ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 17, Number 2, December 2013
Available online at http://acutm.math.ut.ee

On generalized sequence spaces defined by modulus functions

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Abstract. Let $\left(X, |\cdot|\right)$ be a seminormed space, $\Phi = (\phi_k)$ a sequence of moduli, and $\mathcal{B}$ a sequence of infinite scalar matrices $B^i = (b^i_{kj})$. Let (λ, g_λ) and (Λ, g_Λ) be solid F-seminormed (paranormed) spaces of single and double number sequences, respectively. V. Soomer and E. Kolk proved in 1996-1997 that the set of all scalar sequences $\mathbf{u} = (u_k)$ with $\Phi(\mathbf{u}) = (\phi_k(|u_k|)) \in \lambda$ is a linear space which may be topologized by the F-seminorm (paranorm) $g_{\lambda,\Phi}(\mathbf{u}) = g_\lambda(\Phi(\mathbf{u}))$ under certain restrictions on Φ or (λ, g_λ). We generalize this result to the space of all X-valued sequences $\mathbf{x} = (x_k)$ with $\left(\phi_k\left(|\mathcal{B}^i_k \mathbf{x}|\right)\right) \in \Lambda$, where $\mathcal{B}^i_k \mathbf{x} = \sum_j b^i_{kj} x_j$. Applications are given in the case when Λ is the strong summability domain of a non-negative matrix method. Our corollaries and critical remarks outline results from more than thirty previous papers by many different authors.

1. Introduction

Let $\mathbb{N} = \{1, 2, \dots\}$ and let $\mathbb{K}$ be the field of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$. In the following we specify the domains of indices for the symbols lim, sup, inf and $\sum$ only if they are different from $\mathbb{N}$. By ι we denote the identity mapping $\iota(z) = z$. In all definitions which contain infinite series we tacitly assume the convergence of these series.

An F-space is usually understood as a complete metrizable topological vector space over $\mathbb{K}$. The topology of an F-space E can be given by an F-*norm*, i.e., by the functional $g : E \to \mathbb{R}$ with axioms (see [29], p. 13)

(N1) $g(0) = 0$,
(N2) $g(x + y) \le g(x) + g(y) \quad (x, y \in E)$,

Received October 20, 2013.

2010 *Mathematics Subject Classification.* 46A45, 40A05, 40C05, 40F05.

Key words and phrases. Double sequence, F-seminorm, F-norm, matrix method of summability, modulus function, paranorm, seminorm, sequence space.

http://dx.doi.org/10.12097/ACUTM.2013.17.16

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(N3) $|\alpha| \leq 1 \ \ (\alpha \in \mathbb{K}),\ x \in E \implies g(\alpha x) \leq g(x)$,
(N4) $\lim_n \alpha_n = 0 \ \ (\alpha_n \in \mathbb{K}),\ x \in E \implies \lim_n g(\alpha_n x) = 0$,
(N5) $g(x) = 0 \implies x = 0$.

A functional g with axioms (N1)–(N4) is called an F-*seminorm.* A *paranorm* on E is defined as a functional $g : E \to \mathbb{R}$ satisfying axioms (N1), (N2) and

(N6) $g(-x) = g(x) \quad (x \in E)$,
(N7) $\lim_n \alpha_n = \alpha \quad (\alpha_n, \alpha \in \mathbb{K}),\ \lim_n g(x_n - x) = 0 \quad (x_n, x \in E) \implies \lim_n g(\alpha_n x_n - \alpha x) = 0$.

A *seminorm* on E is a functional $g : E \to \mathbb{R}$ with axioms (N1), (N2) and

(N8) $g(\alpha x) = |\alpha| g(x) \quad (\alpha \in \mathbb{K},\ x \in E)$.

An F-seminorm (paranorm, seminorm) g is called *total* if (N5) holds. So, an F-norm (norm) is a total F-seminorm (seminorm).

It is known (see [33], Remark 1) that F-seminorms are precisely the paranorms satisfying axiom (N3).

To avoid confusion with the module $|\cdot|$, following [33], we will often denote the seminorm of an element $x \in E$ by $\dot{|}x\dot{|}$.

Let $(X, \dot{|}\cdot\dot{|})$ be a seminormed linear space over $\mathbb{K}$ and let $\mathbf{X}$ be a sequence of seminormed linear spaces $\left(X_k, \dot{|}\cdot\dot{|}_k\right)$ $(k \in \mathbb{N})$. Then the set $s^2(\mathbf{X})$ of all double sequences $\mathbf{x}^2 = (x_{ki}),\ x_{ki} \in X_k \ \ (k, i \in \mathbb{N})$, and the set $s(\mathbf{X})$ of all sequences $\mathbf{x} = (x_k),\ x_k \in X_k \ \ (k \in \mathbb{N})$, equipped with coordinatewise addition and scalar multiplication, are linear spaces (over $\mathbb{K}$). Any linear subspace of $s^2(\mathbf{X})$ is called a *generalized double sequence space* (GDS *space*) and any linear subspace of $s(\mathbf{X})$ is called a *generalized sequence space* (GS *space*). If $\left(X_k, \dot{|}\cdot\dot{|}_k\right) = \left(X, \dot{|}\cdot\dot{|}\right)$ $(k \in \mathbb{N})$, then we write X instead of $\mathbf{X}$. In the case $X = \mathbb{K}$ we omit the symbol X in our notation. So, for example, s^2 and s denote the linear spaces of all $\mathbb{K}$-valued double sequences $\mathbf{u}^2 = (u_{ki})$ and single sequences $\mathbf{u} = (u_k)$, respectively. As usual, linear subspaces of s^2 are called *double sequence spaces* (DS *spaces*) and linear subspaces of s are called *sequence spaces.* Well-known sequence spaces include the sets ℓ_∞, c, c_0 and $\ell^p \ \ (p > 0)$ of all bounded, convergent, convergent to zero and absolutely p-summable number sequences, respectively. Examples of DS spaces are

$$\mathcal{M} = \{\mathbf{u}^2 \in s^2 : \tilde{u}_k = \sup_i |u_{ki}| < \infty \quad (k \in \mathbb{N})\},$$
$$U\lambda = \{\mathbf{u}^2 \in \mathcal{M} : \tilde{u} = (\tilde{u}_k) \in \lambda\} \quad (\lambda \in \{\ell_\infty,\, c_0,\, \ell^p\}).$$

Let $\mathbb{R}^+ = [0, \infty)$. The idea of a modulus function was shaped by Nakano [37]. Following Ruckle [44] and Maddox [35] we say that a function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a *modulus function* (or, simply, a *modulus*), if

(M1) $\phi(t) = 0 \iff t = 0$,
(M2) $\phi(t + u) \leq \phi(t) + \phi(u) \ \ (t, u \in \mathbb{R}^+)$,

(M3) ϕ is non-decreasing,
(M4) ϕ is continuous from the right at 0.

For example, the function $\iota^p(t) = t^p$ is an unbounded modulus for $p \leq 1$ and the function $\phi(t) = t/(1+t)$ is a bounded modulus.

Since $|\phi(t)-\phi(u)| \leq \phi(|t-u|)$ $(t, u \in \mathbb{R}^+)$ by (M1)–(M3), the moduli are continuous everywhere on $\mathbb{R}^+$. We also remark that the modulus functions are essentially the same concept as the moduli of continuity (see [18], p. 866).

A GS space $\lambda(\mathbf{X}) \subset s(\mathbf{X})$ is called *solid* if $(y_k) \in \lambda(\mathbf{X})$ whenever $(x_k) \in \lambda(\mathbf{X})$ and $\dot{|}y_k\dot{|}_k \leq \dot{|}x_k\dot{|}_k$ $(k \in \mathbb{N})$. Analogously, a GDS space $\Lambda(\mathbf{X}) \subset s^2(\mathbf{X})$ is called *solid* if $(y_{ki}) \in \Lambda(\mathbf{X})$ whenever $(x_{ki}) \in \Lambda(\mathbf{X})$ and $\dot{|}y_{ki}\dot{|}_k \leq \dot{|}x_{ki}\dot{|}_k$ $(k, i \in \mathbb{N})$. For example, it is easy to see that the sets

$$\mathcal{M}(\mathbf{X}) = \left\{ \mathbf{x}^2 \in s^2(\mathbf{X}) : \sup_i \dot{|}x_{ki}\dot{|}_k < \infty \quad (k \in \mathbb{N}) \right\},$$

$$\Lambda(\Phi, \mathbf{X}) = \left\{ \mathbf{x}^2 \in s^2(\mathbf{X}) : \Phi(\mathbf{x}^2) = \left(\phi_k \left(\dot{|}x_{ki}\dot{|}_k \right) \right) \in \Lambda \right\}$$

and $\Lambda(\Phi, \mathcal{M}(\mathbf{X})) = \Lambda(\Phi, \mathbf{X}) \cap \mathcal{M}(\mathbf{X}))$ are solid GDS spaces if $\Lambda \subset s^2$ is a solid DS space and $\Phi = (\phi_k)$ is a sequence of moduli.

Let $B = (b_{kj})$ be an infinite scalar matrix and let $\mathcal{B}$ be a sequence of matrices $B^i = (b^i_{kj})$. For an X-valued sequence $\mathbf{x} = (x_j)$ put $B\mathbf{x} = (B_k\mathbf{x})$ and $\mathcal{B}\mathbf{x} = (\mathcal{B}^i_k\mathbf{x})$, where $B_k\mathbf{x} = \sum_j b_{kj}x_j$ and $\mathcal{B}^i_k\mathbf{x} = \sum_j b^i_{kj}x_j$. Our aim is to determine F-seminorm topologies for the spaces of X-valued sequences $\mathbf{x}$ with $\Phi(\mathcal{B}\mathbf{x})$ in Λ, or $\Phi(B\mathbf{x})$ in λ, if Λ and λ are topologized by absolutely monotone F-seminorms. Main theorems are applicable in the case if λ and Λ are strong summability domains of a non-negative matrix $A = (a_{nk})$. Some special cases of such spaces are considered, for example, in [1]–[4], [6]–[16], [19]–[25], [27], [28], [30], [31], [36], [38]–[41], [43], [45] and [48]–[51]).

2. Main results

The most common summability method is the matrix method defined by an infinite scalar matrix $A = (a_{nk})$. If for a sequence $\mathbf{x} \in s(X)$ the series $A_n\mathbf{x} = \sum_k a_{nk}x_k$ $(n \in \mathbb{N})$ converge and the limit $\lim_n A_n\mathbf{x} = l$ exists in X, then we say that $\mathbf{x}$ *is summable to l by the method A* (briefly, *A-summable to l*) and write $A\text{-}\lim x_k = l$. A summability method (or a matrix) A is called *regular in X* if for all sequences $\mathbf{x} = (x_k)$ convergent in X we have

$$\lim_k x_k = l \implies \lim_n A_n\mathbf{x} = l.$$

A well-known example of a regular matrix method is the Cesàro method C_1 defined by the matrix $C_1 = (c_{nk})$, where, for any $n \in \mathbb{N}$, $c_{nk} = n^{-1}$ if $k \leq n$ and $c_{nk} = 0$ otherwise. A (trivial) regular method is defined by the *unit matrix* $I = (i_{nk})$, where $i_{nn} = 1$ and $i_{nk} = 0$ for $n \neq k$. Recall also that a matrix $A = (a_{nk})$ is called *normal* if, for any $n \in \mathbb{N}$, $a_{nn} \neq 0$ and $a_{nk} = 0$ if $k > n$. For example, the Cesàro matrix C_1 is normal. Every scalar sequence (c_k) defines a *diagonal matrix* $D(c_k) = (d_{ni})$ by the equalities $d_{nn} = c_n$ and $d_{ni} = 0$ if $n \neq i$. Clearly, a diagonal matrix $D(c_k)$ is regular if and only if $\lim_k c_k = 1$, and it is normal if $c_k \neq 0$ for all $k \in \mathbb{N}$.

Another class of summability methods is determined by sequences $\mathcal{B} = (B^i)$ of infinite scalar matrices $B^i = (b^i_{nk})$. Recall (see, for example, [5] and [47]) that a sequence $\mathbf{x} = (x_k) \in s(X)$ is called $\mathcal{B}$-*summable* to the point $l \in X$ if B^i-$\lim x_k = l$ uniformly in i, i.e., if the series $B^i_n\mathbf{x} = \sum_k b^i_{nk} x_k$ $(n, i \in \mathbb{N})$ converge in X and

$$\lim_n \left| B^i_n \mathbf{x} - l \right| = 0 \text{ uniformly in } i.$$

The summability methods $\mathcal{B}$ are also known as the *sequential matrix methods* (SM *methods*) of summability (see [17], p. 19). In the special case

$$b^i_{nk} = \begin{cases} \frac{1}{n}, & \text{if } i \leq k \leq n + i - 1, \\ 0 & \text{otherwise} \end{cases}$$

the $\mathcal{B}$-summability reduces to the so-called *almost convergence* (see [34]). The almost convergence is a non-matrix method of summability. Any matrix method B can be considered as an SM method $\mathcal{B}$ with $B^i = B$ $(i \in \mathbb{N})$. By the unit SM method $\mathcal{I}$ we mean the SM method $\mathcal{B}$ with $B^i = I$ $(i \in \mathbb{N})$.

Let $\mathbf{e}^k = (e^k_j)_{j\in\mathbb{N}}$ $(k \in \mathbb{N})$ be the sequences with the elements $e^k_j = 1$ if $j = k$ and $e^k_j = 0$ otherwise. If we define, for an arbitrary sequence $\mathbf{z} = (z_k)$, the double sequence $\mathbf{z}^{(2)} = (z^{(2)}_{ki})$ with $z^{(2)}_{ki} = z_k$ $(k, i \in \mathbb{N})$, then every sequence $\mathbf{e}^k$ $(k \in \mathbb{N})$ also determines a double sequence $\mathbf{e}^{k(2)} = (e^k_{ji})_{j,i\in\mathbb{N}}$ such that, for all $i \in \mathbb{N}$, $e^k_{ji} = 1$ if $j = k$ and $e^k_{ji} = 0$ if $j \neq k$. An F-seminormed sequence space (λ, g_λ) is called an AK-*space*, if λ contains the sequences $\mathbf{e}^k$ $(k \in \mathbb{N})$ and for any $\mathbf{u} = (u_k) \in \lambda$ we have $\lim_n g_\lambda \left(\mathbf{u} - \mathbf{u}^{[n]}\right) = 0$, where $\mathbf{u}^{[n]} = \sum_{k=1}^n u_k \mathbf{e}^k$. Analogously, an F-seminormed DS space (Λ, g_Λ) is called an AK-*space* (see [42]), if Λ contains the sequences $\mathbf{e}^{k(2)}$ $(k \in \mathbb{N})$ and for any $\mathbf{u}^2 = (u_{ki}) \in \Lambda$ we have $\lim_n g_\Lambda \left(\mathbf{u}^2 - \mathbf{u}^{2[n]}\right) = 0$, where $\mathbf{u}^{2[n]} = \sum_{k=1}^n \mathbf{u}_k \mathbf{e}^{k(2)}$ with $\mathbf{u}_k = (u_{ki})_{i\in\mathbb{N}}$ and $\mathbf{u}_k \mathbf{e}^{k(2)} = (u_{ki} e^k_{ji})_{j,i\in\mathbb{N}}$. Well-known AK-spaces are c_0 and ℓ^p $(p \geq 1)$ with respect to ordinary norms $\|\mathbf{u}\|_\infty = \sup_k |u_k|$ and $\|\mathbf{u}\|_p = (\sum_k |u_k|^p)^{1/p}$. It is not difficult to see that Uc_0 and $U\ell^p$ $(p \geq 1)$, topologized by norms $\|\mathbf{u}\|_{\tilde{\infty}} = \|\tilde{\mathbf{u}}\|_\infty$ and $\|\mathbf{u}\|_{\tilde{p}} = \|\tilde{\mathbf{u}}\|_p$, are examples of normed DS-AK-spaces.

Let $\Phi = (\phi_k)$ be a sequence of moduli. If λ is a solid sequence space, then

$$\lambda(\Phi) = \{\mathbf{u} = (u_k) \in s : \Phi(\mathbf{u}) = (\phi_k(|u_k|)) \in \lambda\}$$

is also a solid sequence space. Soomer [46] and Kolk [32] proved that if λ is topologized by an absolutely monotone F-seminorm g_λ, i.e., $g_\lambda(\mathbf{v}) \leq g_\lambda(\mathbf{u})$ for all $\mathbf{u}, \mathbf{v} \in \lambda$ with $|v_k| \leq |u_k|$ $(k \in \mathbb{N})$, then $\lambda(\Phi)$ may be topologized by the absolutely monotone F-seminorm $g_{\lambda,\Phi}(\mathbf{u}) = g_\lambda(\Phi(\mathbf{u}))$ whenever (λ, g_λ) is an AK-space or the sequence Φ satisfies one of two (equivalent) conditions:

(M5) there exist a function ν and a number $\delta > 0$ such that $\phi_k(ut) \leq \nu(u)\phi_k(t)$ $(k \in \mathbb{N},\ 0 < u < \delta,\ t > 0)$ and $\lim_{u\to 0+} \nu(u) = 0$,

(M6) $\lim\limits_{u\to 0+} \sup\limits_{t>0} \sup\limits_{k} \dfrac{\phi_k(ut)}{\phi_k(t)} = 0.$

In the following we prove the similar statements about the sets

$$\lambda\,(\Phi, B, X) = \left\{\mathbf{x} \in s(X) : \Phi(B\mathbf{x}) = \left(\phi_k\left(\left|B_k\mathbf{x}\right|\right)\right) \in \Lambda\right\},$$
$$\Lambda(\Phi, \mathcal{B}, X) = \left\{\mathbf{x} \in s(X) : \Phi(\mathcal{B}\mathbf{x}) = \left(\phi_k\left(\left|\mathcal{B}_k^i\mathbf{x}\right|\right)\right) \in \Lambda\right\},$$
$$\Lambda(\Phi, \mathcal{B}, \mathcal{M}(X)) = \{\mathbf{x} \in s(X) : \mathcal{B}\mathbf{x} \in \mathcal{M}(X)\} \cap \Lambda(\Phi, \mathcal{B}, X),$$

where the sequence spaces λ, Λ are solid, B is a matrix method, and $\mathcal{B}$ is an SM-method of summability.

Theorem 1. *If λ and Λ are solid sequence space, then the sets $\lambda\,(\Phi, B, X)$, $\Lambda(\Phi, \mathcal{B}, X)$ and $\Lambda(\Phi, \mathcal{B}, \mathcal{M}(X))$ are GS spaces, i.e., linear subsets of $s(X)$. Moreover, $\Lambda(\Phi, \mathcal{B}, X)$ and $\Lambda(\Phi, \mathcal{B}, \mathcal{M}(X))$ are solid if*

$$\left|y_k\right| \leq \left|x_k\right| \implies \left|\mathcal{B}_k^i\mathbf{y}\right| \leq \left|\mathcal{B}_k^i\mathbf{x}\right| \ \ (k, i \in \mathbb{N}), \tag{1}$$

and $\lambda\,(\Phi, B, X)$ is solid if

$$\left|y_k\right| \leq \left|x_k\right| \implies \left|B_k\mathbf{y}\right| \leq \left|B_k\mathbf{x}\right| \ \ (k \in \mathbb{N}). \tag{2}$$

Proof. To prove the linearity of the set $\Lambda(\Phi, \mathcal{B}, X)$, fix $\alpha, \beta \in \mathbb{K}$ and $\mathbf{x}, \mathbf{y} \in \Lambda(\Phi, \mathcal{B}, X)$. Using the linearity of the operators $\mathcal{B}_k^i$, by axioms (M2) and (M3) we have

$$\begin{aligned}\phi_k\left(\left|\mathcal{B}_k^i\left(\alpha\mathbf{x} + \beta\mathbf{y}\right)\right|\right) &\leq \phi_k\left(|\alpha|\left|\mathcal{B}_k^i\mathbf{x}\right|\right) + \phi_k\left(|\beta|\left|\mathcal{B}_k^i\mathbf{y}\right|\right)\\ &\leq ([|\alpha|] + 1)\phi_k\left(\left|\mathcal{B}_k^i\mathbf{x}\right|\right) + ([|\beta|] + 1)\phi_k\left(\left|\mathcal{B}_k^i\mathbf{y}\right|\right)\end{aligned}$$

for all $k, i \in \mathbb{N}$, where $[c]$ denotes the integer part of a number $c \in \mathbb{R}$. But this gives $\alpha\mathbf{x} + \beta\mathbf{y} \in \Lambda(\Phi, \mathcal{B}, X)$ because Λ is linear and solid. The linearity of the subset $\Lambda(\Phi, \mathcal{B}, \mathcal{M}(X))$ of $\Lambda(\Phi, \mathcal{B}, X)$ clearly follows from

$$\sup_i \left|\mathcal{B}_k^i\left(\alpha\mathbf{x} + \beta\mathbf{y}\right)\right| \leq |\alpha| \sup_i \left|\mathcal{B}_k^i\mathbf{x}\right| + |\beta| \sup_i \left|\mathcal{B}_k^i\mathbf{y}\right|.$$

Now let $\mathbf{x} \in \Lambda(\Phi, \mathcal{B}, X)$ and $\mathbf{y} \in s(X)$ be such that $\left|y_k\right| \leq \left|x_k\right|$ $(k \in \mathbb{N})$. Since the moduli ϕ_k are increasing, by (1) we get

$$\phi_k\left(\left|\mathcal{B}_k^i\mathbf{y}\right|\right) \leq \phi_k\left(\left|\mathcal{B}_k^i\mathbf{x}\right|\right) \ \ (k, i \in \mathbb{N}), \tag{3}$$

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and in view of solidity of Λ, the sequence $\Phi(\mathcal{B}\mathbf{y})$ is in Λ. Thus $\mathbf{y} \in \Lambda(\Phi, \mathcal{B}, X)$. Hence, $\Lambda(\Phi, \mathcal{B}, X)$ is solid if (1) holds. The solidity of $\mathbf{x}, \mathbf{y} \in \Lambda(\Phi, \mathcal{B}, \mathcal{M}(X))$ is obvious.

The statements about the set $\lambda(\Phi, B, X)$ follow similarly, with B_k instead of $\mathcal{B}_k^i$. □

An F-seminorm g_Λ on a DS space Λ is said to be *absolutely monotone* if $g_\Lambda(\mathbf{v}^2) \leq g_\Lambda(\mathbf{u}^2)$ for all $\mathbf{u}^2, \mathbf{v}^2 \in \Lambda$ with $|v_{ki}| \leq |u_{ki}|$ $(k, i \in \mathbb{N})$.

Theorem 2. *Let Λ be a solid DS space which is topologized by an absolutely monotone F-seminorm g_Λ.*

a) *If a sequence of moduli $\Phi = (\phi_k)$ satisfies one of two (equivalent) conditions* (M5) *and* (M6)*, then the GS space $\Lambda(\Phi, \mathcal{B}, X)$ may be topologized by the F-seminorm*

$$g_{\Lambda,\mathcal{B}}(\mathbf{x}) = g_\Lambda(\Phi(\mathcal{B}\mathbf{x})).$$

Moreover, if g_Λ is an F-norm on Λ, the space X is normed, and $\mathcal{B}$ satisfies the condition

$$\mathcal{B}\mathbf{x} = 0 \implies \mathbf{x} = 0, \tag{4}$$

then $g_{\Lambda,\mathcal{B}}$ is an F-norm on $\Lambda(\Phi, \mathcal{B}, X)$. The F-seminorm (*or F-norm*) *$g_{\Lambda,\mathcal{B}}$ is absolutely monotone if* (1) *holds.*

b) *If (Λ, g_Λ) is an AK-space, then the GS space $\Lambda(\Phi, \mathcal{B}, \mathcal{M}(X))$ may be topologized by the F-seminorm $g_{\Lambda,\mathcal{B}}$ for an arbitrary sequence of moduli Φ. Moreover, if g_Λ is an F-norm in Λ, the space X is normed, and $\mathcal{B}$ satisfies* (4)*, then $g_{\Lambda,\mathcal{B}}$ is an F-norm on $\Lambda(\Phi, \mathcal{B}, \mathcal{M}(X))$. The F-seminorm* (*or F-norm*) *$g_{\Lambda,\mathcal{B}}$ is absolutely monotone on GS space $\Lambda(\Phi, \mathcal{B}, \mathcal{M}(X))$ whenever $\mathcal{B}$ satisfies* (1).

Proof. a) First, we prove that $g_{\Lambda,\mathcal{B}}$ is an F-seminorm. Since g_Λ is an F-seminorm, (N1) holds by (M1). Because the operator $\mathcal{B}$ is linear, axiom (N2) follows immediately from the subadditivity of ϕ_k and g_Λ. If $|\alpha| \leq 1$, then by (M3) we get

$$\phi_k\left(\left|\mathcal{B}_k^i(\alpha\mathbf{x})\right|\right) = \phi_k\left(|\alpha|\left|\mathcal{B}_k^i\mathbf{x}\right|\right) \leq \phi_k\left(\left|\mathcal{B}_k^i\mathbf{x}\right|\right) \quad (k, i \in \mathbb{N}).$$

Since g_Λ is absolutely monotone,

$$g_{\Lambda,\mathcal{B}}(\alpha\mathbf{x}) = g_\Lambda\left(\left(\phi_k\left(\left|\mathcal{B}_k^i(\alpha\mathbf{x})\right|\right)\right)\right) \leq g_\Lambda\left(\left(\phi_k\left(\left|\mathcal{B}_k^i\mathbf{x}\right|\right)\right)\right) = g_{\Lambda,\mathcal{B}}(\mathbf{x}),$$

i.e., (N3) is true.

To prove (N4), let $\mathbf{x} \in \Lambda(\Phi, \mathcal{B}, X)$. Using the equivalence of (M5) and (M6) (see [32], Remark 1), we may assume that Φ satisfies (M5). Therefore, if $\lim_n \alpha_n = 0$ $(\alpha_n \in \mathbb{K})$, we can fix an index n_0 such that $|\alpha_n| < \delta$ for all $n \geq n_0$. Then by (M5) we obtain

$$\phi_k\left(\left|\mathcal{B}_k^i(\alpha_n\mathbf{x})\right|\right) \leq \nu(|\alpha_n|)\,\phi_k\left(\left|\mathcal{B}_k^i\mathbf{x}\right|\right)$$

for all $k, i \in \mathbb{N}$. So, since g_Λ is absolutely monotone, we get

$$g_\Lambda\left(\Phi\left(\mathcal{B}\left(\alpha_n \mathbf{x}\right)\right)\right) \leq g_\Lambda\left(\nu\left(|\alpha_n|\right)\Phi\left(\mathcal{B}\mathbf{x}\right)\right) \quad (n \geq n_0).$$

But this yields $\lim_n g_{\Lambda,\mathcal{B}}\left(\alpha_n \mathbf{x}\right) = 0$ because $\lim_n \nu\left(|\alpha_n|\right) = 0$. Thus (N4) holds and $g_{\Lambda,\mathcal{B}}$ is an F-seminorm on $\Lambda(\Phi, \mathcal{B}, X)$.

Let g_Λ be an F-norm and let $(X, \|\cdot\|_X)$ be a normed space. If $g_{\Lambda,\mathcal{B}}(\mathbf{x}) = 0$, then, using also (M1), we have

$$\|\mathcal{B}_k^i \mathbf{x}\|_X = 0 \quad (k, i \in \mathbb{N}),$$

which gives $\mathbf{x} = 0$ by (4). So, $g_{\Lambda,\mathcal{B}}$ is an F-norm on $\Lambda\left(\Phi, \mathcal{B}, X\right)$ in this case.

Now, suppose that (1) is satisfied. Then (3) holds, and since g_Λ is absolutely monotone,

$$g_{\Lambda,\mathcal{B}}(\mathbf{y}) = g_\Lambda\left(\left(\phi_k\left(\left|\mathcal{B}_k^i \mathbf{y}\right|\right)\right)\right) \leq g_\Lambda\left(\left(\phi_k\left(\left|\mathcal{B}_k^i \mathbf{x}\right|\right)\right)\right) = g_{\Lambda,\mathcal{B}}\left(\mathbf{x}\right).$$

Consequently, F-seminorm (or F-norm) $g_{\Lambda,\mathcal{B}}$ is absolutely monotone if (1) holds.

b) By the proof of a) it suffices to show that the functional

$$g_{\Lambda,\mathcal{B}} : \Lambda(\Phi, \mathcal{B}, \mathcal{M}(X)) \to \mathbb{K}$$

satisfies axiom (N4). Let $\lim_n \alpha_n = 0$ and let $\mathbf{x}$ be an arbitrary element from $\Lambda(\Phi, \mathcal{B}, \mathcal{M}(X))$. Then $\Phi(\mathcal{B}\mathbf{x}) \in \Lambda$, and since Λ is an AK-space,

$$\lim_n g_\Lambda\left(\Phi(\mathcal{B}\mathbf{x}) - \Phi(\mathcal{B}\mathbf{x})^{[n]}\right) = 0. \tag{5}$$

Using the equality

$$\Phi(\mathcal{B}\mathbf{x}) - \Phi(\mathcal{B}\mathbf{x})^{[n]} = \Phi\left(\mathcal{B}\mathbf{x} - (\mathcal{B}\mathbf{x})^{[n]}\right),$$

by (5) we can find, for fixed $\varepsilon > 0$, an index m such that

$$g_\Lambda\left(\Phi\left(\mathcal{B}\mathbf{x} - (\mathcal{B}\mathbf{x})^{[m]}\right)\right) < \varepsilon/2. \tag{6}$$

The double sequence $\mathcal{B}\mathbf{x} \in \mathcal{M}(X)$ determines the single sequence $(\tilde{z}_k)$ by $\tilde{z}_k = \sup_i \left|\mathcal{B}_k^i \mathbf{x}\right|$ $(k \in \mathbb{N})$. Since

$$\lim_n \phi_k(|\alpha_n \tilde{z}_k|) = 0 \quad (k \in \mathbb{N})$$

and g_Λ satisfies (N4), we have that

$$\lim_n g_\Lambda\left(\phi_k\left(|\alpha_n \tilde{z}_k|\right)\mathbf{e}^{k(2)}\right) = 0 \quad (k \in \mathbb{N}). \tag{7}$$

Further, since g_Λ satisfies (N2) and is absolutely monotone, we may write

$$g_\Lambda\left(\Phi\left(\mathcal{B}\left(\alpha_n \mathbf{x}\right)\right)^{[m]}\right) = g_\Lambda\left(\sum_{k=1}^{m}\left(\phi_k\left(\left|\alpha_n \mathcal{B}_k^i \mathbf{x}\right|\right)\right)_i \mathbf{e}^{k(2)}\right)$$

$$\leq \sum_{k=1}^{m} g_{\Lambda}\left(\left(\phi_k\left(\left|\alpha_n \mathcal{B}_k^i \mathbf{x}\right|\right)\right)_i \mathbf{e}^{k(2)}\right)$$
$$\leq \sum_{k=1}^{m} g_{\Lambda}\left(\phi_k\left(\left|\alpha_n \tilde{z}_k\right|\right) \mathbf{e}^{k(2)}\right).$$

This yields

$$\lim_n g_{\Lambda}\left(\Phi\left(\mathcal{B}\left(\alpha_n \mathbf{x}\right)\right)^{[m]}\right)=0$$

because of (7). Thus there exists an index n_0 such that, for all $n \geq n_0$,

$$\left|\alpha_n\right| \leq 1 \text{ and } g_{\Lambda}\left(\Phi\left(\left|\alpha_n\right|(\mathcal{B} \mathbf{x})^{[m]}\right)\right)<\varepsilon / 2. \tag{8}$$

Now, by (6) and (8) we get

$$\begin{aligned} g_{\Lambda, \mathcal{B}}\left(\alpha_n \mathbf{x}\right) &= g_{\Lambda}\left(\Phi\left(\mathcal{B}\left(\alpha_n \mathbf{x}\right)\right)\right) \\ &\leq g_{\Lambda}\left(\Phi\left(\left|\alpha_n\right|\left(\mathcal{B} \mathbf{x}-(\mathcal{B} \mathbf{x})^{[m]}\right)\right)\right)+g_{\Lambda}\left(\Phi\left(\left|\alpha_n\right|(\mathcal{B} \mathbf{x})^{[m]}\right)\right) \\ &\leq g_{\Lambda}\left(\Phi\left(\mathcal{B} \mathbf{x}-(\mathcal{B} \mathbf{x})^{[m]}\right)\right)+g_{\Lambda}\left(\Phi\left(\left|\alpha_n\right|(\mathcal{B} \mathbf{x})^{[m]}\right)\right) \\ &< \varepsilon / 2+\varepsilon / 2=\varepsilon \end{aligned}$$

for $n \geq n_0$. Hence, $\lim_n g_{\Lambda, \mathcal{B}}\left(\alpha_n \mathbf{x}\right)=0$, i.e., (N4) is true for $g_{\Lambda, \mathcal{B}}$. □

Let $\lambda \subset s$ be a solid sequence space and let $B=\left(b_{kj}\right)$ be an infinite scalar matrix. Denoting by $\lambda^{(2)}$ the set of all double sequences $\mathbf{x}^{(2)}$ with $\mathbf{x} \in \lambda$, and using the sequence $\underline{B}=\left(\underline{B}^i\right)$ of matrices $\underline{B}^i=\left(\underline{b}_{kj}^i\right)$ with the elements $\underline{b}_{kj}^i=b_{kj}$ $(i \in \mathbb{N})$ it is easy to see that $\lambda(\Phi, B, X)$ is isomorphic to the space $\lambda^{(2)}(\Phi, \underline{B}, X)$ of type $\Lambda(\Phi, \mathcal{B}, X)$. In addition, if λ is topologized by an (absolutely monotone) F-seminorm g_{λ}, then the equality

$$g_{\lambda, \underline{B}}\left(\mathbf{x}^{(2)}\right)=g_{\lambda}(\mathbf{x})$$

defines an (absolutely monotone) F-seminorm on $\lambda^{(2)}(\Phi, \underline{B}, X)$. Thus, since $\mathbf{x} \in \mathcal{M}(X)$ for every $\mathbf{x} \in \lambda^{(2)}(\Phi, \underline{B}, X)$ and $\left(\lambda^{(2)}(\Phi, \underline{B}, X), g_{\lambda, \underline{B}}\right)$ is an AK-space if and only if $\left(\lambda(\Phi, B, X), g_{\lambda}\right)$ is, Theorem 2 gives the following topologization theorem for $\lambda(\Phi, B, X)$.

Theorem 3. *Let λ be a solid sequence space topologized by an absolutely monotone F-seminorm g_{λ}.*

a) *If a sequence of moduli $\Phi=\left(\phi_k\right)$ satisfies one of two (equivalent) conditions* (M5) *and* (M6)*, then the GS space $\lambda(\Phi, B, X)$ may be topologized by the F-seminorm*

$$g_{\lambda, B}(\mathbf{x})=g(\lambda(B \mathbf{x})).$$

Moreover, if g_λ *is an F-norm on* λ*, the space* X *is normed, and* B *satisfies the condition*

$$B\mathbf{x} = 0 \implies \mathbf{x} = 0, \tag{9}$$

then $g_{\lambda,B}$ *is an F-norm on* $\lambda\,(\Phi, B, X)$*. The F-seminorm (or F-norm)* $g_{\lambda,B}$ *is absolutely monotone if* (2) *holds.*

b) *If* (λ, g_λ) *is an AK-space, then the GS space* $\lambda(\Phi, X)$ *may be topologized by the F-seminorm* $g_{\lambda,B}$ *for an arbitrary sequence of moduli* Φ*. Moreover, if* g_λ *is an F-norm, the space* X *is normed, and* B *satisfies* (9)*, then* $g_{\lambda,B}$ *is an F-norm on* $\lambda(\Phi, B, X)$*. The F-seminorm (or F-norm)* $g_{\lambda,B}$ *is absolutely monotone whenever* B *satisfies* (2).

Remark 1. It is not difficult to see that in Theorems $1-3$ we may write $\mathbf{X}$ instead of X whenever the matrices B^i $(i \in \mathbb{N})$ and B are diagonal or, more generally, whenever each row of these matrices contains not more than one non-zero element.

Remark 2. Ghosh and Srivastava [27] considered, for one modulus ϕ and for a sequence $\mathbf{X}$ of Banach spaces $(X_k, \|\cdot\|_k)$ $(k \in \mathbb{N})$, the GS space

$$\lambda(\phi, \mathbf{X}) = \{\mathbf{x} : \phi(\mathbf{x}) = (\phi(\|x_k\|_k)) \in \lambda\},$$

where λ is a solid sequence space. They assert (see [27], Theorem 3.1) that if λ is topologized by an absolutely monotone paranorm g, then

$$g_\phi(\mathbf{x}) = g(\phi(\mathbf{x}))$$

is a paranorm on $\lambda(\phi, \mathbf{X})$. But this is not true in general. Indeed, if ϕ is a bounded modulus and the solid sequence space ℓ_∞ is topologized by the absolutely monotone norm $g(\mathbf{u}) = \sup_k |u_k|$, then $\ell_\infty(\phi, \mathbf{X}) = s(\mathbf{X})$, and so, $\ell_\infty(\phi, \mathbf{X})$ contains an unbounded sequence $\mathbf{z} = (z_k)$. If now (z_{k_i}) is a subsequence of $\mathbf{z}$ such that $z_{k_i} \neq 0$ and $\lim_i \|z_{k_i}\|_{k_i} = \infty$, then, defining

$$\alpha_n = \begin{cases} (\|z_{k_i}\|_{k_i})^{-1}, & \text{if } n = k_i \quad (i \in \mathbb{N}), \\ 0 & \text{otherwise}, \end{cases}$$

we get the sequence (α_n) with $\lim_n \alpha_n = 0$. Since

$$\phi\left(\|\alpha_{k_i} z_{k_i}\|_{k_i}\right) = \phi(1) > 0 \quad (i \in \mathbb{N}),$$

we have that

$$\lim_n g_\phi(\alpha_n \mathbf{z}) = \lim_n \sup_k \|\alpha_n z_k\|_k \neq 0.$$

Thus g_ϕ does not satisfy axiom (N4) and, consequently, is not a paranorm on $\ell_\infty(\phi, \mathbf{X})$ if the modulus ϕ is bounded. Theorem 3 a) (for $B = I$) and Remark 1 show that if the solid sequence space λ is topologized by an absolutely monotone F-seminorm (or a paranorm with (N3)) g, then g_ϕ is an absolutely monotone F-seminorm (paranorm) on the GS space $\lambda(\phi, \mathbf{X})$ whenever (λ, g)

is an AK-space or the modulus ϕ satisfies one of the following (equivalent) conditions:

(M5°) there exists a function ν and a number $\delta > 0$ such that $\phi(ut) \leq \nu(u)\phi(t)$ $(0 \leq u < \delta,\ t \geq 0)$ and $\lim_{u\to 0+} \nu(u) = 0$,

(M6°) $\lim\limits_{u\to 0+} \sup\limits_{t>0} \dfrac{\phi(ut)}{\phi(t)} = 0.$

These conditions clearly fail if ϕ is bounded, since by $\sup_{t>0} \phi(t) = M < \infty$ we have

$$\sup_{t>0} \frac{\phi(ut)}{\phi(t)} \geq M^{-1} \sup_{t>0} \phi(ut) = 1$$

for any fixed $u > 0$.

3. Applications related to strong summability domains

Let $A = (a_{nk})$ be a non-negative matrix, i.e., $a_{nk} \geq 0\ \ (n, k \in \mathbb{N})$. We say that A is *column-positive* if for any $k \in \mathbb{N}$ there exists an index n_k such that $a_{n_k,k} > 0$. Obviously, any normal non-negative matrix is column-positive, and a diagonal matrix $D(c_k)$ is column-positive if $c_k > 0$ for all $k \in \mathbb{N}$. A sequence $\mathbf{u} = (u_k) \in s$ is called *strongly A-summable with index $p \geq 1$ to l* if $\lim_n \sum_k a_{nk}|u_k - l|^p = 0$, and *strongly A-bounded with index p* if $\sup_n \sum_k a_{nk}|u_k|^p < \infty$. It is clear that the set $c_0^p[A]$ of all strongly A-summable with index p to zero sequences and the set $\ell_\infty^p[A]$ of all strongly A-bounded with index p sequences are solid linear spaces and $c_0^p[A] \subset \ell_\infty^p[A]$. Moreover, the functional

$$g^p_{[A]}(\mathbf{u}) = \sup_n \left(\sum_k a_{nk}|u_k|^p \right)^{1/p}$$

is a seminorm on $\ell_\infty^p[A]$ and $c_0^p[A]$, and it is a norm if A is column-positive.

Natural generalizations of sequence spaces $c_0^p[A]$ and $\ell_\infty^p[A]$ are related to arbitrary solid F-seminormed sequence spaces (λ, g_λ) and (Λ, g_Λ). It is easy to see that the sets

$$\lambda^p[A] = \left\{ \mathbf{u} \in s : A^{1/p}\left(|\mathbf{u}|^p\right) = \left(\left(\sum_k a_{nk}|u_k|^p \right)^{1/p} \right)_{n\in\mathbb{N}} \in \lambda \right\},$$

$$\Lambda^p[A] = \left\{ \mathbf{u}^2 \in s^2 : A^{1/p}\left(|\mathbf{u}^2|^p\right) = \left(\left(\sum_k a_{nk}|u_{ki}|^p \right)^{1/p} \right)_{n,i\in\mathbb{N}} \in \Lambda \right\}$$

are solid linear subspaces of s and s^2, respectively. In addition, if F-seminorms g_λ and g_Λ are absolutely monotone, then the functionals

$$g^p_{\lambda,[A]}(\mathbf{u}) = g_\lambda\left(A^{1/p}\left(|\mathbf{u}|^p\right)\right) \quad \text{and} \quad g^p_{\Lambda,[A]}(\mathbf{u}^2) = g_\Lambda\left(A^{1/p}\left(|\mathbf{u}^2|^p\right)\right)$$

define F-seminorms, respectively, on $\lambda^p[A]$ and $\Lambda^p[A]$. Moreover, if A is column-positive, then $g^p_{\lambda,[A]}$ (or $g^p_{\Lambda,[A]}$) is an F-norm (a norm) whenever the space λ (or Λ) is F-normed (normed).

As a special case of $\Lambda^p[A]$ we will consider the DS space

$$U\lambda^p[A] = \left\{\mathbf{u}^2 \in s^2 : A^{1/p}\left(|\mathbf{u}^2|^p\right) \in U\lambda\right\},$$

which may be topologized by the F-seminorm

$$g^p_{\lambda,[\tilde{A}]}(\mathbf{u}) = g_\lambda\left(\tilde{A}^{1/p}\left(|\mathbf{u}^2|^p\right)\right)$$

if a solid sequence space λ is topologized by an absolutely monotone F-seminorm g_λ and

$$\tilde{A}^{1/p}\left(|\mathbf{u}^2|^p\right) = \left(\sup_i \left(\sum_k a_{nk}|u_{ki}|^p\right)^{1/p}\right)_{n\in\mathbb{N}}.$$

Let $\mathbf{p} = (p_k)$ be a bounded sequence of positive numbers with $r = \max\{1, \sup_k p_k\}$, let $B = (b_{nk})$ be an infinite scalar matrix, and let $\mathcal{B}$ be an SM method. For a sequence of moduli $\Phi = (\phi_k)$ and solid sequence spaces $\lambda \subset s$, $\Lambda \subset s^2$, we consider, as some generalizations of $\lambda^p[A]$ and $\Lambda^p[A]$, the sets

$$\lambda[A^{1/r}, B, \Phi, \mathbf{p}, X] = \left\{\mathbf{x} \in s(X) : A^{1/r}\left(\Phi^{\mathbf{p}}(B\mathbf{x})\right) \in \lambda\right\},$$
$$\Lambda[A^{1/r}, \mathcal{B}, \Phi, \mathbf{p}, X] = \left\{\mathbf{x} \in s(X) : A^{1/r}\left(\Phi^{\mathbf{p}}(\mathcal{B}\mathbf{x})\right) \in \Lambda\right\},$$

where,

$$A^{1/r}(\Phi^{\mathbf{p}}(B\mathbf{x})) = \left(\left(\sum_k a_{nk}\left(\phi_k\left(\Big|\sum_j b_{kj}x_j\Big|\right)\right)^{p_k}\right)^{1/r}\right)_{n\in\mathbb{N}},$$
$$A^{1/r}(\Phi^{\mathbf{p}}(\mathcal{B}\mathbf{x})) = \left(\left(\sum_k a_{nk}\left(\phi_k\left(\Big|\sum_j b^i_{kj}x_j\Big|\right)\right)^{p_k}\right)^{1/r}\right)_{n,i\in\mathbb{N}}.$$

Using the equalities $p_k = (p_k/r)r$ $(k \in \mathbb{N})$ and denoting by $\Phi^{\mathbf{p}/r}$ the sequence of moduli $\phi_k^{\mathbf{p}/r}(t) = (\phi_k(t))^{p_k/r}$ $(t \in \mathbb{R}^+,\ k \in \mathbb{N})$, we may write

$$\begin{aligned} \lambda[A^{1/r}, B, \Phi, \mathbf{p}, X] &= \left\{\mathbf{x} \in s_B(X) : \Phi^{\mathbf{p}/r}(B\mathbf{x}) \in \lambda^r[A]\right\} \\ &= \lambda^r[A]\left(\Phi^{\mathbf{p}/r}, B, X\right), \end{aligned} \tag{10}$$

$$\begin{aligned} \Lambda[A^{1/r}, \mathcal{B}, \Phi, \mathbf{p}, X] &= \left\{\mathbf{x} \in s_{\mathcal{B}}(X) : \Phi^{\mathbf{p}/r}(\mathcal{B}\mathbf{x}) \in \Lambda^r[A]\right\} \\ &= \Lambda^r[A]\left(\Phi^{\mathbf{p}/r}, \mathcal{B}, X\right). \end{aligned} \tag{11}$$

Thus, since the spaces $\lambda^r[A]$ and $\Lambda^r[A]$ are solid, Theorem 1 shows that $\lambda[A^{1/r}, B, \Phi, \mathbf{p}, X]$ and $\Lambda[A^{1/r}, \mathcal{B}, \Phi, \mathbf{p}, X]$ are GS spaces. Remark 1 shows that we get the GS spaces $\lambda[A^{1/r}, B, \Phi, \mathbf{p}, \mathbf{X}]$ and $\Lambda[A^{1/r}, \mathcal{B}, \Phi, \mathbf{p}, \mathbf{X}]$, for example, in the special case, when B is a diagonal matrix and $\mathcal{B}$ is a sequence of diagonal matrices.

The representations (10) and (11) are also useful for the topologization of sequence spaces $\lambda[A^{1/r}, B, \Phi, \mathbf{p}, X]$ and $\Lambda[A^{1/r}, \mathcal{B}, \Phi, \mathbf{p}, X]$. But first of all, we prove an auxiliary result about the property AK of the spaces $\left(\lambda^p[A], g^p_{\lambda,[A]}\right)$ and $\left(U\lambda^p[A], g^p_{\Lambda,[\tilde{A}]}\right)$.

Lemma 1. *Let $p \geq 1$ and let $A = (a_{nk})$ be a non-negative infinite matrix. Suppose that $\lambda \subset s$ is a solid AK-space with respect to an absolutely monotone F-seminorm g_λ.*

(i) *If*

$$\mathbf{a}_k = ((a_{nk})^{1/p})_{n\in\mathbb{N}} \in \lambda \quad (k \in \mathbb{N}), \tag{12}$$

then $(\lambda^p[A], g^p_{\lambda,A})$ is an AK-space.

(ii) *If the matrix A is row-finite (i.e., for any $n \in \mathbb{N}$ there exists an index k_n with $a_{nk} = 0 \quad (k > k_n)$), and (12) holds, then $(U\lambda^p[A], g^p_{\lambda,\tilde{A}})$ is an AK-space.*

Proof. The proof of statement (i) is quite similar to the proof of Lemma 1 from [33] and therefore it is omitted.

To prove (ii), let $\mathbf{u}^2 \in U\lambda^p[A]$. Thus $\tilde{A}^{1/p}\left(|\mathbf{u}^2|^p\right) \in \lambda$, and since (λ, g_λ) is an AK-space,

$$\begin{aligned} &\lim_m g_\lambda\left(\tilde{A}^{1/p}\left(|\mathbf{u}^2|^p\right) - \tilde{A}^{1/p}\left(|\mathbf{u}^2|^p\right)^{[m]}\right) \\ &\quad = \lim_m g_\lambda\left(\left(\overbrace{0, \ldots, 0}^{m}, \sup_i\left(\sum_k a_{m+1,k}|u_{ki}|^p\right)^{1/p}, \ldots\right)\right) = 0. \end{aligned} \tag{13}$$

By condition (12) and by

$$\tilde{A}^{1/p}\left(|\mathbf{e}^{j(2)}|^p\right) = \left(\sup_i (a_{nj})^{1/p}\right)_{n\in\mathbb{N}} = \mathbf{a}_j \quad (j \in \mathbb{N})$$

we conclude that $U\lambda^p[A]$ contains the sequences $\mathbf{e}^{j(2)}$. To prove the equality $\lim_m \mathbf{u}^{[m]} = \mathbf{u}$ in $U\lambda^p[A]$, we use the inequality

$$\begin{aligned} g^p_{\lambda,[\tilde{A}]}\left(\mathbf{u}^2 - \mathbf{u}^{2[m]}\right) &\leq \sum_{n=1}^{s} g_\lambda\left(\sup_i\left(\sum_{k=m+1}^{\infty} a_{nk}|u_{ki}|^p\right)^{1/p}\mathbf{e}^n\right) \\ &\quad + g_\lambda\left(\left(\overbrace{0,\ldots,0}^{s}, \sup_i\left(\sum_{k=m+1}^{\infty} a_{s+1,k}|u_{ki}|^p\right)^{1/p},\ldots\right)\right) \\ &= G^1_{sm} + G^2_{sm}. \end{aligned}$$

Let $\varepsilon > 0$. As g_λ is absolutely monotone, we have

$$G^2_{mm} \leq g_\lambda\left(\tilde{A}^{1/p}\left(|\mathbf{u}^2|^p\right) - \tilde{A}^{1/p}\left(|\mathbf{u}^2|^p\right)^{[m]}\right),$$

and by (13) we get $\lim_m G^2_{mm} = 0$. Thus, there exists a number $m_0 \in \mathbb{N}$ with

$$G^2_{m_0,m_0} < \varepsilon.$$

Since the matrix A is row-finite, we can find $m_1 \geq m_0$ such that for all $n = 1, 2, \ldots, m_0$ and $i \in \mathbb{N}$ one has

$$\sum_{k=m+1}^{\infty} a_{nk}|u_{ki}|^p = 0 \quad (m \geq m_1),$$

which yields

$$G^1_{m_0,m} = 0 \quad (m \geq m_1).$$

Hence, using the inequalities $G^2_{m_0,m} \leq G^2_{m_0,m_0}$ $(m \geq m_0)$, we have that

$$g^p_{\lambda,[\tilde{A}]}\left(\mathbf{u}^2 - \mathbf{u}^{2[m]}\right) \leq G^1_{m_0,m} + G^2_{m_0,m} < 0 + \varepsilon = \varepsilon$$

if $m \geq m_1$. Consequently, $\lim_m \mathbf{u}^{[m]} = \mathbf{u}$ in $U\lambda^p[A]$. The proof is completed. □

Now we can determine F-seminorms on GS spaces $\lambda[A^{1/r}, B, \Phi, \mathbf{p}, X]$ and $\Lambda[A^{1/r}, \mathcal{B}, \Phi, \mathbf{p}, X]$.

Proposition 1. *Let $\Phi = (\phi_k)$ be a sequence of moduli and let $\mathbf{p} = (p_k)$ be a bounded sequence of positive numbers and $r = \max\{1, \sup_k p_k\}$. Let $A = (a_{nk})$ be a non-negative infinite matrix and let $B = (b_{nk})$ be an infinite matrix of scalars. Suppose that $(X, \dot{|}\cdot\dot{|})$ is a seminormed space, $\mathbf{X}$ is a sequence of seminormed spaces $(X_k, \dot{|}\cdot\dot{|}_k)$ $(k \in \mathbb{N})$, and $\lambda \subset s$ is a solid sequence space topologized by an absolutely monotone F-seminorm g_λ.*

a) *If the sequence of moduli* $\Phi^{\mathbf{p}/r}$ *satisfies one of conditions* (M5) *and* (M6)*, then the GS space* $\lambda[A^{1/r}, B, \Phi, \mathbf{p}, X]$ *may be topologized by the F-seminorm*

$$g_{\lambda,A,B}^{\Phi,\mathbf{p}}(\mathbf{x}) = g_{\lambda}\left(A^{1/r}\left(\Phi^{\mathbf{p}}\left(B\mathbf{x}\right)\right)\right).$$

b) *If* (λ, g_{λ}) *is an AK-space and condition* (12) *holds with* $p = r$*, then* $g_{\lambda,A,B}^{\Phi,\mathbf{p}}$ *is an F-seminorm on* $\lambda[A^{1/r}, B, \Phi, \mathbf{p}, X]$ *for an arbitrary sequence of moduli* Φ*.*

If, in a) *and* b)*,* g_{λ} *is an F-norm,* X *is normed,* A *is column-positive, and* (9) *holds, then* $g_{\lambda,A,B}^{\Phi,\mathbf{p}}$ *is an F-norm on* $\lambda[A^{1/r}, B, \Phi, \mathbf{p}, X]$*. Moreover, for a diagonal matrix* $B = D(c_k)$*, we get the absolutely monotone F-seminorm* (*or F-norm*) $g_{\lambda,A,D(c_k)}^{\Phi,\mathbf{p}}$ *on the GS space* $\lambda[A^{1/r}, D(c_k), \Phi, \mathbf{p}, \mathbf{X}]$*.*

Proof. Statement a) follows from (10) and Theorem 3 a) because

$$g_{\lambda[A]^r,B}(\mathbf{x}) = g_{\lambda}\left(A^{1/r}\left(\left(\Phi^{\mathbf{p}/r}\left(B\mathbf{x}\right)\right)^r\right)\right) = g_{\lambda}\left(A^{1/r}\left(\Phi^{\mathbf{p}}\left(B\mathbf{x}\right)\right)\right)$$

for any $\mathbf{x} \in \lambda^r[A]\left(\Phi^{\mathbf{p}/r}, B, X\right)$. Analogously, we deduce statement b) from (11) and Theorem 3 b) in view of Lemma 1(i). □

Let us investigate the topologization of spaces of type $\Lambda[A^{1/r}, \mathcal{B}, \Phi, \mathbf{p}, X]$ in the case $\Lambda = U\lambda$.

Proposition 2. *Let* Φ*,* $\mathbf{p}$*,* A*,* X*, and* $\mathbf{X}$ *be the same as in Proposition 1 and let* $\mathcal{B}$ *be a sequence of infinite matrices* $B^i = (b_{nk}^i)$*.*

a) *If* $\Phi^{\mathbf{p}/r}$ *satisfies one of conditions* (M5) *and* (M6)*, then the GS space* $U\lambda[A^{1/r}, \mathcal{B}, \Phi, \mathbf{p}, X]$ *may be topologized by the F-seminorm*

$$\begin{aligned} g_{U\lambda,A,\mathcal{B}}^{\Phi,\mathbf{p}}(\mathbf{x}) &= g_{\lambda}\left(\tilde{A}^{1/r}\left(\Phi^{\mathbf{p}}\left(\mathcal{B}\mathbf{x}\right)\right)\right) \\ &= g_{\lambda}\left(\left(\left(\sup_i\left(\sum_k a_{nk}\left(\phi_k\left(\left|\sum_j b_{kj}^i x_j\right|\right)\right)^{p_k}\right)\right)^{1/r}\right)_{n\in\mathbb{N}}\right). \end{aligned}$$

b) *Suppose that* (λ, g_{λ}) *is an AK-space and the moduli* ϕ_k $(k \in \mathbb{N})$ *are unbounded. If the matrix* A *is row-finite and column-positive, and* (12) *holds with* $p = r$*, then* $g_{U\lambda,A,\mathcal{B}}^{\Phi,\mathbf{p}}$ *is an F-seminorm on* $U\lambda[A^{1/r}, \mathcal{B}, \Phi, \mathbf{p}, X]$*.*

If, in a) *and* b)*,* g_{λ} *is an F-norm,* X *is normed, and* (4) *is true, then* $g_{U\lambda,A,\mathcal{B}}^{\Phi,\mathbf{p}}$ *is an F-norm on* $U\lambda[A^{1/r}, \mathcal{B}, \Phi, \mathbf{p}, X]$*. Moreover,* $g_{U\lambda,A,\mathcal{B}}^{\Phi,\mathbf{p}}$ *is an absolutely monotone F-seminorm* (*or F-norm*) *on the GS space* $U\lambda[A^{1/r}, \mathcal{B}, \Phi, \mathbf{p}, \mathbf{X}]$ *if* $\mathcal{B}$ *is a sequence of diagonal matrices.*

Proof. Statement a) follows from Theorem 2 a) in view of (11).

b) Under our assumptions, the space $\left(U\lambda^r[A], g^r_{\lambda,[\tilde{A}]}\right)$ has the property AK by Lemma 1 (ii). In addition, since our non-negative matrix A is column-positive, for any fixed k there exists an index n_k such that $a_{n_k,k} > 0$. Thus, using the inequality

$$\left(\left(\phi_k\left(\left|\sum_j b^i_{kj}x_j\right|\right)\right)^{p_k}\right)^{1/r} \le (a_{n_k,k})^{-1/r}\sup_i\left(\sum_k a_{nk}\left(\phi_k\left(\left|\sum_j b^i_{kj}x_j\right|\right)\right)^{p_k}\right)^{1/r},$$

by $\tilde{A}^{1/r}\left(\Phi^{\mathbf{p}}\left(\mathcal{B}\mathbf{x}\right)\right) \in \mathcal{M}$ we get

$$\sup_i\left(\left(\phi_k\left(\left|\sum_j b^i_{kj}x_j\right|\right)\right)^{p_k}\right)^{1/r} < \infty.$$

But this yields

$$\sup_i\left|\sum_j b^i_{kj}x_j\right| < \infty \quad (k \in \mathbb{N})$$

because the moduli $\phi_k^{\mathbf{p}/r}(t) = (\phi_k(t))^{p_k/r} \quad (k \in \mathbb{N})$ are unbounded. Consequently, for any $\mathbf{x} \in U\lambda[A^{1/r}, \mathcal{B}, \Phi, \mathbf{p}, X]$ we have $\mathcal{B}\mathbf{x} \in \mathcal{M}(X)$. Hence, by equality (11) we have

$$U\lambda[A^{1/r}, \mathcal{B}, \Phi, \mathbf{p}, X] = U\lambda^r[A]\left(\Phi^{\mathbf{p}/r}, \mathcal{B}, \mathcal{M}\left(X\right)\right),$$

and b) follows from Theorem 2 b). □

4. Some special cases

In the following we apply Propositions 1 and 2 for the topologization of GS spaces

$$\lambda[A, B, \Phi, \mathbf{p}, X] = \left\{\mathbf{x} \in s(X) : A\left(\Phi^{\mathbf{p}}(B\mathbf{x})\right) \in \lambda\right\},$$
$$U\lambda[A, \mathcal{B}, \Phi, \mathbf{p}, X] = \left\{\mathbf{x} \in s(X) : A\left(\Phi^{\mathbf{p}}(\mathcal{B}\mathbf{x})\right) \in U\lambda\right\},$$

where

$$A(\Phi^{\mathbf{p}}(B\mathbf{x})) = \left(\sum_k a_{nk}\left(\phi_k\left(\left|\sum_j b_{kj}x_j\right|\right)\right)^{p_k}\right)_{n\in\mathbb{N}},$$

$$A(\Phi^{\mathbf{p}}(\mathcal{B}\mathbf{x})) = \left(\sum_k a_{nk}\left(\phi_k\left(\left|\sum_j b^i_{kj}x_j\right|\right)\right)^{p_k}\right)_{n,i\in\mathbb{N}}.$$

In the special case $p_k = 1 \quad (k \in \mathbb{N})$, the following corollaries of Propositions 1 and 2 are obvious.

Corollary 1. *Let A, $\mathcal{B}$, X, $\mathbf{X}$, and λ be the same as in Proposition 1.*

a) *If the sequence of moduli Φ satisfies one of conditions* (M5) *and* (M6)*, then the GS space*

$$\lambda[A, B, \Phi, X] = \{\mathbf{x} \in s(X) : A(\Phi(B\mathbf{x})) \in \lambda\}$$

may be topologized by the F-seminorm

$$\begin{aligned} g^{\Phi}_{\lambda,A,B}(\mathbf{x}) &= g_{\lambda}(A(\Phi(B\mathbf{x}))) \\ &= g_{\lambda}\left(\left(\sum_k a_{nk}\left(\phi_k\left(\left|\sum_j b_{kj}x_j\right|\right)\right)\right)_{n\in\mathbb{N}}\right). \end{aligned}$$

b) *If (λ, g_{λ}) is an AK-space and condition* (12) *holds with $p = 1$, then $g^{\Phi}_{\lambda,A,B}$ is an F-seminorm on $\lambda[A, B, \Phi, X]$ for an arbitrary sequence of moduli Φ.*

If, in a) *and* b)*, g_{λ} is an F-norm, X is normed, A is column-positive, and B satisfies* (9)*, then $g^{\Phi}_{\lambda,A,B}$ is an F-norm on $\lambda[A, B, \Phi, X]$. Moreover, $g^{\Phi}_{\lambda,A,B}$ is an absolutely monotone F-seminorm* (*or F-norm*) *on the GS space $\lambda[A, B, \Phi, \mathbf{X}]$ if B is a diagonal matrix.*

Corollary 2. *Let Φ, A, X, $\mathbf{X}$, and $\mathcal{B}$ be the same as in Proposition 2.*

a) *If Φ satisfies one of conditions* (M5) *and* (M6)*, then the sequence space $U\lambda[A, \mathcal{B}, \Phi, X]$ may be topologized by the F-seminorm*

$$\begin{aligned} g^{\Phi}_{U\lambda,A,\mathcal{B}}(\mathbf{x}) &= g_{\lambda}\left(\tilde{A}(\Phi(\mathcal{B}\mathbf{x}))\right) \\ &= g_{\lambda}\left(\left(\sup_i \sum_k a_{nk}\left(\phi_k\left(\left|\sum_j b_{kj}x_j\right|\right)\right)\right)_{n\in\mathbb{N}}\right). \end{aligned}$$

b) *Suppose that (λ, g_{λ}) is an AK-space and the moduli $\phi_k \quad (k \in \mathbb{N})$ are unbounded. If the matrix A is row-finite and column-positive, and* (12) *holds with $p = 1$, then $g^{\Phi}_{U\lambda,A,\mathcal{B}}$ is an F-seminorm on $U\lambda[A^{1/r}, \mathcal{B}, \Phi, X]$.*

If, in a) *and* b)*, g_{λ} is an F-norm, X is normed, and $\mathcal{B}$ satisfies* (4)*, then $g^{\Phi}_{U\lambda,A,\mathcal{B}}$ is an F-norm on $U\lambda[A^{1/r}, \mathcal{B}, \Phi, X]$. Moreover, $g^{\Phi}_{U\lambda,A,\mathcal{B}}$ is an absolutely monotone F-seminorm* (*or F-norm*) *on the GS space $U\lambda[A^{1/r}, \mathcal{B}, \Phi, \mathbf{X}]$ if $\mathcal{B}$ is a sequence of diagonal matrices.*

First investigations of spaces of type $\lambda[A, B, \Phi, X]$ are related to the case $B = I$ and $\phi_k = \phi \quad (k \in \mathbb{N})$. Ruckle [44] considered the space

$$\ell[I, \phi] = \{\mathbf{u} \in s : \sum_k \phi(|u_k|) < \infty\}$$

and Maddox [35] introduced the sequence spaces $c_0[C_1, \phi]$ and $\ell_\infty[C_1, \phi]$. The spaces $\lambda[A, \phi]$ and $\lambda[C_1, \phi, X]$ (X is a Banach space) are studied, respectively, in [10] and [11]. Corollary 1 allows to determine F-seminorm topologies for sequence spaces from [15] and [20]. It also extends Theorem 2.6 of [9], which determines the paranorm $g^{\phi}_{D(k^{-s}),I}$ on $\lambda[D(k^{-s}), I, \phi, \mathbf{X}]$ if $s > 0$ and λ is a Banach space with the property AK.

Further corollaries of Propositions 1 and 2 deal with the sequence $\mathbf{p}$ and are related to $\lambda \in \{\ell_\infty, c_0, \ell^r\}$. It is clear that ℓ_∞ and c_0 are solid sequence spaces with the absolutely monotone norm $\|\mathbf{u}\|_\infty = \sup_k |u_k|$. Since, moreover, $(c_0, \|\cdot\|_\infty)$ is an AK-space, and for $\lambda \in \{\ell_\infty, c_0\}$ we have

$$(|u_k|) \in \lambda \iff (|u_k|^q) \in \lambda \quad (q > 0),$$

Proposition 1 immediately yields the following corollary.

Corollary 3. *Let Φ, $\mathbf{p}$, A, X, $\mathbf{X}$, and B be the same as in Proposition 1.*

a) *If the sequence of moduli $\Phi^{\mathbf{p}/r}$ satisfies one of conditions* (M5) *and* (M6)*, then the GS space $\ell_\infty[A, B, \Phi, \mathbf{p}, X]$ may be topologized by the F-seminorm*

$$g^{\Phi,\mathbf{p}}_{\infty,A,B}(\mathbf{x}) = \sup_n \left(A\left(\Phi^{\mathbf{p}}\left(B\mathbf{x}\right)\right)\right) = \sup_n \left(\sum_k a_{nk}\left(\phi_k\left(\Big|\sum_j b_{kj}x_j\Big|\right)\right)^{p_k}\right)^{1/r}.$$

b) *If the matrix A is such that*

$$\lim_n a_{nk} = 0 \ \ (k \in \mathbb{N}), \tag{14}$$

then $g^{\Phi,\mathbf{p}}_{\infty,A,B}$ is an F-seminorm on $c_0[A, B, \Phi, \mathbf{p}, X]$ for an arbitrary sequence of moduli Φ.

If, in a) *and* b)*, the space X is normed, A is column-positive, and B satisfies* (9)*, then $g^{\Phi,\mathbf{p}}_{\infty,A,B}$ is an F-norm. Moreover, $g^{\Phi,\mathbf{p}}_{\infty,A,D(c_k)}$ is an absolutely monotone F-seminorm* (*F-norm*) *on $c_0[A, D(c_k), \Phi, \mathbf{p}, \mathbf{X}]$.*

We may consider the space $\ell[A, B, \Phi, \mathbf{p}, X]$ as the space $\ell^r[A^{1/r}, B, \Phi, \mathbf{p}, X]$. So, since ℓ^r is solid AK-space with respect to the norm $\|\mathbf{u}\|_r = (\sum_k |u_k|^r)^{1/r}$, Proposition 1 b) gives the following corollary.

Corollary 4. *Let Φ, $\mathbf{p}$, A, X, $\mathbf{X}$, and B be the same as in Proposition 1. If the matrix A is such that*

$$\sum_n |a_{nk}| < \infty \quad (k \in \mathbb{N}),$$

then

$$g_{1,A,B}^{\Phi,\mathbf{p}} = \left(\sum_n |A^{1/r}\left(\Phi^{\mathbf{p}}\left(B\mathbf{x}\right)\right)|^r\right)^{1/r}$$
$$= \left(\sum_n \sum_k a_{nk}\left(\phi_k\left(\left|\sum_j b_{kj}x_j\right|\right)\right)^{p_k}\right)^{1/r}$$

is an F-seminorm on $\ell[A, B, \Phi, \mathbf{p}, X]$ *for an arbitrary sequence of moduli* Φ.

If the space X *is normed,* A *is column-positive, and* (9) *holds, then* $g_{1,A,B}^{\Phi,\mathbf{p}}$ *is an F-norm. Moreover,* $g_{1,A,D(c_k)}^{\Phi,\mathbf{p}}$ *is an absolutely monotone F-seminorm* (*or F-norm*) *on* $\ell[A, D(c_k), \Phi, \mathbf{p}, \mathbf{X}]$.

The GS spaces from Corollaries 3 and 4 have been studied earlier in the special cases when the role of matrix A was played not only by C_1 and different diagonal matrices, but also by the matrix of de la Vallée-Poussin and by the matrix of lacunary strong convergence. Recall that if $d = (d_k)$ is a non-decreasing sequence of positive numbers tending to ∞ with $d_1 = 1$ and $d_{n+1} \leq d_n + 1$, then the *matrix of de la Vallée-Poussin* $V_d = (v_{nk})$ is defined by the equalities $v_{nk} = 1/d_n$ if $k \in [n - d_n + 1, n]$ and $v_{nk} = 0$ otherwise. Further, a sequence of integers $\theta = (k_j)$ is called *lacunary* if $k_0 = 0$, $0 < k_j < k_{j+1}$ and $h_j = k_j - k_{j-1} \to \infty$ as $j \to \infty$. A sequence $\mathbf{u} = (u_k)$ is said to be *lacunary strongly convergent* to a number l if (see [26])

$$\lim_j 1/h_j \sum_{i \in (k_{j-1}, k_j]} |u_i - l| = 0.$$

Thus, given the matrix $N_\theta = (w_{ji})$ with $w_{ji} = 1/h_j$ if $i \in (k_{j-1}, k_j]$ and by $w_{ji} = 0$ otherwise, the lacunary strong convergence is precisely the strong N_θ-summability. It is clear that both matrices V_d and N_θ are regular and column-positive. Moreover, V_d is normal and reduces to C_1 for $d_n = n$.

Corollary 3 permits to define, for example, an F-seminorm on the sequence spaces $c_0[V_d, B, \phi, \mathbf{p}]$ and $\ell_\infty[V_d, B, \phi, \mathbf{p}]$ from [16], and an F-norm on the GS space $c_0[N_\theta, I, \Phi, X]$ which is considered in [41] for a Banach space X. Corollary 3 also contains, as special cases, the results about the topologization of some sequence spaces of type $c_0[A, I, \phi, \mathbf{p}]$ from [10], [14], and [39].

In Theorem 1 of [14] it was asserted that for any non-negative regular matrix A the space $\ell_\infty[A, I, \phi; \mathbf{p}]$ may be topologized by the paranorm

$$g_{\infty,A}^{\phi,\mathbf{p}}(\mathbf{u}) = \sup_n \left(\sum_k a_{nk}\left(\phi\left(|u_k|\right)\right)^{p_k}\right)^{1/r}.$$

if $\inf_k p_k > 0$. But it is possible to prove, as in Remark 2, that this is not true for a bounded modulus ϕ if $A = I$, and $p_k = 1$ $(k \in \mathbb{N})$. By

Corollary 3 a) we can say that $g^{\phi,\mathbf{p}}_{\infty,A}$ is an F-seminorm (or a paranorm) on $\ell_\infty[A, I, \phi, \mathbf{p}]$ for any non-negative matrix A whenever the sequence of moduli $\phi^{p_k/r}(t) = (\phi(t))^{p_k/r}$ $(k \in \mathbb{N})$ satisfies one of conditions (M5) and (M6). If $\inf_k p_k > 0$, then it suffices to assume that the modulus ϕ satisfies one of conditions (M5°) and (M6°). In the case $p_k = 1$ $(k \in \mathbb{N})$ this result completes Theorem 10 (ii) of [10]. We also remark that Corollary 3 b) gives, for $p_k = 1$ $(k \in \mathbb{N})$, an F-seminorm on the sequence space $c_0[N_\theta, B, \Phi]$ from [15].

Corollary 4 generalizes the results from [6], [12], and [13], where the paranorm topologies are defined on $\ell[A, B, \Phi, \mathbf{p}, X]$ provided $\phi_k = \phi$ $(k \in \mathbb{N})$ and $A = D(k^{-s})$, $s \geq 0$.

Many papers from the mathematical literature are devoted to the investigation of sequence spaces from Corollaries 3 and 4 in the case when the matrix B is determined by various differences of sequences. For fixed $m, n \in \mathbb{N}$ the *difference operator* Δ^m_n is defined by (see [49])

$$\Delta^m_n \mathbf{x} = (\Delta^m_n x_k), \;\; \Delta^m_n x_k = \Delta^{m-1}_n x_k - \Delta^{m-1}_n x_{k+n}, \;\; \Delta^0_n x_k = x_k \quad (k \in \mathbb{N}).$$

The difference operator $\Delta^m = \Delta^m_1$ was introduced already in [31] (m=1) and [25]. If $v = (v_k)$ is a fixed sequence of nonzero numbers, then $v\Delta^m_n$ denotes the difference operator defined by $v\Delta^m_n \mathbf{x} = (\Delta^m_n v_k x_k)$.

Since

$$v\Delta^m_n x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} v_{k+ni} x_{k+ni} \quad (k \in \mathbb{N}),$$

$v\Delta^m_n$ is the summability operator defined by the *difference matrix* $v\Delta^m_n = (v\delta_{kj})$, where $v\delta_{kj} = (-1)^i \binom{m}{i} v_j$ if $j = k + ni$, $(0 \leq i \leq m, k \in \mathbb{N})$ and $v\delta_{kj} = 0$ otherwise. It is not difficult to see that (9) fails if B is a difference operator $v\Delta^m_n$. Therefore, by means of Corollaries 3 and 4 it is not possible to determine F-norms in the case $B = v\Delta^m_n$. To overcome this difficulty we use a new class of summability matrices which contains all difference matrices.

Let $m \geq 0$ be a fixed integer. Following [33], we say that an infinite scalar matrix $B = (b_{ki})$ is *m-normal* if, for any $k \in \mathbb{N}$, $b_{k,k+m} \neq 0$ and $b_{ki} = 0$ if $i > k + m$. By this definition, 0-normal matrices are just normal matrices. For example, the difference matrix $v\Delta^m_n$ is nm-normal and $v\Delta^m$ is m-normal. Now, if the matrix B is m-normal, then $B\mathbf{x} = 0$ and $\mathbf{x} \in s_B(X)$ imply $\mathbf{x} = 0$ whenever $x_1 = \cdots = x_m = 0$. This approach and the definitions of norms from [31] and [25] lead us to the following proposition which complements Proposition 1 and Corollaries 3 and 4.

Proposition 3. *Let Φ, $\mathbf{p}$, A, and X be the same as in Proposition 1. Assume that B is an m-normal infinite matrix with $m \geq 1$.*

a) *If the solid sequence space* λ *is topologized by an absolutely monotone F-seminorm* g_λ, *then statements* a) *and* b) *of Proposition 1 hold with*

$$\hat{g}^{\Phi,\mathbf{p}}_{\lambda,A,B}(\mathbf{x}) = \sum_{i=1}^{m} |x_i| + g^{\Phi,\mathbf{p}}_{\lambda,A,B}(\mathbf{x})$$

instead of $g^{\Phi,\mathbf{p}}_{\lambda,A,B}(\mathbf{x})$. *Then* $\hat{g}^{\Phi,\mathbf{p}}_{\lambda,A,B}$ *is an F-norm on* $\lambda[A,B,\Phi,\mathbf{p},X]$ *whenever* g_λ *is F-norm,* X *is normed, and* A *is column-positive.*

b) *Statements of Corollaries 3 and 4 are true with*

$$\hat{g}^{\Phi,\mathbf{p}}_{\nu,A,B}(\mathbf{x}) = \sum_{i=1}^{m} |x_i| + g^{\Phi,\mathbf{p}}_{\nu,A,B}(\mathbf{x}), \quad \nu \in \{\infty,\, 1\}$$

instead of $g^{\Phi,\mathbf{p}}_{\nu,A,B}(\mathbf{x}), \quad \nu \in \{\infty,\, 1\}$. *If* X *is normed and* A *is column-positive, then these functionals determine F-norms, respectively, on* $\ell_\infty[A,B,\Phi,\mathbf{p},X]$, $c_0[A,B,\Phi,\mathbf{p},X]$, *and* $\ell[A,B,\Phi,\mathbf{p},X]$.

In 1) *and* 2), *the term* $\sum_{i=1}^m |x_i|$ *may also be replaced with the expression* $\max_{i=1,\dots,m} |x_i|$ *or, more generally, with the expressions* $\sum_{i=1}^m \varphi_i\left(|x_i|\right)$ *or* $\max_{i=1,\dots,m} \varphi_i\left(|x_i|\right)$, *where* $\varphi_i \ \ (i=1,\dots,m)$ *are moduli.*

For example, the authors of [1], [21], and [22] determine paranorms of type $\hat{g}^{\Phi,\mathbf{p}}_{0,A,B}$ on some GS spaces $c_0[A,B,\Phi,\mathbf{p},X]$ with $B = v\Delta^m$. At the same time the various spaces of type $c_0[A,v\Delta^m,\Phi,\mathbf{p},X]$ and $\ell[A,v\Delta^m,\Phi,\mathbf{p},X]$ from [2], [3], [4], [8], [20], [24], [43], and [48] are topologized, as in Corollaries 3 and 4, by the paranorms $g^{\Phi,\mathbf{p}}_{\nu,A,v\Delta^m}(\mathbf{x}) \ \ (\nu \in \{\infty,\, 1\})$. Proposition 3 b) allows us to define alternative paranorms (or F-seminorms) in the form $\hat{g}^{\Phi,\mathbf{p}}_{\nu,A,v\Delta^m}$ $(\nu \in \{1,\, \infty\})$ on all these spaces. In addition, Corollary 3 and Proposition 3 b) determine F-seminorm (or paranorm) topologies on the spaces of type $\ell_\infty[A,v\Delta^m,\Phi,\mathbf{p},X]$ from the papers [1], [2], [4], [8], [20], [21], [22], [24], and [43].

Tripathy, Mahanta, and Et [50] consider the generalized sequence space $m(\psi,p)[I,\Delta^n,\phi,X]$, where $\left(m(\psi,p), g_{m(\psi,p)}\right) \quad (1 \le p < \infty)$ is the solid Banach space defined in [51] by means of a special non-decreasing sequence $\psi = (\psi_k)$. Theorem 2 of [50] asserts that $\hat{g}^{\phi}_{m(\psi,p),I,\Delta^n}$ is a paranorm on $m(\psi,p)[I,\Delta^n,\phi]$ for any modulus ϕ. Besides this, Tripathy and Chandra ([48], Theorem 3.2) assert that the sequence space $\ell_\infty[I,D(c_k)\Delta^1_n,\phi,\mathbf{p}]$ may be topologized by the paranorm $g^{\phi,\mathbf{p}}_{\infty,I,D(c_k)\Delta^1_n}$ for every modulus ϕ. But these assertions are not true in general. Indeed, if $p = n = 1$ and $\psi_k = k \quad (k \in \mathbb{N})$, then (see [51], Corollary 11) $m(\psi,p) = \ell_\infty$ with $g_{m(\psi,p)} = \|\cdot\|_\infty$. Hence $m(\psi,p)[I,\Delta^n,\phi,\mathbb{K}]$ reduces to the space

$$\ell_\infty[I,\Delta^1,\phi] = \left\{\mathbf{u} = (u_k) \in s : \Delta^1\mathbf{u} \in \ell_\infty(\phi)\right\},$$

and

$$\hat{g}^{\phi}_{m(\psi,p),I,\Delta^n}(\mathbf{u}) = |u_1| + \sup_k \phi(|\Delta^1 u_k|).$$

Analogously, for $n = 1$ and $p_k = c_k = 1 \quad (k \in \mathbb{N})$, $\ell_\infty[I, D(c_k)\Delta^1_n, \phi, \mathbf{p}]$ reduces also to $\ell_\infty[I, \Delta^1, \phi]$ with

$$g^{\phi,\mathbf{p}}_{\infty,I,D(c_k)\Delta^1_n} = \sup_k \phi(|\Delta^1 u_k|).$$

Therefore, if the modulus ϕ is bounded, then $\ell_\infty[I, \Delta^1, \phi] = s$ and we can prove, as in Remark 2, that $\hat{g}^{\phi}_{m(\psi,p),I,\Delta^n}$ and $g^{\phi,\mathbf{p}}_{\infty,I,\Delta^1_n}$ are not paranorms. Proposition 3 a) and Corollary 3 a) show that Theorem 2 of [50], and Theorem 3.2 (about $\ell_\infty[I, D(c_k)\Delta^1_n, \phi, \mathbf{p}]$) from [48], are true whenever the modulus ϕ satisfies one of conditions (M5°) and (M6°).

Let us apply Proposition 2 and Corollary 2 to define F-seminorms and F-norms on the GS spaces

$$U\ell_\infty[A, \mathcal{B}, \Phi, \mathbf{p}, X] = \left\{ \mathbf{x} \in s(X) : \sup_{n,i} \sum_k a_{nk} \left(\phi_k \left(\left| \sum_j b^i_{kj} x_j \right| \right) \right)^{p_k} < \infty \right\}$$

and

$$\begin{aligned} Uc_0[A, \mathcal{B}, \Phi, \mathbf{p}, X] &= \{\mathbf{x} \in s(X) : A(\Phi^{\mathbf{p}}(\mathcal{B}\mathbf{x})) \in Uc_0\} \\ &= \mathcal{M}c_0[A, \mathcal{B}, \Phi, \mathbf{p}, X] \cap uc_0[A, \mathcal{B}, \Phi, \mathbf{p}, X], \end{aligned}$$

where

$$\mathcal{M}c_0[A, \mathcal{B}, \Phi, \mathbf{p}, X] = \{\mathbf{x} \in s(X) : A(\Phi^{\mathbf{p}}(\mathcal{B}\mathbf{x})) \in \mathcal{M}(X)\},$$

$$uc_0[A, \mathcal{B}, \Phi, \mathbf{p}, X] = \left\{ \mathbf{x} \in s(X) : \lim_n \sum_k a_{nk} \left(\phi_k \left(\left| \sum_j b^i_{kj} x_j \right| \right) \right)^{p_k} = 0 \right. \\ \left. \text{uniformly in } i \right\}.$$

Taking into account the inclusion

$$Uc_0[A, \mathcal{B}, \Phi, \mathbf{p}, X] \subset U\ell_\infty[A, \mathcal{B}, \Phi, \mathbf{p}, X], \tag{15}$$

from Proposition 2 we get the following result.

Corollary 5. *Let* Φ, $\mathbf{p}$, A, $\mathcal{B}$, X, *and* $\mathbf{X}$ *be the same as in Proposition 2.*

a) *If* $\Phi^{\mathbf{p}/r}$ *satisfies one of conditions* (M5) *and* (M6), *then on the GS space* $U\ell_\infty[A, \mathcal{B}, \Phi, \mathbf{p}, X]$ *we may define the F-seminorm*

$$g^{\Phi,\mathbf{p}}_{\infty,A,\mathcal{B}}(\mathbf{x}) = \sup_{n,i} \sum_k a_{nk} \left(\phi_k \left(\left| \sum_j b^i_{kj} x_j \right| \right) \right)^{p_k}.$$

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b) *If the moduli* $\phi_k \quad (k \in \mathbb{N})$ *are unbounded, and the row-finite and column-positive matrix* A *satisfies* (14), *then* $g^{\Phi,\mathbf{p}}_{\infty,A,\mathcal{B}}$ *is an F-seminorm on the space* $Uc_0[A,\mathcal{B},\Phi,\mathbf{p},X]$.

If, in a) *and* b), X *is normed and* $\mathcal{B}$ *satisfies* (4), *then* $g^{\Phi,\mathbf{p}}_{\infty,A,\mathcal{B}}$ *is an F-norm. Moreover,* $g^{\Phi,\mathbf{p}}_{\infty,A,\mathcal{B}}$ *is an absolutely monotone F-seminorm* (*or F-norm*) *on* $Uc_0[A,\mathcal{B},\Phi,\mathbf{p},\mathbf{X}]$ *if* $\mathcal{B}$ *is a sequence of diagonal matrices.*

Corollary 5 permits to determine F-seminorms (or paranorms), for example, on the spaces $U\ell_\infty[V_\lambda,\mathcal{F}_{C_1},\Phi,X]$ and $Uc_0[V_\lambda,\mathcal{F}_{C_1},\Phi,X]$ from [28], and also on similar spaces from [30].

For an infinite matrix $B=(b_{nk})$ let $\widehat{B}$ be the sequence of matrices $\widehat{B}^i = (b_{n+i,k})_{n,k\in\mathbb{N}} \quad (i\in\mathbb{N})$. In this case we have $\widehat{B}\mathbf{x}=(B_{n+i}\mathbf{x})_{n,i\in\mathbb{N}}$, which, for $B=I$, gives $\widehat{I}\mathbf{x}=(x_{n+i})_{n,i\in\mathbb{N}}$. We can prove a stronger variant of Corollary 5 b) under the assumption that $p_k=1$, $\phi_k=\phi \quad (k\in\mathbb{N})$, $A=C_1$ and $\mathcal{B}=\widehat{B}$. Then the GS space $uc_0[A,\mathcal{B},\Phi,\mathbf{p},X]$ reduces to (for the case $B=I$ and $X=\mathbb{K}$ see [40])

$$uc_0[C_1,\widehat{B},\phi,X]=\left\{\mathbf{x}\in s(X): \lim_n n^{-1}\sum_{k=1}^{n}\phi\left(\left|B_{k+i-1}\mathbf{x}\right|\right)=0\right.$$
$$\left.\text{uniformly in } i\right\}.$$

Proposition 4. *Let* $B=(b_{nk})$ *be an m-normal infinite matrix such that* $K=\inf_n|b_{nn}|>0$ *and there exists an index* $j_0>m$ *with* $b_{nk}=0 \quad (k\le n+m-j_0,\ n>j_0-m)$. *The functional*

$$g^{\phi}_{\infty,C_1,\widehat{B}}(\mathbf{x})=\sup_{n,i} n^{-1}\sum_{k=1}^{n}\phi\left(\left|B_{k+i-1}\mathbf{x}\right|\right)=\sup_k\phi\left(\left|B_k\mathbf{x}\right|\right)$$

defines an F-seminorm (*F-norm if* X *is normed and* $m=0$) *on the GS space* $uc_0[C_1,\widehat{B},\phi,X]$ *if and only if the modulus* ϕ *is unbounded.*

Proof. Since $\mathbf{x}\in uc_0[C_1,\widehat{B},\phi,X]$ means that the sequence $\phi(B_k\mathbf{x}) = \left(\phi\left(\left|B_k\mathbf{x}\right|\right)\right)$ is almost convergent to zero, but every almost convergent sequence is bounded (see [17], Theorem 1.2.18), we clearly have

$$uc_0[C_1,\widehat{B},\phi,X]=Uc_0[C_1,\widehat{B},\phi,X]. \tag{16}$$

Moreover, by

$$\phi\left(\left|B_i\mathbf{x}\right|\right)\le\sup_n n^{-1}\sum_{k=1}^{n}\phi\left(\left|B_{k+i-1}\mathbf{x}\right|\right)\le\sup_k\phi\left(\left|B_k\mathbf{x}\right|\right)\quad(i\in\mathbb{N}),$$

we get

$$g^{\phi}_{\infty,C_1,\widehat{B}}(\mathbf{x})=\sup_k\phi\left(\left|B_k\mathbf{x}\right|\right). \tag{17}$$

Sufficiency. If the modulus ϕ is unbounded, then $g^{\phi}_{\infty,C_1,\widehat{B}}$ is an F-seminorm on $uc_0[C_1,\widehat{B},\phi,X]$ by Proposition 3 b) because the matrix C_1 is normal and regular. In particular, since 0-normal matrix is normal and every normal matrix B satisfies (9), the functional $g^{\phi}_{\infty,C_1,\widehat{B}}$ defines an F-norm on $uc_0[C_1,\widehat{B},\phi,X]$ if X is normed and $m=0$.

Necessity. Assume that $g^{\phi}_{\infty,C_1,\widehat{B}}$ is an F-seminorm on $uc_0[C_1,\widehat{B},\phi,X]$ and define (see [35])

$$\hat{w}_0=\{\mathbf{u}=(u_k)\in s:\lim_n n^{-1}\sum_{k=1}^{n}|u_{k+i-1}|=0\ \text{ uniformly in }\ i\}.$$

If $\mathbf{v}=(v_k)$ is the sequence of numbers

$$v_k=\begin{cases}1, & \text{if }\ k=2^j \quad (j\in\mathbb{N})\\ 0 & \text{otherwise,}\end{cases}$$

then for $2^j\le n<2^{j+1}$ we have

$$\sup_i n^{-1}\sum_{k=1}^{n}|v_{k+i-1}|\le n^{-1}\sum_{k=1}^{n}|v_{k+1}|<\frac{j+1}{2^j}\to 0\ \text{ as }\ j\to\infty,$$

and so, $\mathbf{v}\in\hat{w}_0$. By means of $\mathbf{v}$, using a fixed element $y_0\in X$ with $\dot{|}y_0\dot{|}=1$, we consider the X-valued sequence $\mathbf{y}=(y_k)$, $y_k=kv_ky_0\quad(k\in\mathbb{N})$. Now, assuming that the modulus ϕ is bounded and $M=\sup_{t>0}\phi(t)$, by the inequalities

$$\phi\left(\dot{|}y_k\dot{|}\right)\le\phi(k)\le Mv_k\quad(k\in\mathbb{N})$$

we get $\phi(\mathbf{y})\in\hat{w}_0$ because $\hat{w}_0$ is solid sequence space. Further, the equality $B\mathbf{z}=\mathbf{y}$ clearly determines a new X-valued sequence $\mathbf{z}=(z_k)$ with $z_1=\cdots=z_m=0$. This sequence $\mathbf{z}$ is unbounded, since we can find an index i_0 such that, for $i>i_0$,

$$B_{2^i}\mathbf{z}=b_{2^i,2^i}z_{2^i}=y_{2^i}=2^iy_0,$$

and so, $\dot{|}z_{2^i}\dot{|}=2^i|b_{2^i,2^i}|^{-1}\ge 2^i/K$ if $i>i_0$. Moreover, $\mathbf{z}\in uc_0[C_1,\widehat{B},\phi,X]$ by the representation

$$uc_0[C_1,\widehat{B},\phi,X]=\hat{w}_0(B,\phi,X).$$

Thus, as in Remark 2, using equality (17) we can show, that $g^{\phi}_{\infty,C_1,\widehat{B}}$ does not satisfy axiom (N4), i.e., it is not an F-seminorm. □

Assumptions of Proposition 4 are clearly satisfied for $B=I$ and $B=\Delta^m$.

Corollary 6. *The functional*

$$g^{\phi}_{\infty,C_1,\widehat{I}}(\mathbf{x}) = \sup_{n,i} n^{-1} \sum_{k=1}^{n} \phi\left(\left|x_{k+i-1}\right|\right) = \sup_{k} \phi\left(\left|x_k\right|\right)$$

defines an F-seminorm (F-norm if X is normed) on the GS space

$$uc_0[C_1, \widehat{I}, \phi, X] = \{\mathbf{x} \in s(X) : \lim_n n^{-1} \sum_{k=1}^{n} \phi(\left|x_{k+i-1}\right|) = 0 \text{ uniformly in } i\}$$

if and only if the modulus ϕ is unbounded.

Et ([23], Theorem 2.3) asserts that the sequence space

$$[\hat{c}, \phi, \mathbf{p}](\Delta^m) = \left\{\mathbf{u} \in s : \lim_n 1/n \sum_{k=1}^{n} \left(\phi\left(|\Delta^m u_{k+i}|\right)\right)^{p_k} = 0 \text{ uniformly in } i\right\}$$

may be topologized by the paranorm

$$g_{\Delta}(\mathbf{u}) = \sup_{n,i} \left(\sum_{k=1}^{n} \left(\phi\left(|\Delta^m u_{k+i}|\right)\right)^{p_k}\right)^{1/r}$$

for any modulus ϕ. Proposition 4 (with $B = \Delta^m$) shows that this is not true if ϕ is bounded, since the space $[\hat{c}, \phi, \mathbf{p}](\Delta^m)$ reduces, for $p_k = 1 \ (k \in \mathbb{N})$, to $uc_0[C_1, \widehat{\Delta^m}, \phi, \mathbb{K}]$ with $g_{\Delta} = g^{\phi}_{\infty, C_1, \widehat{\Delta^m}}$. Corollary 6 allows us to say that similar inaccuracies may be found in theorems about the topologization of various spaces of type $uc_0[A, \mathcal{B}, \Phi, \mathbf{p}, X]$ from [7], [19], [36], [38], and [45], because all these spaces contain $uc_0[C_1, \widehat{I}, \phi, X]$ as a special case.

Remark 3. For the topologization of GS spaces $uc_0[A, \mathcal{B}, \Phi, \mathbf{p}, X]$ by F-seminorms (or paranorms) $g^{\Phi,\mathbf{p}}_{\infty,A,\mathcal{B}}$ it is necessary that

$$uc_0[A, \mathcal{B}, \Phi, \mathbf{p}, X] \subset U\ell_{\infty}[A, \mathcal{B}, \Phi, \mathbf{p}, X] \tag{18}$$

or, equivalently,

$$uc_0[A, \mathcal{B}, \Phi, \mathbf{p}, X] = Uc_0[A, \mathcal{B}, \Phi, \mathbf{p}, X].$$

The following example shows that (18) is not true in general. Let $A_1 = (a_{nk})$ be the Cesàro matrix $C_1 = (c_{nk})$ which is modified by setting $c_{1k} = 0 \ (k \geq 2)$, and let $\Phi = (\phi_1, \phi, \phi, \dots)$ be the sequence of moduli, where $\phi_1(t) = t$ and ϕ is a bounded modulus. Then the unbounded sequence $\mathbf{y}$, defined in the proof of Proposition 4, belongs to $uc_0[A_1, \widehat{I}, \Phi, X]$ because, for $n \geq 2$, we have

$$\sum_k a_{nk} \phi_k \left(\left|y_{k+i-1}\right|\right) = n^{-1} \sum_{k=2}^{n} \phi\left(\left|y_{k+i-1}\right|\right).$$

But since (for $n = 1$)

$$\sup_i \sum_k a_{1k} \phi_k \left(|y_{k+i-1}| \right) = \sup_i |y_i| = \infty,$$

$\mathbf{y}$ is not in $U\ell_\infty[A_1, \widehat{I}, \Phi, X]$. This example allows us to state that the proofs of inclusions (18) from [19], [38], and [45] are not convincing, and the correctness of the definition of functional $g^{\Phi,\mathbf{p}}_{\infty,A,\mathcal{B}}$ in [7] remains actually open.

Acknowledgement

Research was partially supported by Estonian Targeted Financing Project SF0180039s08.

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