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Some remarks on $H^{p,s}$ analytic function spaces in the unit disk and related estimates

Miloš Arsenović and Romi F. Shamoyan

Abstract. We prove new projection theorems involving certain Hardy–Lorentz analytic function spaces based on Lorentz classes on the unit circle which are related to p-Carleson type measures. We also study the action of fractional derivative in related analytic spaces in the unit disk partially extending previously known assertions.

1. Definitions and preliminaries

The goal of this note is to provide new estimates for Bergman-type projections and the fractional derivative acting in certain Hardy–Lorentz analytic function spaces based on Lorentz classes on the unit circle. Here we introduce notation and recall definitions which are well known in literature. Then we list preliminary results which will be used in proofs of main results of this paper.

We denote, as usual, Borel measures on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ by μ, normalized Lebesgue measure on $\mathbb{D}$ by $dA(z)$ and the Lebesgue measure on $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ by dm or $d\xi$. If $(X, d\lambda)$ is a measure space and $0 < p, q \leq \infty$, $L^{p,q}(\lambda)$ denotes the Lorentz space on X with respect to measure $d\lambda$. These spaces are complete metric spaces, if $p > 1$, $q \geq 1$ they are Banach spaces, see [5] for details on these spaces. In the special case $X = \mathbb{T}$ with measure dm we use notation $L^{p,q}(\mathbb{T}) = L^{p,q}$. For $0 < r < q$ we will need the following equivalent quasinorm on $L^{q,\infty}$ (see [5]):

$$\|f\|_{q,\infty} = \sup_{I \subset \mathbb{T}} \frac{1}{|I|^{1-r/q}} \int_I |f(\xi)|^r d\xi. \tag{1}$$

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Also, we have

$$\left|\int_X fg d\lambda\right| \le \|f\|_{L^{p,1}(\lambda)}\|g\|_{L^{p',\infty}(\lambda)},$$

where $1 < p < \infty$, p' denotes the exponent conjugate to p and $(X, d\lambda)$ is a measure space (see [5]). The convolution of functions $f, g \in L^1(\mathbb{T})$ is denoted by $f * g$. We note, for future use, Young's inequality

$$\|f * g\|_r \le C\|f\|_p\|g\|_q, \quad 1 < p, q, r < \infty, \ 1/p + 1/q = 1 + 1/r,$$

and its generalization to Lorentz spaces (see [10]),

$$\|f * g\|_{r,s} \le C\|f\|_{p,s_1}\|g\|_{q,s_2}, \quad 1/p + 1/q = 1 + 1/r, \ 1/s_1 + 1/s_2 = 1/s, \ (2)$$

where, again, $1 < p, q, r < \infty$.

The following definitions are standard, these can be found, for example, in [17], [4], [16], [14]. For an arc $I \subset \mathbb{T}$ we denote the tent over I by $T(I)$, the Carleson box over I by $\Box I$ and the length of I by $|I|$. The Stolz angle, of aperture $t > 1$ at $\xi \in \mathbb{T}$ is defined by

$$\Gamma_t(\xi) = \{z \in \mathbb{D} : |1 - \overline{\xi}z| < t(1 - |z|)\}.$$

The space of all holomorphic functions on $\mathbb{D}$ is denoted by $H(\mathbb{D})$. The non-tangential maximal function of a function $f \in H(\mathbb{D})$ is defined by

$$A_\infty f(\xi) = \sup_{z \in \Gamma_t(\xi)} |f(z)|, \qquad \xi \in \mathbb{T}.$$

To any holomorphic function f with Taylor coefficients a_k we associate another holomorphic function g with coefficients $(k+1)^\alpha a_k$, where α is real. This function g is a fractional derivative of f of order α and it will be denoted by $D^\alpha f$. This definition can be easily extended also to functions analytic in the unit polydisk (see, for example, [3] and references therein). Clearly, $D^\alpha f \in H(\mathbb{D})$ for $f \in H(\mathbb{D})$, $\alpha \in \mathbb{R}$.

H^p as usual stands for the classical Hardy space on the unit disc, $0 < p \le \infty$. We set also for all $0 < p \le \infty$ and $0 < s \le \infty$

$$H^{p,s}(\mathbb{T}) = \{f \in H(\mathbb{D}) : A_\infty f \in L^{p,s}(\mathbb{T})\}.$$

Spaces $h^{p,s}(\mathbb{T})$ are defined analogously, the only difference is that non-tangential supremum is replaced by the radial supremum.

Clearly $H^p(\mathbb{T}) \subset H^{p,p}(\mathbb{T})$, for all positive p. This follows directly from the maximal theorem for Hardy spaces (see, for example, [14]).

Definition 1. A positive Borel measure μ in the unit disk is a p-Carleson measure, $0 < p \le 1$, if

$$\|\mu\|_p = \left\|\sup_{\xi \in I} \frac{1}{|I|^p} \int_{\Box I} d\mu(z)\right\|_{L^\infty(\mathbb{T})} < \infty.$$

Definition 2. We define, for $0 < q < \infty$, $s < k$, $k \in \mathbb{N}$, $s \in \mathbb{R}$ and $0 < p \le 1$, the following space of analytic functions on the unit disc:

$$F^{\infty,q}_{s,p,k} = \Big\{ f \in H(\mathbb{D}) : |D^k f(z)|^q (1-|z|)^{(k-s)q-1} dA(z) \\ \text{is a } p\text{-Carleson measure} \Big\},$$

and, for $f \in F^{\infty,q}_{s,p,k}$, $\|f\|_{F^{\infty,q}_{s,p,k}}$ is the norm of the corresponding p-Carleson measure.

We often write $F^{\infty,q}_{s,p}$ instead of $F^{\infty,q}_{s,p,k}$ when k is clear from the context. The spaces $F^{\infty,q}_{s,p,k}$ include as special cases many important function spaces. For example, $F^{\infty,2}_{0,p,1} = Q_p(\mathbb{D})$, and Q_p classes where studied by several authors (see [16] and [3]). Also, $F^{\infty,2}_{0,1} = BMOA(\mathbb{D})$. The relation

$$\|f\|_{F^{\infty,\rho}_{s,p,k}} \asymp \sup_{I \subset \mathbb{T}} \frac{1}{|I|^p} \int_I \int_{\Gamma_\alpha(\xi)} |D^k f(z)|^\rho (1-|z|)^{(k-s)\rho-2} dA(z) dm(\xi)$$

follows from the following relations, valid for all $0 < p < 1$ and all positive Borel measures μ in $\mathbb{D}$:

$$\begin{aligned} \sup_{I \subset \mathbb{T}} \frac{1}{|I|^p} \int_{\Box I} d\mu(z) &\asymp \sup_{w \in \mathbb{D}} (1-|w|)^p \int_{\mathbb{D}} \frac{d\mu(z)}{|1-\overline{w}z|^{2p}} \\ &\asymp \sup_{I \subset \mathbb{T}} \frac{1}{|I|^p} \int_I \int_{\Gamma_\alpha(\xi)} \frac{d\mu(z)}{1-|z|} dm(\xi). \end{aligned} \tag{3}$$

These relations provide direct connection between p-Carleson measures when $p < 1$ and the so-called Luzin area integral, in fact, they provide characterizations of p-Carleson measures. Various other characterizations of p-Carleson measures are known (see [16]). We would like to mention that various spaces like mixed norm spaces, Bergman spaces and Hardy spaces can also be characterized using the mentioned area integral (see [12] and references therein).

The proofs of these relations can be found in [2] or [16]. We will also need the following assertion (see [12]):

$$\int_{\mathbb{D}} |f(z)|^p (1-|z|)^\alpha d\mu(z) \asymp \int_{\mathbb{T}} \int_{\Gamma_\alpha(\xi)} |f(z)|^p (1-|z|)^{\alpha-1} d\mu(z) d\xi, \tag{4}$$

valid for all $f \in H(\mathbb{D})$ and $0 < p, \alpha < \infty$. We also have, for $f \in H(\mathbb{D})$,

$$\left(\int_{\mathbb{D}} |f(z)|(1-|z|)^\alpha dA(z) \right)^p \le C \int_{\mathbb{D}} |f(z)|^p (1-|z|)^{\alpha p + 2p - 2} dA(z), \tag{5}$$

where $\alpha > \frac{1-2p}{p}$ and $0 < p \leq 1$. Next, for $\beta > 0$, $q \geq 1$, $\alpha > -1$, $\alpha > -\frac{1}{q}$ and $\epsilon > 0$ we have

$$\left(\int_{\mathbb{D}} \frac{|f(w)|(1-|w|)^{\alpha}}{|1-\overline{w}z|^{\beta+2}} dA(w)\right)^{q} \leq C \int_{\mathbb{D}} \frac{|f(w)|^{q}(1-|w|)^{\alpha q}(1-|z|)^{-\epsilon q}}{|1-\overline{w}z|^{(\beta-\epsilon)q+2}} dA(w) \tag{6}$$

for $f \in H(\mathbb{D})$. For the above two inequalities we refer to [12], [3]. Note that the estimate (5) is valid also if f is only harmonic. Namely, the proof of this estimate is based only on subharmonicity of $|f|$, combined with properties of so-called dyadic decomposition of the unit disk (see [3]). This remark is relevant for our discussion after the proof of our first result below.

One of our aims is to use results on p-Carleson measures to obtain results on continuity for Bergman-type projections on Hardy–Lorentz analytic function spaces with quasinorms which are based on Lorentz spaces on the unit circle. We present proofs of some new results on p-Carleson measures which were known before for particular values of parameters. Let us note that characterizations of p-Carleson measures via Luzin's area operator are crucial for our considerations. We alert the reader to the fact that characterizations of the classical analytic Hardy spaces and mixed norm spaces $A_s^{p,q}(\mathbb{D})$ and $F_s^{p,q}(\mathbb{D})$ via Luzin's area operator are well known and have numerous applications in the theory of analytic function spaces (see [12]). The action of Bergman-type projections on various spaces of analytic functions is a topic of great interest in function theory (see, for example, [12], [3], [14] and references therein).

We will use also the following estimate from [2] (see also [12]). For $f, g \in H(\mathbb{D})$ we have

$$\int_{\mathbb{D}} |f(z)|^{p_1}|g(z)|^{p_2}(1-|z|)^{\alpha} dA(z) \asymp \int_{\mathbb{T}} \int_{\Gamma(\xi)} |f(z)|^{p_1}|g(z)|^{p_2}(1-|z|)^{\alpha-1} dA(z) d\xi, \tag{7}$$

where $\alpha > 0$ and $0 < p_1, p_2 < \infty$.

Let us note that (4) and (7) are special cases of the following relation:

$$\int_{\mathbb{D}} G(z) d\mu(z) \asymp \int_{\mathbb{T}} \left(\int_{\Gamma_{\alpha}(\xi)} \frac{G(z)}{1-|z|} d\mu(z)\right) d\xi, \tag{8}$$

valid for all Borel measures μ on $\mathbb{D}$ and all positive measurable functions G on $\mathbb{D}$. Under the same assumptions this yields the following one-sided estimate

$$\int_{\mathbb{D}} |f(z)g(z)| d\mu(z) \leq C \int_{\mathbb{D}} \sup_{\Gamma_t(\xi)} |f(z)| \int_{\Gamma_t(\xi)} \frac{|g(z)|}{1-|z|} d\mu(z) d\xi. \tag{9}$$

There is also a "pointwise" version of (8):

$$\int_0^1 |G(r\xi)|d\mu(r) \le C \int_{\Gamma_t(\xi)} |G(z)| \frac{d\tilde{\mu}(z)}{1-|z|}, \quad z \in \mathbb{D}, \ G \in H(\mathbb{D}), \tag{10}$$

where $d\mu$ is a measure on $[0,1]$ and $d\tilde{\mu}$ is the product of measures $d\mu(r)$ and $\frac{1}{2\pi} r d\theta$.

Definition 3 ([16])**.** Let $0 < p, q, r < \infty$. $T^{\infty,q}_{r,p}(\mathbb{D}) = T^{\infty,q}_{r,p}$ denotes the space of all measurable functions f on $\mathbb{D}$ such that

$$\sup_{I \subset \mathbb{T}} \frac{1}{|I|^r} \int_{T(I)} \frac{|f(z)|^q}{(1-|z|)^p} dA(z) < \infty.$$

For particular values of parameters these classes were extensively studied (see, for example, [17], [16]).

The Bergman-type operator is one of the main objects on which we will focus our attention in this paper. The action of various versions of operators of this type in various domains in $\mathbb{C}^n$ was studied by many authors during past several decades (see [17], [3], [16], [12], [14] and various references contained in these papers). We are interested in the following Bergman-type operator. Let

$$L_{k,s}g(w) = D^{-k} \int_{\mathbb{D}} \frac{(1-|z|)^{k-s-1} g(z)}{(1-w\bar{z})^{1+2(k-s)}} dA(z) = D^{-k} \tilde{L}_{k,s} g(w), \quad w \in \mathbb{D},$$

where D^{-k} denotes fractional integration of order k, where k is an integer. This Bergman-type operator was investigated in [12]. It was proved there that $L_{k,s}$ maps $T^{\infty,q}_{1,1}$ into $F^{\infty,q}_{s,1}$ provided $s \in \mathbb{R}$, $k > s$ and $1 < q < \infty$. This was used in [12] to establish important duality between $F^{\infty,q'}_s$ and $F^{1,q}_s$, $s \in \mathbb{R}$, $1/q + 1/q' = 1$. One of the main results of the present paper is a partial extension of this result, moreover we provide a new approach to estimates of such type. In addition, this approach allows to extend our result below to the case of the unit polydisk in a standard way. We note that in order to define such operators in a polydisk one has to increase the number of variables in the definition above in a standard manner, see the last chapters of [3].

Theorem 1 below, which can be found in [1], will be under attention in this note. We will partially extend it in the next section. This theorem describes how the operator of fractional integration acts on the scale of Hardy–Lorentz analytic function spaces on the unit disc based on Lorentz spaces on the unit circle.

Theorems of this kind have a long history, they are well known not only in the spaces of analytic functions in the unit disk and higher dimension, but also in various function spaces in Euclidean space $\mathbb{R}^n$. For this topic we refer reader to a classical book [15] and an extensive survey article [14].

Results of this type have various interesting applications in function theory, for example, in various embedding theorems. Note that the first results of this kind in Bergman and Hardy and mixed norm analytic function spaces in the unit disk can be found in the early work of Hardy and Littlewood (see [4], [3]). Later these results were extended in various directions to various other spaces and even various regions in $\mathbb{C}^n$.

Theorem 1. 1) *Let $0 < p < r < \infty$, $\alpha = 1/p - 1/r$. Then the operator $D^{-\alpha}$ of fractional integration is bounded from H^{p,s_1} to H^{r,s_2} for $s_1 = s_2$, that is $\|D^{-\alpha} f\|_{H^{r,s}} \leq C\|f\|_{H^{p,s}}$ for $0 < s \leq \infty$.*

2) *Let $0 < p < q < \infty$ and $t = 1/p - 1/q$. Then the operator D^{-t} is bounded from $h_1^{p,\infty}(\mathbb{T})$ to $H^{q,\infty}(\mathbb{T})$, where*

$$h_1^{p,s}(\mathbb{T}) = \left\{ f \in H(\mathbb{D}) : \sup_{r<1} \|f(r\xi)\|_{L^{p,s}} < \infty \right\} \quad 0 < s \leq \infty.$$

We alert the reader that our results can be considered as a continuation of intensive investigation started relatively recently in [7] and [8] of analytic Hardy–Lorentz spaces in the unit disk, though the approach we present below is completely different from those in the cited papers. Some new results on these spaces can be found in a recent note [6].

2. Main results

In this section we collected all main results of this paper. We begin this section with a continuity result for the Bergman-type operator $L_{k,s}$. In the case of the unit ball and all values of k, but with special values of parameters $p = 1$ and $r = 1$, this result can be found in [12], with a proof different from ours. On the other hand, the fact that Bergman-type projections are continuous in spaces of r-Carleson measures for $0 < r < 1$ is known in the context of Q_p spaces and can be found, for example, in [18] and [17]. Nevertheless we present a different, simpler proof which allows generalizations.

Theorem 2. *Let $1 < p < 2$, $1 < q < \infty$, $0 < r < 1$ and $k > s + 1 - \frac{1}{q}$ and $k > \frac{1}{q} + s$ Then the operator $L_{k,s}$ maps $T^{\infty,q}_{r,p}$ continuously into $F^{\infty,q}_{s,r}$.*

Proof. Let $g \in T^{\infty,q}_{r,p}$ and set $\phi = L_{k,s} g$. Then we have

$$\begin{aligned}\|\phi\|_{F^{\infty,q}_{s,r}} &= \|L_{k,s} g\|_{F^{\infty,q}_{s,r}} = \|D^{-k}\tilde{L}_{k,s} g\|_{F^{\infty,q}_{s,r}} \\ &= \|\, |D^k[D^{-k}\tilde{L}_{k,s} g](w)|^q (1-|w|)^{(k-s)q-1} dA(w)\|_r \\ &= \|\, |\tilde{L}_{k,s} g(w)|^q (1-|w|)^{(k-s)q-1} dA(w)\|_r.\end{aligned}$$

Using (6) we obtain

$$|\tilde{L}_{k,s} g(w)|^q \leq C \int_{\mathbb{D}} \frac{|g(z)|^q (1-|w|)^{-\epsilon q} (1-|z|)^{(k-s-1)q}}{|1-\overline{w}z|^{(2(k-s)-1-\epsilon)q+2}} dA(z)$$

for every $\epsilon > 0$, and our task is to prove that

$$d\mu(w) = \int_{\mathbb{D}} \frac{|g(z)|^q(1-|w|)^{-\epsilon q}(1-|z|)^{(k-s-1)q}}{|1-\overline{w}z|^{(2(k-s)-1-\epsilon)q+2}} dA(z)(1-|w|)^{(k-s)q-1}dA(w)$$

is an r-Carleson measure. We fix an aperture $t > 1$ and set $\Gamma(\xi) = \Gamma_t(\xi)$. Fubini's theorem and (8) give, for any $\tau \in \mathbb{T}$, the following pointwise estimate:

$$\begin{aligned}\psi(\tau) &= \int_{\Gamma(\tau)} \frac{d\mu(w)}{1-|w|}\\ &= \int_{\mathbb{D}} |g(z)|^q(1-|z|)^{q(k-s-1)} \int_{\Gamma(\tau)} \frac{(1-|w|)^{(k-s)q-2-\epsilon q}}{|1-w\overline{z}|^{q(-1-\epsilon+2(k-s))+2}} dA(w)dA(z)\\ &\le C \int_{\mathbb{T}} \int_{\Gamma(\xi)} |g(z)|^q(1-|z|)^{q(k-s-1)-1}\\ &\quad \times \int_{\Gamma(\tau)} \frac{(1-|w|)^{(k-s)q-2-\epsilon q}dA(w)}{|1-w\overline{z}|^{q(-1-\epsilon+2(k-s))+2}} dA(z)d\xi.\end{aligned}$$

Set $t_1 = p + (k-s-1)q$ and $t_2 = -\epsilon q + (k-s)q + 2 - p$ so that $t_1 + t_2 = (2(k-s)-1-\epsilon)q + 2$. Using a well-known estimate of the integral of the Bergman kernel over Luzin's cone (see [3]) and elementary inequality $|1-w\overline{z}| \ge 1-|z|$, $z, w \in \mathbb{D}$, we conclude that $\psi(\tau)$ is estimated by

$$\begin{aligned}&\int_{\mathbb{T}}\int_{\Gamma(\xi)} |g(z)|^q(1-|z|)^{q(k-s-1)-1-t_1} \int_{\Gamma(\tau)} \frac{(1-|w|)^{-\epsilon q+(k-s)q+2-p}dA(w)}{|1-w\overline{z}|^{t_2}} dA(z)d\xi\\ &\qquad \le C \int_{\mathbb{T}} \int_{\Gamma(\xi)} \frac{|g(z)|^q}{(1-|z|)^{p+1}} \frac{1}{|1-\overline{\tau}z|^{2-p}} dA(z)d\xi\\ &\qquad \le C \int_{\mathbb{T}} \sup_{z\in\Gamma(\xi)} \frac{1}{|1-\overline{\tau}z|^{2-p}} \int_{\Gamma(\xi)} \frac{|g(z)|^q dA(z)}{(1-|z|)^{p+1}} d\xi\\ &\qquad = C\ u * v(\tau),\end{aligned}$$

where

$$u(\xi) = \sup_{z\in\Gamma(\xi)} \frac{1}{|1-z|^{2-p}}, \quad \xi \in \mathbb{T},$$

and

$$v(\xi) = \int_{\Gamma(\xi)} \frac{|g(z)|^q dA(z)}{(1-|z|)^{p+1}}, \quad \xi \in \mathbb{T}.$$

Therefore we proved the estimate

$$\psi(\tau) = \int_{\Gamma(\tau)} \frac{d\mu(w)}{1-|w|} \le Cu * v(\tau), \quad \tau \in \mathbb{T}. \tag{11}$$

Since $2 - p < 1$, we have $u \in L^1$, and since $g \in T^{\infty,q}_{r,p}$, we have $v \in L^{\rho,\infty}$, where $\rho = 1/(1-r) > 1$ due to $0 < r < 1$. Next we use an estimate from

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the Lorentz space function theory (see [5]),

$$\|u * v\|_{L^{\rho,\infty}} \le C\|u\|_{L^1}\|v\|_{\rho,\infty},$$

which, together with (11), implies

$$\|\psi\|_{\rho,\infty} \le C\|u\|_1\|v\|_{\rho,\infty} \le C\|g\|_{T^{\infty,q}_{r,p}}.$$

Now one can apply (1), with ρ in place of q and $r = 1 < \rho$, to deduce that the third quantity in (3) is finite for our measure $d\mu$. Therefore $d\mu$ is an r-Carleson measure and, moreover, $\|d\mu\|_r \le C\|g\|_{T^{\infty,q}_{r,p}}$. This completes the proof. □

The above theorem, in various forms and for special values of parameters, appeared in [12], [18] and later in books on Q_p spaces we mentioned, it was stated for functions in the unit ball as well. The above proof is based on completely different ideas related to weak Lorentz spaces. This proof admits generalizations to polydisk and, moreover, to various spaces defined by the expressions

$$\left\|\left(\int_{\Gamma_t(\xi)} |f(z)|^r(1-|z|)^s dA(z)\right)^{1/r}\right\|_{L^{p,q}}, \quad \left\|\sup_{\Gamma_t(\xi)} |f(z)|(1-|z|)^s\right\|_{L^{p,q}},$$

where $0 < r, s, p, q < \infty$. In fact, at the last step of the proof one uses various substitutes for the inequality $\|f * g\|_{L^{v,\infty}} \le C\|g\|_{L^{v,\infty}}\|f\|_{L^1}$.

Next we can consider the same $L_{k,s}$ Bergman-type operator acting on harmonic subspaces of $T^{\infty,q}_{r,p}$ classes. These type projection theorems are also well known in the literature (see, for example, [3]), where such projections from harmonic spaces to analytic spaces are considered for classical Bergman spaces. In the harmonic case we can get results analogous to the theorem we just proved, using (5) instead of (6) at the first step of the proof and remarks after (6).

The next two theorems contain some new results related to Theorem 1, their proofs are based on ideas used in the proof of Theorem 2. Namely, we extend certain known results on the action of fractional derivatives in the classical Hardy and Bergman spaces to some new analytic spaces in the unit disk based on Lorentz classes on the unit circle. We remark that some of these results were stated without proofs in [13]. The idea here again is to combine (3) with (1), i.e., to combine characterization of p-Carleson measures via Luzin's area integral with description of an equivalent quasinorm in weak Lorentz spaces.

Theorem 3. 1) *Let* $0 < s \le \infty$, $1 < v, q, r < \infty$, $1/v + 1/q = 1 + 1/r$ *and* $\beta \ge 0$. *Then for* $f \in H(\mathbb{D})$ *we have*

$$\|\sup_{z\in\Gamma_t(\xi)} |D^{\beta+1/q-1}f(z)|(1-|z|)^{\beta}\|_{L^{r,s}} \le C\|f\|_{H^{v,s}}. \tag{12}$$

2) *If* $1 < p < r$, $0 < s, s_1 \le \infty$, $1/s - 1/s_1 \ge 1/r - 1/p + 1$ *and* $1/p - 1/r < \alpha < 1$, *then for* $f \in H(\mathbb{D})$ *we have*

$$\|D^{-\alpha} f\|_{H^{r,s}} \le C\|f\|_{H^{p,s_1}}. \tag{13}$$

Theorem 4. 1) *Let* $1 < p < q < \infty$, $\alpha \ge 0$, $\tau > 0$ *and* $t = 1/p - 1/q$. *Then for* $f \in H(\mathbb{D})$ *we have*

$$\left\| \sup_{z\in\Gamma_s(\xi)} |D^{\alpha-t} f(z)|(1-|z|)^{\alpha} \right\|_{L^{q,\tau}} \le C\|f\|_{h^{p,\tau}(\mathbb{T})}. \tag{14}$$

2) *Let* $0 < s \le \infty$, $\theta > 0$ *and* $1 < v < r$. *Then for* $f \in H(\mathbb{D})$ *we have*

$$\begin{aligned} &\left\| \sup_{z\in\Gamma_t(\xi)} |D^{1/r-1/v} f(z)| \right\|_{L^{r,s}} \\ &\le C \left\| \left(\int_{\Gamma_t(\xi)} |D^{\theta} f(z)|^2 (1-|z|)^{2\theta-2} dA(z) \right)^{1/2} \right\|_{L^{v,s}}. \end{aligned} \tag{15}$$

Outline of proofs. We first focus our attention to estimates (12) and (14). These estimates are proved using similar arguments so we will provide a detailed proof of (12) and indicate small changes needed for the proof of (14). We use dilates $f_\rho(z) = f(\rho z)$, where $\rho < 1$, in order to avoid difficulties with boundary values of f which may not exist. Applying the Littlewood–Paley inequality to f_ρ we pass from integration over the unit circle to integration over the unit disk:

$$\begin{aligned} &|D^{-\alpha+\tau} f_\rho(z)|(1-|z|)^{\tau} \\ &\quad = C\left| \int_{\mathbb{T}} f_\rho(\xi) \left[D^{-\alpha+\tau}\left(\frac{1}{1-\bar{\xi}z}\right)\right] (1-|z|)^{\tau} d\xi \right| dA(w) \\ &\quad \le C \int_{\mathbb{D}} |D^{-\beta} f_\rho(w)| \frac{(1-|z|)^{\tau}(1-|w|)^{2\alpha-1-\beta}}{|1-\overline{w}z|^{\alpha+\tau+1}} dA(w). \end{aligned} \tag{16}$$

Now using (2) and (7) and letting $\rho \to 1$ we obtain the estimate

$$\begin{aligned} &\left\| \sup_{z\in\Gamma_t(\xi)} |D^{-\alpha+\tau} f(z)|(1-|z|)^{\tau} \right\|_{L^{r,s}} \\ &\le C \left\| \sup_{w\in\Gamma_t(\xi)} |D^{-\beta} f(w)| \right\|_{L^{p,s_1}} \times \left\| \int_{\Gamma_t(\xi)} \frac{(1-|w|)^{2\alpha-2-\beta}}{|1-\overline{w}\xi|^{\alpha+1}} dA(w) \right\|_{L^{q,s_2}} \\ &= CI \times J, \end{aligned} \tag{17}$$

where $1/p + 1/q = 1 + 1/r$ and $1/s_1 + 1/s_2 = 1/s$. Clearly, $J = J(\alpha, \beta, s_2, q)$ and our next step is to show that $J < \infty$ for a suitable choice of parameters involved.

Let us prove (12). Set, in (17), $\beta = 0$ and $\alpha = 1 - 1/q$. Now we apply (17), with p replaced by v, s_2 by $+\infty$, τ by β and s_1 by s, to obtain

$$\| \sup_{z\in\Gamma_t(\xi)} |D^{\beta+1/q-1} f(z)|(1-|z|)^{\beta}\|_{L^{r,s}} \le C\|f\|_{H^{v,s}} J,$$

where J is the $L^{q,\infty}$ norm of the function

$$\phi(\xi) = \int_{\Gamma_t(\xi)} \frac{(1-|w|)^{2\alpha-2}}{|1-\overline{w}|^{\alpha+1}} dA(w) = \int_{\Gamma_t(1)} \frac{(1-|w|)^{2\alpha-2}}{|1-\overline{w}\xi|^{\alpha+1}} dA(w), \quad \xi \in \mathbb{T}. \tag{18}$$

Since $\alpha = 1 - 1/q$, we obtain, by elementary estimates, $\phi(\xi) \le C|1-\xi|^{-1/q}$. Therefore we have $\|\phi\|_{q,\infty} < +\infty$ and (12) is proved.

It is of some interest to note that there is another proof of (12). Namely, one can set $\beta = \alpha - 1 + 1/q$ in (17) where $2\alpha > \beta$. Then, using (3) and so-called composition formula, see Lemma 2.5 in [11], we obtain

$$J \le C \sup_{w,\tilde{w}\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|z|)^{2\alpha-\beta-1}(1-|\tilde{w}|)^{1-1/q} dA(z)}{|1-\overline{z}w|^{\alpha+1}|1-\overline{z}\tilde{w}|^{2-2/q}} \le C < \infty,$$

and this suffices to establish (12).

Note that the proof of (14) in Theorem 4 is similar to the first of the above proofs of (12) in Theorem 3. Indeed, the integration over Luzin's cone should be replaced by integration over the unit interval in the above arguments. Using (10) the integral over the unit interval can be estimated from above by an integral over Luzin's cone (see [12], [11]). Therefore we obtain an analogue of (17) and proceed as in the above proof of (12), we leave details to the interested reader.

Next we consider the estimate (13). Since $H^{p,s_1'} \hookrightarrow H^{p,s_1''}$ for $s_1' < s_1''$, we can assume that $\frac{1}{s} - \frac{1}{s_1} = \frac{1}{r} - \frac{1}{p} + 1$. Let us set, in (17), $\tau = \beta = 0$ and choose q and s_2 so that $\frac{1}{q} = 1 - (\frac{1}{p} - \frac{1}{r}) = \frac{1}{s_2}$. Note that $q = s_2$. This choice of parameters satisfies conditions required for (17), and we obtain

$$\|D^{-\alpha} f\|_{H^{r,s}} \le C\|f\|_{H^{p,s_1}} J,$$

where J is the L^{q,s_2} norm of the function ϕ appearing in (18). As above, we have an estimate $\phi(\xi) \le C|1-\xi|^{\alpha-1}$, $\xi \in \mathbb{T}$, and the L^{q,s_2} norm is simply the L^q norm due to equality $q = s_2$. Since $\alpha - 1 > -1/q$, we see that $J < \infty$, and this proves (13). Note that the limit case $\alpha = 1/p - 1/r$ is excluded due to the fact that here J is the L^q norm of ϕ, not a weak L^q norm of ϕ as in the proof of (12).

The remaining estimate (15) can be proved in a similar manner. Let us set $a = 1/v - 1/r > 0$. Letting $\rho \to 1$ we obtain, from (16) and (8),

$$|D^{-a} f(z)| \le C \int_{\mathbb{T}} \int_{\Gamma_t(\xi)} |D^{\theta} f(w)| \frac{(1-|w|)^{2a-2+\theta} dA(w)}{|1-\overline{w}z|^{a+1}} d\xi, \quad z \in \mathbb{D}. \tag{19}$$

Next we set

$$\psi(\xi) = \left(\int_{\Gamma_t(\xi)} |D^\theta f(w)|^2 (1-|w|)^{2\theta-2} dA(w) \right)^{1/2}, \quad \xi \in \mathbb{T}$$

and deduce from (19), using Cauchy–Schwarz–Bunyakowsky inequality,

$$|D^{-a} f(z)| \le C \int_{\mathbb{T}} \psi(\xi) \left(\int_{\Gamma_t(\xi)} \frac{(1-|w|)^{4a-2} dA(w)}{|1-\overline{w}z|^{2(a+1)}} \right)^{1/2} d\xi, \quad z \in \mathbb{D}.$$

The inner integral can be estimated using well-known estimates of the integrals of weighted Bergman kernels over Luzin's cones (see [3]), as in the proof of Theorem 2:

$$\sup_{z \in \Gamma_t(\eta)} \left(\int_{\Gamma_t(\xi)} \frac{(1-|w|)^{4a-2} dA(w)}{|1-\overline{w}z|^{2(a+1)}} \right)^{1/2} \le C|\eta - \xi|^{a-1}, \quad \xi, \eta \in \mathbb{T}.$$

Therefore,

$$h(\eta) = \sup_{z \in \Gamma_t(\eta)} |D^{-a} f(z)| \le C(\psi * \phi)(\eta), \quad \eta \in \mathbb{T},$$

where $\phi(\xi) = |1-\xi|^{a-1}$ is in $L^{\frac{1}{1-a},\infty}$. Now the desired estimate follows immediately from (2). □

These two theorems extend some known results from the case of Hardy spaces or weighted Hardy spaces to the more general case of analytic Hardy–Lorentz classes. Indeed, if in our results one replaces Lorentz spaces and Lorentz (quasi) norms by standard L^p spaces and L^p norms, then these results can be found in literature (see, for example, [14], [12], [3] and references therein). Part 1 of the first theorem in the special case $\beta = 0$ is contained in Theorem 1. Also, the part 2 of the second theorem for $r = v = s$ is contained in [12] in the case of the unit ball.

We note also that some of the above estimates can be partially extended to the case of several variables, namely to functions defined on the unit polydisc in $\mathbb{C}^n$. In fact, most of the arguments can be applied to each of the variables separately. This type of procedure is quite common and appeared in literature in various topics in function theory on polydomains. We omit details. Also, some similar estimates using the same approach can be obtained for the action of the operator D^α acting *into* (not from as we had above) spaces with (quasi) norms

$$\left\| \int_{\Gamma_t(\xi)} |F(w)|(1-|w|)^{r-2} dA(w) \right\|_{L^{p,s}(\mathbb{T})},$$

where all parameters p, r, s are positive.

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University of Belgrade, Studentski Trg. 16, 11000 Belgrade, Serbia
E-mail address: arsenovic@matf.bg.ac.rs

Bryansk University, Bryansk, Russia
E-mail address: rshamoyan@gmail.com