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New integral results using Pólya–Szegö inequality

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Abstract. In this paper, we use the Riemann–Liouville fractional integral to present recent integral results using new inequalities of Pólya–Szegö type.

1. Introduction

The integral inequalities involving functions of independent variables play a fundamental role in the theory of differential equations. Significant development in this area has been achieved in the last two decades. For more details, we refer to [3, 11, 14, 15] and the references therein. The study of fractional type inequalities is also of great importance. We refer the reader to [1, 16] for further information and applications.

Let us now introduce some results that have inspired our work. We consider the functional (see [2])

$$T(f,g) := \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \frac{1}{(b-a)^2}\int_a^b f(x)\,dx \int_a^b g(x)\,dx,$$

where f and g are two integrable functions on $[a,b]$. Grüss [10] proved the inequality

$$|T(f,g)| \leq \frac{(M-m)(N-n)}{4},$$

provided that f and g are two integrable functions on $[a,b]$ satisfying

$$m \leq f(x) \leq M,\ n \leq g(x) \leq N, \quad m, M, n, N \in \mathbb{R},\ x \in [a,b].$$

Using the Pólya–Szegö inequality (see [17], pp. 213–214)

$$\frac{\int_a^b f^2(x)\,dx \int_a^b g^2(x)\,dx}{\left(\int_a^b f(x)\,dx \int_a^b g(x)\,dx\right)^2} \leq \frac{1}{4}\left(\sqrt{\frac{MN}{mn}} + \sqrt{\frac{mn}{MN}}\right)^2, \tag{1}$$

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Dragomir and Diamond [8] proved that

$$|T(f,g)| \leq \frac{(M-m)(N-n)}{4(b-a)^2\sqrt{mMnN}} \int_a^b f\,(x)\,dx \int_a^b g\,(x)\,dx, \tag{2}$$

where f and g are two positive integrable functions on $[a,b]$ such that

$$0 < m \leq f(x) \leq M < \infty,\ 0 < n \leq g(x) \leq N < \infty.$$

For other Grüss type inequalities, see [5, 4, 6, 7, 12, 13, 18] and the references cited therein.

The main aim of this paper is to establish some new fractional integral inequalities of Pólya–Szegö type. Then, we use these results to generate some other fractional integral inequalities.

2. Description of the fractional calculus

We introduce some definitions and properties concerning the Riemann–Liouville fractional integral operator.

Definition 1. A real valued function f is said to be in the space $C_\mu([0,\infty))$, $\mu \in \mathbb{R}$, if there exists a real number $p > \mu$ such that $f(t) = t^p f_1(t)$, where $f_1 \in C([0,\infty))$.

Definition 2. A function f is said to be in the space $C_\mu^n([0,\infty))$, $n \in \mathbb{N}$, if $f^{(n)} \in C_\mu([0,\infty))$.

Definition 3. The Riemann–Liouville fractional integral operator of order $\alpha \geq 0$, for a function $f \in C_\mu([0,\infty))$, $\mu \geq -1$, is defined as

$$\begin{aligned} &J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad \alpha > 0, t > 0, \\ &J^0 f(t) = f(t). \end{aligned} \tag{3}$$

For the convenience of establishing the results, we give the following property:

$$J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t).$$

For the expression (3), when $f(t) = t^\beta$, we get another expression that will be used later:

$$J^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta}.$$

For more details, see [9, 16].

3. Main results

Our first result is the following theorem.

Theorem 1. *Let f and g be two positive integrable functions on $[0,\infty)$. Suppose that there exist positive real numbers m, M, n, N such that*

$$0 < m \leq f(\tau) \leq M < \infty,\ 0 < n \leq g(\tau) \leq N < \infty, \quad \tau \in [0,t], t > 0. \tag{4}$$

Then we have

$$\begin{aligned} &\left| \frac{t^{\alpha}}{\Gamma(\alpha+1)} J^{\alpha}\Big(f(t)\,g(t)\Big) - J^{\alpha} f(t)\, J^{\alpha} g(t) \right| \\ &\qquad \leq \frac{(M-m)(N-n)}{4\sqrt{mMnN}} J^{\alpha} f(t)\, J^{\alpha} g(t), \end{aligned} \tag{5}$$

for any $\alpha > 0,\ t > 0$.

Remark 2. If we take $\alpha = 1$, then we obtain (2) on $[0,t]$.

To prove Theorem 1, we need the following lemma.

Lemma 3. *Suppose that h and l are two positive integrable functions on $[0,\infty)$ such that*

$$0 < m_1 \leq h(\tau) \leq M_1 < \infty,\ 0 < n_1 \leq l(\tau) \leq N_1 < \infty, \quad \tau \in [0,t], t > 0. \tag{6}$$

Then for any $\alpha > 0,\ t > 0$ we have

$$\frac{\left(J^{\alpha} h^2(t)\right)\left(J^{\alpha} l^2(t)\right)}{\left(J^{\alpha} h(t)\, l(t)\right)^2} \leq \frac{1}{4}\left(\sqrt{\frac{M_1 N_1}{m_1 n_1}} + \sqrt{\frac{m_1 n_1}{M_1 N_1}}\right)^2.$$

Proof. From the condition

$$\frac{h(\tau)}{l(\tau)} \leq \frac{M_1}{n_1}, \quad \tau \in [0,t], t > 0,$$

we have

$$\frac{n_1}{M_1}\left(h^2(\tau)\right) \leq h(\tau) l(\tau), \quad \tau \in [0,t], t > 0. \tag{7}$$

Multiplying both sides of (7) by $\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}$, $\tau \in (0,t)$, and integrating the resulting inequality with respect to τ over $(0,t)$, we get

$$\frac{n_1}{M_1 \Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} (h(\tau))^2\, d\tau \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} h(\tau) l(\tau) d\tau.$$

Therefore,

$$\frac{n_1}{M_1} J^{\alpha} h^2(t) \leq J^{\alpha}\Big(h(t) l(t)\Big). \tag{8}$$

Now, using the condition

$$\frac{m_1}{N_1} \leq \frac{h(\tau)}{l(\tau)}, \quad \tau \in [0,t],\ t > 0,$$

we can write

$$\frac{m_1}{N_1}J^{\alpha}l^2(t) \leq J^{\alpha}\Big(h(t)l(t)\Big). \tag{9}$$

Multiplying (8) and (9), we obtain

$$\left(\frac{n_1}{M_1}\frac{m_1}{N_1}\right)J^{\alpha}h^2(t)J^{\alpha}l^2(t) \leq \Big(J^{\alpha}\left(l(\tau)h(\tau)\right)\Big)^2.$$

This implies that

$$\frac{J^{\alpha}h^2(t)J^{\alpha}l^2(t)}{\Big(J^{\alpha}l(t)h(t)\Big)^2} \leq \frac{M_1N_1}{m_1n_1} \leq \frac{1}{4}\left(\sqrt{\frac{M_1N_1}{m_1n_1}}+\sqrt{\frac{m_1n_1}{M_1N_1}}\right)^2.$$

Lemma is thus proved. □

Remark 4. If we take $\alpha = 1$, then we obtain Pólya–Szegö inequality on $[0,t]$.

Proof of Theorem 1. Denoting

$$Q(\tau,\rho) := \left(f(\tau) - f(\rho)\right)\left(g(\tau) - g(\rho)\right),$$

we have

$$\begin{aligned}\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}Q(\tau,\rho)d\tau &= \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\Big(f(\tau)g(\tau)+f(\rho)g(\rho)\\ &\quad - f(\tau)g(\rho) - f(\rho)g(\tau)\Big)d\tau\\ &= J^{\alpha}\Big(\left(f(t)\right)g(t)\Big) + \frac{t^{\alpha}}{\Gamma(\alpha+1)}f(\rho)g(\rho)\\ &\quad - g(\rho)J^{\alpha}\left(f(t)\right) - f(\rho)J^{\alpha}\left(g(t)\right).\end{aligned}$$

Consequently,

$$\begin{aligned}&\int_0^t\int_0^t \frac{(t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}}{\Gamma^2(\alpha)}Q(\tau,\rho)d\tau d\rho\\ &\quad = 2\frac{t^{\alpha}}{\Gamma(\alpha+1)}J^{\alpha}\Big(\left(f(t)\right)g(t)\Big) - 2J^{\alpha}f(t)J^{\alpha}g(t).\end{aligned} \tag{10}$$

On the other hand, using the Cauchy–Schwartz inequality for double integrals, we can write

$$\begin{aligned}&\left|\int_0^t\int_0^t \frac{(t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}}{\Gamma^2(\alpha)}Q(\tau,\rho)d\tau d\rho\right|\\ &\leq \Bigg[\int_0^t\int_0^t \frac{(t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}}{\Gamma^2(\alpha)}f^2(\tau)d\tau d\rho\\ &\quad + \int_0^t\int_0^t \frac{(t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}}{\Gamma^2(\alpha)}f^2(\rho)d\tau d\rho\end{aligned}$$

$$
\begin{aligned}
& -2\int_0^t\int_0^t \frac{(t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}}{\Gamma^2(\alpha)} f(\rho)f(\tau)\,d\tau d\rho\Bigg]^{\frac{1}{2}} \\
\times & \left[\int_0^t\int_0^t \frac{(t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}}{\Gamma^2(\alpha)} g^2(\tau)d\tau d\rho\right. \\
& +\int_0^t\int_0^t \frac{(t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}}{\Gamma^2(\alpha)} g^2(\rho)d\tau d\rho \\
& \left.-2\int_0^t\int_0^t \frac{(t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}}{\Gamma^2(\alpha)} g(\rho)g(\tau)\,d\tau d\rho\right]^{\frac{1}{2}}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\left|\int_0^t\int_0^t \frac{(t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}}{\Gamma^2(\alpha)} Q(\tau,\rho)d\tau d\rho\right| \\
&\qquad \le 2\left[\frac{t^\alpha}{\Gamma(\alpha+1)}J^\alpha f^2(t)-\Big(J^\alpha f(x)\Big)^2\right]^{\frac{1}{2}} \\
&\qquad\quad \times\left[\frac{t^\alpha}{\Gamma(\alpha+1)}J^\alpha g^2(t)-\Big(J^\alpha g(x)\Big)^2\right]^{\frac{1}{2}}.
\end{aligned}
\tag{11}
$$

Using Lemma 3, we get

$$
\begin{aligned}
\frac{t^\alpha}{\Gamma(\alpha+1)}J^\alpha f^2(x) &\le \frac{1}{4}\left(\sqrt{\frac{M}{m}}+\sqrt{\frac{m}{M}}\right)^2 (J^\alpha f(x))^2 \\
&= \frac{(M+m)^2}{4mM}(J^\alpha f(x))^2.
\end{aligned}
$$

Hence

$$
\frac{t^\alpha}{\Gamma(\alpha+1)}J^\alpha f^2(x)-(J^\alpha f(x))^2 \le (\frac{(M+m)^2}{4mM}-1)(J^\alpha f(x))^2, \tag{12}
$$

that is

$$
\frac{t^\alpha}{\Gamma(\alpha+1)}J^\alpha f^2(x)-(J^\alpha f(x))^2 \le \frac{(M-m)^2}{4mM}(J^\alpha f(x))^2.
$$

In a similar fashion, we obtain

$$
\frac{t^\alpha}{\Gamma(\alpha+1)}J^\alpha g^2(x)-(J^\alpha g(x))^2 \le \frac{(N-n)^2}{4nN}(J^\alpha g(x))^2. \tag{13}
$$

Combining (10), (11), (12) and (13), we deduce the desired inequality (5). □

Our second result is the following.

Theorem 5. *Let f and g be two positive integrable functions on $[0,\infty)$. Suppose that there exist positive real numbers m,n,M,N such that (4) holds. Then we have*

$$\begin{aligned}\Bigg|\frac{t^{\alpha}}{\Gamma(\alpha+1)}J^{\beta}\Big(f(t)\,g(t)\Big)+\frac{t^{\beta}}{\Gamma(\beta+1)}J^{\alpha}\Big(f(t)\,g(t)\Big)&\\ -J^{\alpha}f(t)\,J^{\beta}g(t)-J^{\beta}f(t)\,J^{\alpha}g(t)\Bigg|&\\ \leq\frac{(M-m)(N-n)}{2\sqrt{mMnN}}J^{\alpha}f(t)\,J^{\beta}g(t)&\end{aligned}\tag{14}$$

for any $\alpha>0,\ \beta>0,\ t>0$.

To prove this theorem, we need the following lemma which is another generalization of Pólya–Szegö inequality (1).

Lemma 6. *Suppose that h and l are two positive integrable functions on $[0,\infty)$ such that (6) holds. Then for any $\alpha>0,\ \beta>0,\ t>0$, we have*

$$\frac{\big(J^{\alpha}h^{2}(t)\big)\big(J^{\beta}l^{2}(t)\big)}{\left(J^{\alpha}h(t)\,l(t)\right)\left(J^{\beta}h(t)\,l(t)\right)}\leq\frac{1}{4}\left(\sqrt{\frac{M_1N_1}{m_1n_1}}+\sqrt{\frac{m_1n_1}{M_1N_1}}\right)^{2}.\tag{15}$$

Proof. By the condition

$$\frac{m_1}{N_1}\leq\frac{h(\tau)}{l(\tau)},\qquad\tau\in[0,t],\,t>0,$$

we get

$$\frac{m_1}{N_1}J^{\beta}l^{2}(t)\leq J^{\beta}\Big(h(t)l(t)\Big).\tag{16}$$

Thanks to (8) and (16) we obtain (15). □

Proof of Theorem 5. We have

$$\begin{aligned}\int_{o}^{t}\int_{0}^{t}\frac{(t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}Q(\tau,\rho)d\tau d\rho&\\ =\frac{t^{\alpha}}{\Gamma(\alpha+1)}J^{\beta}\Big((f(t))\,g(t)\Big)+\frac{t^{\beta}}{\Gamma(\beta+1)}J^{\alpha}\Big((f(t))\,g(t)\Big)&\\ -J^{\alpha}f(t)\,J^{\beta}g(t)-J^{\beta}f(t)\,J^{\alpha}g(t)\,.&\end{aligned}\tag{17}$$

As in the proof of Theorem 1, using Cauchy–Schwartz inequality for double integrals, we find that

$$\left|\int_{0}^{t}\int_{0}^{t}\frac{(t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}Q(\tau,\rho)d\tau d\rho\right|$$

$$\leq \left[\int_0^t \int_0^t \frac{(t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} f^2(\tau) d\tau d\rho \right.$$
$$+ \int_0^t \int_0^t \frac{(t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} f^2(\rho) d\tau d\rho$$
$$\left. -2\int_0^t \int_0^t \frac{(t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} f(\rho) f(\tau) d\tau d\rho \right]^{\frac{1}{2}}$$
$$\times \left[\int_0^t \int_0^t \frac{(t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} g^2(\tau) d\tau d\rho \right.$$
$$+ \int_0^t \int_0^t \frac{(t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} g^2(\rho) d\tau d\rho$$
$$\left. -2\int_0^t \int_0^t \frac{(t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} g(\rho) g(\tau) d\tau d\rho \right]^{\frac{1}{2}}.$$

Thus

$$\left| \int_0^t \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)} Q(\tau,\rho) d\tau d\rho \right|$$
$$\leq \left[\frac{t^\alpha}{\Gamma(\alpha+1)} J^\beta f^2(t) + \frac{t^\beta}{\Gamma(\beta+1)} J^\alpha f^2(t) - 2J^\alpha f(t) J^\beta f(t) \right]^{\frac{1}{2}} \tag{18}$$
$$\times \left[\frac{t^\alpha}{\Gamma(\alpha+1)} J^\beta g^2(t) + \frac{t^\beta}{\Gamma(\beta+1)} J^\alpha g^2(t) - 2J^\alpha g(t) J^\beta g(t) \right]^{\frac{1}{2}}.$$

Using Lemma 6, we may state that

$$\frac{t^\alpha}{\Gamma(\alpha+1)} J^\beta f^2(t) - J^\alpha f(t) J^\beta f(t) \leq \frac{(M-m)^2}{4mM} J^\alpha f(t) J^\beta f(t) \tag{19}$$

and

$$\frac{t^\beta}{\Gamma(\beta+1)} J^\alpha f^2(t) - J^\alpha f(t) J^\beta f(t) \leq \frac{(M-m)^2}{4mM} J^\alpha f(t) J^\beta f(t). \tag{20}$$

For the function g, we have

$$\frac{t^\alpha}{\Gamma(\alpha+1)} J^\beta g^2(t) - J^\alpha g(t) J^\beta g(t) \leq \frac{(N-n)^2}{4nN} J^\alpha g(t) J^\beta g(t) \tag{21}$$

and

$$\frac{t^\beta}{\Gamma(\beta+1)} J^\alpha g^2(t) - J^\alpha g(t) J^\beta g(t) \leq \frac{(N-n)^2}{4nN} J^\alpha g(t) J^\beta g(t). \tag{22}$$

Now, in view of (17) and (18)–(22), we obtain (14). □

Remark 7. Applying Theorem 5, for $\alpha = \beta$ we obtain Theorem 1, and for $\alpha = \beta = 1$ we obtain (4) on $[0, t]$.

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