cdot\|)$ is said to be *statistically* λ*-convergent* (briefly, S_{δ_λ}*-convergent*) to $l \in X$ with respect to the 2-norm if for every $\varepsilon > 0$ the set

$$K_\varepsilon(\lambda, z) = \{m \in \mathbb{N} : \|t_m(x) - l, z\| \geq \varepsilon\}$$

has natural density zero for each $\varepsilon > 0$ and nonzero $z \in X$. In this case we write $S_{\delta_\lambda}^{2N}\text{-}\lim x_k = l$.

The method proving the following theorem is similar to the one given by Mursaleen and Alotaibi [19].

Theorem 2.1. *Let X be a 2-normed space and $\lambda = (\lambda_n) \in \Delta$. If (x_k) is a bounded sequence and $S_\lambda^{2N}\text{-}\lim x_k = l$, then $S_{\delta_\lambda}^{2N}\text{-}\lim x_k = l$.*

Proof. Let $x = (x_k)$ be bounded and S_λ^{2N}-$\lim x_k = l$. For each nonzero $z \in X$ and $\varepsilon > 0$ we write $K_{\varepsilon,\lambda} = \{k \in I_m : \|x_k - l, z\| \geq \varepsilon\}$. Then

$$\begin{aligned}\|t_m(x) - l, z\| &= \|\frac{1}{\lambda_m} \sum_{k \in I_m} x_k - l, z\| \\ &\leq \|\frac{1}{\lambda_m} \sum_{k \in K_{\varepsilon,\lambda}} x_k - l, z\| \\ &\leq \frac{1}{\lambda_m} (\sup_k \|x_k - l, z\|) K_{\varepsilon,\lambda} \to 0 \text{ as } m \to \infty.\end{aligned}$$

This implies that x is (V, λ)-summable to l and hence $S_{\delta_\lambda}^{2N}$- $\lim x_k = l$. □

Remark 2.2. The converse of the above result is not true in general. Consider $\lambda_m = m$ and define the sequence $x = (x_k)$ by

$$x_k = \begin{cases} 2 & \text{if } k \text{ is even,} \\ -2 & \text{if } k \text{ is odd.} \end{cases}$$

Then x is not λ-statistically convergent, but x is (V, λ)-summable to 0 and hence statistically λ-convergent to 0.

The proof of the following theorem is straightforward, thus we omit it.

Theorem 2.3. *The set $S_\lambda^{2N} \cap \ell_\infty$ is a closed subset of ℓ_∞, where ℓ_∞ denotes the space of all bounded sequences of elements in a 2-normed space X.*

Theorem 2.4. *Let X be a 2-normed space and let $\lambda = (\lambda_m) \in \Delta$. Then*

(i) $x_k \to l([V, \lambda]^{2N}) \Rightarrow x_k \to l(S_\lambda^{2N})$,
(ii) $[V, \lambda]^{2N}$ *is a proper subset of* S_λ^{2N},
(iii) *if* $x \in \ell_\infty$ *and* $x_k \to l(S_\lambda^{2N})$, *then* $x_k \to l([V, \lambda]^{2N})$ *and hence* $x_k \to l([C, 1]^{2N})$, *provided* $x = (x_k)$ *is not eventually constant,*
(iv) $S_\lambda^{2N} \cap \ell_\infty = [V, \lambda]^{2N} \cap \ell_\infty$.

Proof. (i) If $\varepsilon > 0$ and $x_k \to \ell([V, \lambda]^{2N})$, then we can write

$$\begin{aligned}\sum_{k \in I_m} \|x_k - l, z\| \geq & \sum_{k \in I_m, \|x_k - l, z\| \geq \varepsilon} \|x_k - l, z\| \\ & \geq \varepsilon |\{k \in I_m : \|x_k - l, z\| \geq \varepsilon\}|\end{aligned}$$

and so

$$\frac{1}{\varepsilon \lambda_m} \sum_{k \in I_m} \|x_k - l, z\| \geq \frac{1}{\lambda_m} |\{k \in I_m : \|x_k - l, z\| \geq \varepsilon\}|.$$

This proves the statement (i).

(ii) In order to establish that the inclusion $[V,\lambda]^{2N} \subset S_\lambda^{2N}$ is proper, let $X = \mathbb{R}$. We define a sequence $x = (x_k)$ by

$$x_k = \begin{cases} k & \text{if } m - [\sqrt{\lambda_m}] + 1 \le k \le m, \\ 0 & \text{otherwise.} \end{cases}$$

Then $x \notin \ell_\infty$ and for every $\varepsilon > 0$, $0 < \varepsilon < 1$,

$$\frac{1}{\lambda_m} \sum_{k \in I_m} \|x_k - 0, z\| \le \frac{[\sqrt{\lambda_m}]}{\lambda_m} \to 0 \text{ as } m \to \infty,$$

i.e., $x_k \to 0(S_\lambda^{2N})$. On the other hand,

$$\frac{1}{\lambda_m} | \{k \in I_m : \|x_k - 0, z\| \ge \varepsilon\} | \to \infty \text{ as } m \to \infty,$$

and so, $x_k \nrightarrow 0[V,\lambda]^{2N}$.

(iii) Suppose that $x_k \to l(S_\lambda^{2N})$ and $x \in \ell_\infty$. Then there exists an $M > 0$ such that $\|x_k - l, z\| \le M$ for all $k \in \mathbb{N}$ and $z \in X$. Given $\varepsilon > 0$, we have

$$\frac{1}{\lambda_m} \sum_{k \in I_m} \|x_k - l, z\| = T_1(m) + T_2(m),$$

where

$$T_1(m) = \frac{1}{\lambda_m} \sum_{k \in I_m, k \in K_\varepsilon(z)} \|x_k - l, z\|, \quad T_2(m) = \frac{1}{\lambda_m} \sum_{k \in I_m, k \notin K_\varepsilon(z)} \|x_k - l, z\|.$$

Then $T_2(m) < \varepsilon$ and

$$T_1(m) \le (\sup_k \|x_k - l, z\|) \frac{1}{\lambda_m} |K_\varepsilon(z)| \le \frac{M}{\lambda_m} |K_\varepsilon(z)| \to 0 \text{ as } m \to \infty.$$

This shows that $x_k \to l([V,\lambda]^{2N})$.

Again, we have

$$\begin{aligned} \frac{1}{m} \sum_{k=1}^{m} \|x_k - l, z\| &= \frac{1}{m} \sum_{k=1}^{m-\lambda_m} \|x_k - l, z\| + \frac{1}{m} \sum_{k \in I_m} \|x_k - l, z\| \\ &\le \frac{1}{\lambda_m} \sum_{k=1}^{m-\lambda_m} \|x_k - l, z\| + \frac{1}{\lambda_m} \sum_{k \in I_m} \|x_k - l, z\| \\ &\le \frac{2}{\lambda_m} \sum_{k \in I_m} \|x_k - l, z\|. \end{aligned}$$

Hence $x_k \to l([C,1]^{2N})$, because $x_k \to l([V,\lambda]^{2N})$.

(iv) This is an immediate consequence of (i), (ii) and (iii). □

Theorem 2.5. *Let X be a 2-normed space and let $\lambda = (\lambda_m) \in \Delta$. Then $S^{2N} \subset S_\lambda^{2N}$ if and only if* $\liminf_m \frac{\lambda_m}{m} > 0$.

Proof. Suppose first that $\liminf_m \frac{\lambda_m}{m} > 0$. Then for given $\varepsilon > 0$ we have

$$\begin{aligned}\frac{1}{m}|\{k \le m : \|x_k - l, z\| \ge \varepsilon\}| &\ge \frac{1}{m}|\{k \in I_m : \|x_k - l, z\| \ge \varepsilon\}| \\ &\ge \frac{\lambda_m}{m}.\frac{1}{\lambda_m}|\{k \in I_m : \|x_k - l, z\| \ge \varepsilon\}|.\end{aligned}$$

It follows that $x_k \to l(S^{2N})$ implies $x_k \to l(S_\lambda^{2N})$. Hence $S^{2N} \subset S_\lambda^{2N}$.

Conversely, suppose that $\liminf_m \frac{\lambda_m}{m} = 0$. Then we can select a subsequence $(m(j))_{j=1}^{\infty}$ such that

$$\frac{\lambda_{m(j)}}{m(j)} < \frac{1}{j}.$$

We define a sequence $x = (x_k)$ by

$$x_k = \begin{cases} 1 & \text{if } k \in I_{m(j)},\ j \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Then x is statistically convergent, so $x \in S^{2N}$. But $x \notin [V, \lambda]^{2N}$. Theorem 2.4(iii) implies that $x \notin S_\lambda^{2N}$. This completes the proof. □

Theorem 2.6. *Let X be a 2-normed space. If $\lambda = (\lambda_m) \in \Delta$ is such that $\lim_m \frac{\lambda_m}{m} = 1$, then $S_\lambda^{2N} \subset S^{2N}$.*

Proof. Since $\lim_m \frac{\lambda_m}{m} = 1$, for $\varepsilon > 0$ we observe that

$$\begin{aligned}\frac{1}{m}|\{k \le m : \|x_k - l, z\| \ge \varepsilon\}| &\le \frac{1}{m}|\{k \le m - \lambda_m : \|x_k - l, z\| \ge \varepsilon\}| \\ &\quad + \frac{1}{m}|\{k \in I_m : \|x_k - l, z\| \ge \varepsilon\}| \\ &\le \frac{m - \lambda_m}{m} + \frac{1}{m}|\{k \in I_m : \|x_k - l, z\| \ge \varepsilon\}| \\ &= \frac{m - \lambda_m}{m} + \frac{\lambda_m}{m}\frac{1}{\lambda_m}|\{k \in I_m : \|x_k - l, z\| \ge \varepsilon\}|.\end{aligned}$$

This implies that (x_k) is statistically convergent if (x_k) is λ-statistically convergent. Hence $S_\lambda^{2N} \subset S^{2N}$. □

The method proving the following theorem is similar to the one given by Šalát [20].

Theorem 2.7. *Let X be a 2-normed space and let $x = (x_k)$ be a sequence in X. Then $S_{\delta_\lambda}^{2N}$-$\lim x_k = l$ if and only if there exists a set $K = \{k_1 < k_2, \cdots < k_m < \dots\} \subseteq \mathbb{N}$ such that $\delta(K) = 1$ and λ-$\lim x_{k_m} = l$.*

Proof. Suppose that there exists a set $K = \{k_1 < k_2, \cdots < k_m < \dots\} \subseteq \mathbb{N}$ such that $\delta(K) = 1$ and λ-$\lim x_{k_m} = l$. For each $z \in X$, there is a positive integer N such that

$$\|t_m(x) - l, z\| < \varepsilon \text{ for } m > N.$$

Write $K_\varepsilon(\lambda) = \{m \in \mathbb{N} : \|t_{k_m}(x) - l, z\| \geq \varepsilon\}$ and $K^c = \{k_{N+1}, k_{N+2}, \dots\}$. Then $\delta(K^c) = 1$ and $K_\varepsilon(\lambda) \subseteq \mathbb{N} - K^c$ which implies that $\delta(K_\varepsilon(\lambda)) = 0$. Hence $S^{2N}_{\delta_\lambda}$-$\lim x_k = l$.

Conversely, suppose that $S^{2N}_{\delta_\lambda}$-$\lim x_k = l$. For $i \in \mathbb{N}$ write $K_i(\lambda) = \{m \in \mathbb{N} : \|t_{k_m}(x) - l, z\| \geq \frac{1}{i}\}$ and $M_i(\lambda) = \{m \in \mathbb{N} : \|t_{k_m}(x) - l, z\| < \frac{1}{i}\}$. Then $\delta(K_i(\lambda)) = 0$,

$$M_1(\lambda) \supset M_2(\lambda) \supset \dots \supset M_i(\lambda) \supset M_{i+1}(\lambda) \supset \dots \tag{2.1}$$

and

$$\delta(M_i(\lambda)) = 1, \quad i \in \mathbb{N}. \tag{2.2}$$

Now we have to show that for $m \in M_i(\lambda)$, λ-$\lim x_{k_m} = l$. Suppose that x_{k_m} is not (V, λ)-summable to l. Then there is $\varepsilon > 0$ such that $\|t_{k_m}(x) - l, z\| \geq \varepsilon$. Let $M_\varepsilon(\lambda) = \{m \in \mathbb{N} : \|t_{k_m}(x) - l, z\| < \varepsilon\}$ and $\varepsilon > \frac{1}{i}$ $(i \in \mathbb{N})$. Then

$$\delta(M_\varepsilon(\lambda)) = 0.$$

By (2.1) we have $M_i(\lambda) \subset M_\varepsilon(\lambda)$. Hence $\delta(M_i(\lambda)) = 0$, which contradicts (2.2). Thus λ-$\lim x_{k_m} = l$. This completes the proof. □

Corollary 2.8. *Let X be a 2-normed space and let $x = (x_k)$ be a sequence in X. Then S^{2N}_λ-$\lim x_k = l$ if and only if there exists a set $K = \{k_1 < k_2, \dots < k_m < \dots\} \subseteq \mathbb{N}$ such that $\delta_\lambda(K) = 1$ and $\lim x_{k_m} = l$.*

Acknowledgements

The author expresses his heartfelt gratitude to the anonymous reviewer for such excellent comments and suggestions which have enormously enhanced the quality and presentation of this paper.

References

[1] H. Cakalli, *A study on statistical convergence*, Funct. Anal. Approx. Comput. **1** (2009), 19–24.

[2] A. Caserta, G. Di Maio, and Lj. D. R. Kočinac, *Statistical convergence in function spaces*, Abstr. Appl. Anal. **2011**, Art. ID 420419, 11 pages.

[3] A. Caserta and Lj. D. R. Kočinac, *On statistical exhaustiveness*, Appl. Math. Letters **25** (2012), 1447–1451.

[4] J. Connor and M. A. Swardson, *Measures and ideals of $C^*(X)$*, Ann. New York Acad. Sci. **704** (1993), 80–91.

[5] G. Di Maio and Lj. D. R. Kočinac, *Statistical convergence in topology*, Topology Appl. **156** (2008), 28–45.

[6] H. Fast, *Sur la convergence statistique*, Colloq. Math. **2** (1951), 241–244.

[7] A. R. Freedman, J. J. Sember, and M. Raphael, *Some Cesàro-type summability spaces*, Proc. London Math. Soc. **37** (1978), 508–520.

[8] J. A. Fridy, *On statistical convergence*, Analysis **5** (1985), 301–313.

[9] S. Gähler, *2-metrische Räume und ihre topologische Struktur*, Math. Nachr. **26** (1963), 115–148.

[10] S. Gähler, *Lineare 2-normierte Räume*, Math. Nachr. **28** (1965), 1–43.

[11] H. Gunawan and M. Mashadi, *On finite dimensional 2-normed spaces*, Soochow J. Math. **27** (2001), 147–169.
[12] M. Gürdal and S. Pehlivan, *The statistical convergence in 2-Banach spaces*, Thai J. Math. **2** (2004), 107–113.
[13] M. Gürdal and S. Pehlivan, *Statistical convergence in 2-normed spaces*, Southeast Asian Bull. Math. **33** (2009), 257–264.
[14] L. Leindler, *Über die de la Vallee-Pousinsche Summierbarkeit allgenmeiner Othogonalreihen*, Acta Math. Acad. Sci. Hungar, **16** (1965), 375–387.
[15] Z. Lewandowska, *On 2-normed sets*, Glas. Mat. **38** (2003), 99–110.
[16] I. J. Maddox, *Statistical convergence in a locally convex space*, Math. Proc. Cambridge Philos. Soc. **104** (1988), 141–145.
[17] H. I. Miller, *A measure theoretical subsequence characterization of statistical convergence*, Trans. Amer. Math. Soc.**347** (1995), 1811–1819.
[18] M. Mursaleen, *λ-statistical convergence*, Math. Slovaca **50** (2000), 111–115.
[19] M. Mursaleen and A. Alotaibi, *Statistical summability and approximation by de la Vallée Poussin mean*, Appl. Math. Letters **24** (2011), 320–324.
[20] T. Šalát, *On statistical convergence of real numbers*, Math. Slovaca **30** (1980), 139–150.
[21] I. J. Schoenberg, *The integrability of certain functions and related summability methods*, Amer. Math. Monthly **66** (1959), 361–375.

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