A remark about the positivity problem of fourth order linear recurrence sequences

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Abstract. Consider a fourth order linear recurrence with integer coefficients whose characteristic polynomial has two distinct real and a complex conjugate pair of roots. A new proof showing that its positivity problem is decidable is given for the case where there is exactly one real root having the same absolute value as the two complex conjugate roots.

1. Introduction

By a linear recurrence sequence $(u_n)_{n\geq 0}$ of order $h \in \mathbb{N}, h \geq 2$, we mean here a sequence satisfying

$$u_n = a_1 u_{n-1} + a_2 u_{n-2} + \cdots + a_h u_{n-h} \quad (n \geq h), \qquad (1.1)$$

where $a_1, a_2, \ldots, a_h \ (\neq 0)$ and the initial values $u_0, u_1, \ldots, u_{h-1}$ are given *integers*. An important decision problem related to linear recurrence sequences is the Positivity Problem: are all the terms of such a linear recurrence sequence positive? At present, this problem remains open. However, certain partial results have already appeared, viz., the Positivity Problem for sequences satisfying a second order linear recurrence relation was shown to be decidable by Halava–Harju–Hirvensalo [5] in 2006. The Positivity Problem for sequences satisfying third and fourth order linear recurrences was shown to be decidable in [6], [11] and [12], respectively, see also [9, 10]. As pointed out by Professor J. Ouaknine in a private communication, there is a gap in the proof of Claim 2 on page 141 of [11]. Our objective here is to repair this gap and a few loose arguments by giving a new proof, supplementing the works in [11, 12], that the positivity problem for a fourth order linear recurrence sequence is decidable for one of the hardest cases where its characteristic polynomial has two distinct real and a complex conjugate pair of

roots, and exactly one real root has the same absolute value as the two complex conjugate roots. The proof, though different in some details, uses the same approach as in [10], i.e., invoking upon a deep result about linear forms in logarithms, and was suggested by Professor J. Ouaknine.

Let us recall some facts about recurrence sequences; for general references, see [8] or [4]. The characteristic polynomial associated with the relation (1.1) is

$$\text{Char}(z) := z^h - a_1 z^{h-1} - \cdots - a_{h-1}z - a_h.$$

Let $\lambda_k \in \mathbb{C} \setminus \{0\}$ ($k = 1, \ldots, m$) be all the distinct roots with multiplicities $\ell_1, \ldots, \ell_m$, respectively, of $\text{Char}(z)$, so that $\ell_1 + \cdots + \ell_m = h$. Each sequence element satisfying (1.1) can be written as

$$u_n = \sum_{k=1}^{m} P_k(n)\lambda_k^n \quad (n \geq 0)$$

with $P_k(n) \in \mathbb{C}[n] \setminus \{0\}$, $\deg P_k = \ell_k - 1$ ($k = 1, \ldots, m$). The roots of $\text{Char}(z)$ having the largest absolute value are called *dominating roots*. Such roots play a crucial role in the positivity of the sequence (u_n) as witnessed in the following result of Bell–Gerhold, [2, Theorem 2], which helps reducing considerably the number of cases to be considered.

Lemma 1.1. *Let (u_n) be a nonzero recurrence sequence with no positive dominating characteristic root. Then the sets $\{n \in \mathbb{N}\colon u_n > 0\}$ and $\{n \in \mathbb{N}\colon u_n < 0\}$ have positive density, and so both sets contain infinitely many elements.*

Some auxiliary results are also needed and we list them now.

Lemma 1.2 (see [11], Lemma 2.2). *Let $\varphi, \theta \in [-\pi, \pi)$ with $\theta \notin \{-\pi, 0\}$.*
   I. *If $\theta = s\pi/t$ is a rational multiple of π where s, t (> 0) $\in \mathbb{Z} \setminus \{0\}$, $\gcd(s, t) = 1$, then, as n varies over $\mathbb{N} \cup \{0\}$, the function $\cos(\varphi + n\theta)$ is periodic and takes at most $2t$ explicitly computable distinct values corresponding to $n = 0, 1, \ldots, 2t - 1$.*
  II. *If θ is not a rational multiple of π, then, as n varies over $\mathbb{N} \cup \{0\}$, the range of values of $\cos(\varphi + n\theta)$ is dense in $[-1, 1]$.*
 III. *The function $\cos(\varphi + n\theta)$ takes both positive and negative values for infinitely many $n \in \mathbb{N} \cup \{0\}$.*

Lemma 1.3 (see [11], Claim 1, p. 140). *Let $\theta, \varphi \in [-\pi, \pi]$, $\theta \notin \{-\pi, 0\}$. If θ is not a rational multiple of π, then there is at most one integer $N \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ such that*

$$1 + \cos(\varphi + N\theta) = 0. \tag{1.2}$$

Lemma 1.4 (see [1]). *Let $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$ be algebraic numbers different from 0 or 1, and let $b_1, \ldots, b_m \in \mathbb{Z}$. Write*

$$\Lambda = b_1 \log \alpha_1 + \cdots + b_m \log \alpha_m.$$

Let $A_1, \ldots, A_m$, $B \geq e$ be real numbers such that, for each $j \in \{1, \ldots, m\}$, A_j is an upper bound for the height of α_j, and $B := \max\{|b_r|; r = 1, \ldots, m\}$. Let d be the degree of the extension field $\mathbb{Q}(\alpha_1, \ldots, \alpha_m)$ over $\mathbb{Q}$. If $\Lambda \neq 0$, then

$$\log |\Lambda| > -(16md)^{2(m+2)} \log A_1 \cdots \log A_m \log B.$$

2. The result

Our purpose is to prove the following theorem, which is the case $C(r_1 r_2 z \bar{z})$ in [11].

Theorem 2.1. *Let* $A, B \in \mathbb{R} \setminus \{0\}$, *let* $C \in \mathbb{C}$, *let* $\lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0\}$, *let* $\lambda_3 \in \mathbb{C} \setminus \mathbb{R}$, *and assume that all these numbers are algebraic numbers. If*

$$u_n = A\lambda_1^n + B\lambda_2^n + C\lambda_3^n + \bar{C}\bar{\lambda}_3^n$$

and $\lambda_1 = |\lambda_3| > |\lambda_2|$, *then the Positivity Problem of the sequence* (u_n) *can be effectively solved.*

Proof. This is one of the hardest cases in our earlier work [11]. Let $\lambda_3 = |\lambda_3|e^{i\theta}$, $C = |C|e^{i\varphi}$ where $\theta, \varphi \in [-\pi, \pi)$, $\theta \notin \{-\pi, 0\}$ so that

$$u_n = \lambda_1^n \left\{ A + 2|C| \cos(\varphi + n\theta) + B(\lambda_2/\lambda_1)^n \right\}.$$

The sequence (u_n) is nonnegative if and only if

$$A \geq -2|C| \cos(\varphi + n\theta) - B(\lambda_2/\lambda_1)^n \quad (n \geq 0). \tag{2.1}$$

We consider two possibilities depending on whether θ is a rational multiple of π.

POSSIBILITY 1: θ is a rational multiple of π. The same arguments as in the last paragraph of [11, page 139] show that this case is decidable.

POSSIBILITY 2: θ is not a rational multiple of π. Rewrite the terms of the sequence as

$$u_n = |\lambda_2|^n \left\{ (A + 2|C| \cos(\varphi + n\theta)) (\lambda_1/|\lambda_2|)^n + B(\lambda_2/|\lambda_2|)^n \right\}.$$

The sequence (u_n) is nonnegative if and only if

$$\{A + 2|C| \cos(\varphi + n\theta)\} (\lambda_1/|\lambda_2|)^n + B(\lambda_2/|\lambda_2|)^n \geq 0 \quad (n \geq 0). \tag{2.2}$$

If $|C| = 0$, then $u_n = A\lambda_1^n + B\lambda_2^n$, which is of the form (HHH1) in [11, Lemma 2.3], and so is decidable.

Assume henceforth that $|C| > 0$.

(a) If $A \leq 0$, then the same arguments as in [11, part (a), p. 140] show that this case is decidable.

(b) If $0 < A < 2|C|$, then the same arguments as in [11, part (b), p. 140] show that this case is decidable.

(c) If $A > 2|C| > 0$, then the same arguments as in [11, part (c), p. 140] show that this case is decidable.

(d) If $A = 2\,|C|$, then (2.2) becomes

$$2\,|C|\left\{1 + \cos(\varphi + n\theta)\right\}(\lambda_1/\,|\lambda_2|)^n + B(\lambda_2/\,|\lambda_2|)^n \geq 0 \quad (n \geq 0).$$

If there is $N_0 \in \mathbb{N} \cup \{0\}$ such that $1 + \cos(\varphi + N_0\theta) = 0$ holds, then for (2.2) to hold we must have $B(\lambda_2/\,|\lambda_2|)^{N_0} \geq 0$. Note in addition that N_0 is unique by Lemma 1.3. Since $T_1(n) := 2\,|C|\left\{1 + \cos(\varphi + n\theta)\right\} \geq 0$ and $T_2(n) := (\lambda_1/\,|\lambda_2|)^n \to \infty$ $(n \to \infty)$, using Lemma 2.2 below, which assures us that even in the worst situation where the terms $T_1(n)$ tend to zero, the product $T_1(n)T_2(n) \to \infty$ $(n \to \infty)$, we deduce that

$$2\,|C|\left\{1 + \cos(\varphi + n\theta)\right\}(\lambda_1/\,|\lambda_2|)^n + B(\lambda_2/\,|\lambda_2|)^n \to \infty \quad (n \to \infty).$$

Thus, there is an explicitly computable least integer $N_1 \in \mathbb{N} \cup \{0\}$, depending on $B, C, \varphi, \theta, \lambda_1, \lambda_2$, such that

$$2\,|C|\left\{1 + \cos(\varphi + n\theta)\right\}(\lambda_1/\,|\lambda_2|)^n + B(\lambda_2/\,|\lambda_2|)^n \geq 0 \quad \text{for all } n \geq N_1.$$

Using all the obtained information, we conclude that (2.2) holds if and only if $N_1 = 0$. $\square$

There remains to prove the following lemma, which is Claim 2 in [11]. The proof given here makes use of Lemma 1.4.

Lemma 2.2. *Keeping the notation in the statement and in the proof of Theorem 2.1, and in particular the condition of Case* (d), *if θ is not a rational multiple of π and if $(n_k) \subset \mathbb{N} \cup \{0\}$ is an increasing sequence of integers such that*

$$1 + \cos(\varphi + n_k\theta) \to 0 \quad (k \to \infty),$$

then

$$2|C|\left\{1 + \cos(\varphi + n_k\theta)\right\}\left(\frac{\lambda_1}{|\lambda_2|}\right)^{n_k} + B\left(\frac{\lambda_2}{|\lambda_2|}\right)^{n_k} \to \infty \quad (k \to \infty). \quad (2.3)$$

Proof. For each $n_k \in \mathbb{N} \cup \{0\}$, since

$$2\ell_k\pi < \varphi + n_k\theta \leq (2\ell_k + 2)\pi \quad \text{for some suitable } \ell_k \in \mathbb{Z},$$

using $\theta, \varphi \in [-\pi, \pi)$, $\theta \notin \{-\pi, 0\}$, we have that there exists $\ell_k \in \mathbb{Z}$ with

$$-\frac{n_k + 1}{2} - 1 \ \leq \ell_k \ < \frac{n_k + 1}{2} \qquad (2.4)$$

such that $\varphi + n_k\theta - (2\ell_k + 1)\pi \in (-\pi, \pi]$. Since

$$1 + \cos(x + \pi) \geq \frac{|x|^2}{2!} - \frac{|x|^4}{4!} \geq \frac{1}{20}\,|x|^2 \quad \text{for all } x \in (-\pi, \pi],$$

we have, for k sufficiently large,

$$
\begin{aligned}
1 + \cos(\varphi + n_k\theta) &= 1 + \cos(\varphi + n_k\theta - (2\ell_k + 1)\pi + \pi) \\
&\geq \frac{1}{20} \left|\varphi + n_k\theta - (2\ell_k + 1)\pi\right|^2 \\
&= \frac{1}{20} \left|\log\left(\frac{C}{|C|}\right) + n_k \log\left(\frac{\lambda_3}{|\lambda_3|}\right)\right. \\
&\quad \left. - (2\ell_k + 1)\log(-1)\right|^2 .
\end{aligned}
\tag{2.5}
$$

Since the coefficients A, B, C and the roots λ_1, λ_3 are nonzero algebraic numbers with $\lambda_3/|\lambda_3| = \lambda_3/\lambda_1 \neq 1$, by Lemma 1.4, we have

$$
\left|\log\left(\frac{C}{|C|}\right) + n_k \log\left(\frac{\lambda_3}{|\lambda_3|}\right) - (2\ell_k + 1)\log(-1)\right| > \max(n_k, 2\ell_k + 1)^{-c}
\tag{2.6}
$$

for some positive constant c independent of k. Since $\lambda_1/|\lambda_2| > 1$ and

$$
B(\lambda_2/|\lambda_2|)^{n_k} = \pm B,
$$

using (2.5), (2.6) and (2.4), the desired result (2.3) follows. $\qquad\square$

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