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Interaction between the Fourier transform and the Hilbert transform

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Dedicated to the memory of my friend and colleague S. Baron

Abstract. Well-known and recently observed situations where the two main transforms in harmonic analysis, the Fourier transform and the Hilbert transform, show their adjacency and interplay in a specific interesting manner are overviewed. Some relations of that kind are new, while in other cases well-known formulas are considered in a different setting.

1. Introduction

In this expository paper we overview well-known and recently observed situations where the two main transforms in harmonic analysis, the Fourier transform and the Hilbert transform, show their adjacency and interplay in a specific interesting manner. Some relations of that kind are new, while in other cases well-known formulas being considered in a different setting turn out to show up hidden sides of their nature.

We have chosen the following topics and/or problems for comparative study and discussion: the problem of re-expanding a function with the integrable cosine (sine) Fourier transform into the integrable sine (cosine) Fourier transform; the Paley–Wiener theorem on the integrability of the Hilbert transform of an odd integrable monotone function; the Hardy–Littlewood theorem on the absolute convergence of the Fourier series of a function which is of bounded variation along with its conjugate, and its generalizations; and the problem of the integrability of the Fourier transform of a function of bounded variation.

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In this order we organize the sections which follow the one, in which certain preliminaries are given, that goes immediately after the introduction. The paper is completed with some concluding remarks.

2. Preliminaries

In this section we introduce certain well-known facts and notions for both transforms, which we shall systematically use in the sequel. In what follows $a \ll b$ means that $a \le Cb$ for some absolute constant C but we are not interested in explicit indication of this constant.

2.1. Fourier transform weakly generates Hilbert transform. Let us begin with presenting a natural formal way how the Hilbert transform appears in close connection with the Fourier transform, following Chapter V of Titchmarsh's celebrated book [23].

Let

$$\widehat{g}(x) = \int_{-\infty}^{\infty} g(t)e^{-ixt}dt \tag{1}$$

be the Fourier transform of g; let it exist in that or another sense, no need in greater accuracy so far.

The Fourier integral formula can be written as an analog of the Fourier series in the following manner:

$$g(t) = \int_0^{\infty} [a(x)\cos tx + b(x)\sin tx]\,dx, \tag{2}$$

where

$$a(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} g(t)\cos xt\,dt, \quad b(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} g(t)\sin xt\,dt.$$

Formally, the integral in (2) is the limit, as $u \to 0$, of the integral

$$\int_0^{\infty} [a(x)\cos tx + b(x)\sin tx]e^{-ux}\,dx = U(t,u),$$

while the latter is the real part of the function

$$\int_0^{\infty} [a(x) - ib(x)]e^{izx}\,dx = \Phi(z),$$

with $z = t + iu$. The imaginary part of the function $\Phi(z)$ is

$$-\int_0^{\infty} [b(x)\cos tx - a(x)\sin tx]e^{-ux}\,dx = V(t,u).$$

Denoting $V(t,0)=\mathcal{H}g(t)$, we obtain

$$\begin{aligned}\mathcal{H}g(t)&=-\int_0^\infty [b(x)\cos tx-a(x)\sin tx]\,dx\\&=\frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty \sin(t-v)x\,g(v)\,dv\,dx.\end{aligned} \tag{3}$$

The integral on the right-hand side of (3) is called the conjugate integral of the Fourier integral. It can formally be derived from (2) by replacing a and b with $-b$ and a, respectively.

It can be formally established that

$$\widehat{\mathcal{H}g}(-x)=i\,\mathrm{sign}\,x\,\widehat{g}(x). \tag{4}$$

Further, and once more formally,

$$\mathcal{H}g(t)=\frac{1}{\pi}\int_0^\infty \frac{g(t-v)-g(t+v)}{v}\,dv=\frac{1}{\pi}\int_{-\infty}^\infty \frac{g(v)}{v-t}\,dv. \tag{5}$$

In parallel,

$$g(t)=-\frac{1}{\pi}\int_0^\infty \frac{\mathcal{H}g(t-v)-\mathcal{H}g(t+v)}{v}\,dv=-\frac{1}{\pi}\int_{-\infty}^\infty \frac{\mathcal{H}g(v)}{v-t}\,dv. \tag{6}$$

This duality was first noticed by Hilbert, hence the pair of these transforms are called the Hilbert transforms. What is given above is, in general, much of the formal theory, at least initial, of the Hilbert transform. However, the change of the order of such operations as limit or integration has never been justified during that presentation. Of course, no universal way exists for this, all depends upon the setting in which the game takes place. There are several of them which proved to be useful. Justifying the above operations in an appropriate setting is, in a sense, the corresponding "genuine" theory.

One more thing that may vary from setting to setting and is sometimes a matter of taste is the sign before the integral and sometimes i or absence of it before the integral. In the next subsection we will specify the above consideration in the L^1 setting.

2.2. Hilbert transform and Hardy space. Thus, the Hilbert transform of an integrable function g is defined by

$$\mathcal{H}g(x)=\frac{1}{\pi}\int_{\mathbb{R}}\frac{g(t)}{x-t}\,dt,$$

where the integral is understood in the improper (principal value) sense, as $\lim\limits_{\delta\to 0+}\int_{|t-x|>\delta}$. It is well known that this limit exists almost everywhere.

It is not necessarily integrable, and when it is, we say that g is in the (real) Hardy space $H^1(\mathbb{R})$. If $g \in H^1(\mathbb{R})$, then it satisfies the cancelation property (has mean zero)

$$\int_{\mathbb{R}} g(t)\,dt = 0. \tag{7}$$

It was apparently first mentioned by in [12].

An odd function always has mean zero. However, not every odd integrable function belongs to $H^1(\mathbb{R})$: in [17] an example of an odd function with non-integrable Hilbert transform is given. To this end, take $g(t) = (t-1)^{-1}|\ln^{-2}(t-1)|$ on $(1, \frac{3}{2})$, $g(t) = -g(-t)$ on $(-\frac{3}{2}, -1)$, and 0 otherwise. Then for $x \in (\frac{1}{2}, 1)$

$$\begin{aligned} |\mathcal{H}g(x)| &\ge \left| \int_1^{1+(1-x)} \frac{1}{(t-1)\ln^2(t-1)} \frac{dt}{t-x} \right| - \frac{2}{3\ln 2} \\ &\ge \frac{1}{2(1-x)|\ln(1-x)|} - \frac{2}{3\ln 2}, \end{aligned}$$

which is obviously non-integrable. Similarly, an example in the even case is a modification of Pitt's example given in [12, Theorem 1 (b)]: taking $g_1(t) = t^{-1}\ln^{-2} t$ and $g_2(t) = 2(\ln 2)^{-1}$ in $(0, 1/2)$, $g_1(t) = g_2(t) = 0$ otherwise, $g(t) = g_1(t) - g_2(t)$. This function satisfies (7), is integrable on $\mathbb{R}$ and, by routine calculations as above, its Hilbert transform does not belong to $L_1(-\frac{1}{2}, 0)$. It remains to extend it even and take into account that the even extension possesses the same properties (see [4, Chapter III, Lemma 7.40, p. 354]).

When in the definition of the Hilbert transform the function g is odd, we will denote this transform by $\mathcal{H}_o$, and it is equal to

$$\mathcal{H}_o g(x) = \frac{2}{\pi} \int_0^\infty \frac{tg(t)}{x^2 - t^2}\,dt.$$

If this transform is integrable, we will denote the corresponding Hardy space by $H^1_o(\mathbb{R})$.

Correspondingly, when in the definition of the Hilbert transform the function g is even, we will denote this transform by $\mathcal{H}_e$, and it is equal to

$$\mathcal{H}_e g(x) = \frac{2}{\pi} \int_0^\infty \frac{xg(t)}{x^2 - t^2}\,dt.$$

The real Hardy space can be characterized in a different way, via the so-called atomic decomposition. Let $a(x)$ denote an atom (a $(1, \infty, 0)$-atom), a function that is of compact support:

$$\operatorname{supp} a \subset I = [x_0 - r, x_0 + r]; \tag{8}$$

and satisfies the following size condition (L^∞ normalization)

$$\|a\|_\infty \le \frac{1}{|I|}; \tag{9}$$

and the cancelation condition

$$\int_{\mathbb{R}} a(x)\, dx = 0. \tag{10}$$

It is well known (see, e.g., [4] or [22]) that

$$\|f\|_{H^1} \sim \inf\{\sum_k |c_k| < \infty : f(x) = \sum_k c_k a_k(x)\}, \tag{11}$$

where a_k are the above-described atoms and $\sum\limits_k |c_k| < \infty$ ensures that the sum $\sum\limits_k c_k a_k(x)$ converges in the L^1 norm.

2.3. Functions of bounded variation. As for the definition of bounded variation, we are not going to concentrate on various details. On the contrary, following Bochner [2, Chapter 1], where the Fourier transform of a monotone function is studied, we will mainly restrict ourselves to functions with Lebesgue integrable derivative. Indeed, every such function is of bounded variation in the sense that it is representable as a linear combination (generally, with complex coefficients) of monotone functions. Of course, the usual definition that applies to the uniform boundedness of the sums of oscillations of a function over all possible systems of non-overlapping intervals might be helpful.

3. Re-expansion

It is not a novelty that relations between cosine and sine Fourier expansions are similar to those between the function and its conjugate. In the 1950s (see, e.g., [8] or in more detail [10, Chapters II and VI]), the following problem in Fourier Analysis attracted much attention.

Let $\{a_k\}_{k=0}^\infty$ be the sequence of the Fourier coefficients of the absolutely convergent sine (cosine) Fourier series of a function $f : \mathbb{T} = [-\pi, \pi) \to \mathbb{C}$, that is $\sum |a_k| < \infty$. Under which conditions on $\{a_k\}$ the re-expansion of $f(t)$ ($f(t) - f(0)$, respectively) in the cosine (sine) Fourier series will also be absolutely convergent?

The obtained condition is quite simple and is the same in both cases:

$$\sum_{k=1}^{\infty}|a_k|\ln(k+1)<\infty.$$

This condition is sufficient and sharp on the whole class.

We study (see [15]) a similar problem for Fourier transforms defined on $\mathbb{R}_+=[0,\infty)$. Let

$$\int_0^\infty |F_c(x)|\,dx<\infty,$$

correspondingly,

$$f(t)=\frac{1}{\pi}\int_0^\infty F_c(x)\cos tx\,dx;$$

or, alternatively,

$$\int_0^\infty |F_s(x)|\,dx<\infty$$

and hence

$$f(t)=\frac{1}{\pi}\int_0^\infty F_s(x)\sin tx\,dx.$$

Under which (additional) conditions on F_c we get the integrability of F_s, or, in the alternative case,

under which (additional) conditions on F_s we get the integrability of F_c?

Theorem 1. *In order that the re-expansion F_s of f with the integrable cosine Fourier transform F_c be integrable, it is necessary and sufficient that its Hilbert transform $\mathcal{H}F_c(x)$ be integrable and $\int_0^\infty F_c(t)\,dt=0$ hold.*

Similarly, in order that the re-expansion F_c of f with the integrable sine Fourier transform F_s be integrable, it is necessary and sufficient that its Hilbert transform $\mathcal{H}F_s(x)$ be integrable.

In fact, we prove that

$$F_s(x)=\mathcal{H}F_c(x)\qquad\text{and}\qquad F_c(x)=-\mathcal{H}F_s(x).$$

These formulas are known (see, e.g., the monograph [11]) for, say, square integrable functions. But that proof reduces to (Carleson's solution of) Lusin's conjecture.

In our L^1 setting this is by no means applicable. And, indeed, our proof is different and rests on the less restrictive approach (see [25, Volume II, Chapter XVI, Theorem 1.22]; even more general result can be found in [23, Theorem 107]).

Theorem A. *If* $\frac{|f(t)|}{1+|t|}$ *is integrable on* $\mathbb{R}$*, then the* $(C,1)$ *means*

$$-\frac{1}{\pi}\int_{-\infty}^{\infty} f(x+t)\left[\frac{1}{t}-\frac{\sin Nt}{Nt^2}\right]dt$$

converge to the Hilbert transform $\mathcal{H}f(x)$ *almost everywhere as* $N\to\infty$*.*

The above results naturally give rise to the problem of conditions which ensure the integrability of the Hilbert transform. Let us give a new condition of that sort.

Theorem 2. *Suppose* g *is a bounded function on* $\mathbb{R}$ *such that for some non-negative function* φ

$$|g(t)| \le \frac{\varphi(t)}{1+|t|} \tag{12}$$

with

$$\sum_{k=0}^{\infty} k\lambda_k = \sum_{k=0}^{\infty} k \sup_{2^{k-1}<|t|\le 2^k} \varphi(t) < \infty. \tag{13}$$

If it has mean zero, then $g \in H^1(\mathbb{R})$*.*

Proof. The scheme of the proof is the same as in [22, Chapter 2, §7] for the case where $g(t) = O(\frac{1}{1+t^2})$. We denote $g_0(t) = g(t)$ when $|t| \le 1$ and $g_0(t) = 0$ otherwise. Further, for $k = 1, 2, ...$ define

$$g_k(t) = \begin{cases} g(t), & 2^{k-1} < |t| \le 2^k, \\ 0, & \textit{otherwise} \end{cases}$$

and

$$c_k = \int_{\mathbb{R}} g_k(t)\,dt = \int_{2^{k-1}<|t|\le 2^k} g(t)\,dt.$$

Note that under the assumptions of the theorem g is an integrable function, since

$$|c_k| \le \int_{2^{k-1}<|t|\le 2^k} \frac{\lambda_k}{t}\,dt. \tag{14}$$

We have

$$g(t) = \sum_{k=0}^{\infty} g_k(t).$$

Let us also denote

$$S_k = \sum_{j \ge k} c_j;$$

it follows from the assumption of the theorem that $S_0 = 0$.

Let us now take a bounded function η supported in $\{|t| \le 1\}$ and such that $\int_{\mathbb{R}} \eta(t)\,dt = 1$. Denoting $\eta_k(t) = \frac{1}{2^k}\eta(\frac{t}{2^k})$, we obtain

$$\int_{\mathbb{R}} \eta_k(t)\,dt = 1$$

and

$$g(t) = \sum_{k=0}^{\infty}[g_k(t) - c_k\eta_k(t)] + \sum_{k=0}^{\infty} c_k\eta_k(t). \tag{15}$$

The first sum is

$$\sum_{k=0}^{\infty}[g_k(t) - c_k\eta_k(t)] = \sum_{k=0}^{\infty}\frac{1}{2^k}\int_{2^{k-1}<|x|\le 2^k}[g_k(t) - g(x)\eta(\frac{t}{2^k})]\,dx = \sum_{k=0}^{\infty} B_k(t).$$

We have

$$\int_{\mathbb{R}} B_k(t)\,dt = \int_{2^{k-1}<|x|\le 2^k} g(x)\,dx\left[1 - \int_{\mathbb{R}}\eta_k(t)\,dt\right] = 0.$$

The function $B_k(t)$ is supported in $\{|t| \le 2^k\}$. Finally, for $2^{k-1} < |t| \le 2^k$, there holds $|g_k(t)| \le C\frac{\lambda_k}{2^k}$. Combining this with (14), we obtain

$$\|B_k(t)\|_\infty \le C\frac{\lambda_k}{2^k}.$$

By this, $B_k(t) = C\lambda_k A_k(t)$, where $A_k(t)$ is an atom supported in $\{|t| \le 2^k\}$. Here, the numbers λ_k are the coefficients of an atomic decomposition of g.

Concerning the second sum in (15), it can be represented in an equivalent form due to the mean zero property ($S_0 = 0$):

$$\sum_{k=1}^{\infty} S_k(\eta_k - \eta_{k-1}).$$

Each value $\eta_k(t) - \eta_{k-1}(t)$ is a multiple of an analogous atom. The numbers S_k are appropriate for an atomic decomposition, since

$$\sum_{j=0}^{\infty}|S_j| = \sum_{j=0}^{\infty}\left|\sum_{k=j}^{\infty} c_k\right| \le \sum_{j=0}^{\infty}\sum_{k=j}^{\infty}\int_{2^{k-1}<|t|\le 2^k}|g(t)|\,dt$$
$$\le C\sum_{j=0}^{\infty}\sum_{k=j}^{\infty}\lambda_k = C\sum_{k=1}^{\infty}k\lambda_k,$$

and (13) completes the proof. □

A representative example of a condition that satisfies Theorem 2 is a function that behaves as $|g(t)| = O(\frac{1}{|t|\ln^{\alpha}|t|})$ as $|t|\to\infty$, with $\alpha > 2$.

4. A Paley–Wiener theorem

Let us treat the problem of the integrability of the Hilbert transform in a different way. It is well known that for the Hilbert transform $\mathcal{H}g(x)$ and the weight $w(x) = |x|^{\alpha}$ with $-1 < \alpha < p-1$, there holds $\|\mathcal{H}g\|_{L^p(w)} \ll \|g\|_{L^p(w)}$, $1 < p < \infty$. In [7], Hardy and Littlewood showed that for even functions g, this inequality also holds for $-p-1 < \alpha < p-1$. Later, Flett [3] proved the same results for odd functions provided $-1 < \alpha < 2p-1$.

For $p = 1$, it is known that only weak type inequalities for the Hilbert transform hold. On the other hand, the Paley–Wiener theorem asserts that for an odd and monotone decreasing on $\mathbb{R}_+$ function $g \in L^1$ one has $\mathcal{H}g \in L^1$. This theorem was extended in [18] to general monotone functions. Further, in [17] the weighted analogues of the Paley–Wiener theorem for odd and even (general monotone) functions are proved. In other words, it was an extension of Hardy–Littlewood's [7], Flett's [3] and Andersen's [1] results to the case $p = 1$ under the assumption of (general) monotonicity for an even/odd function.

Besides the initial proof in [19] (for series) and additional study in [24], a different proof of the initial Paley–Wiener theorem can be found in [21, Chapter IV, 6.2]. It was A. Lerner who brought our attention to this fact. Since much will be based on that proof, let us give more details. First of all, the integrability of the Hilbert transform of an integrable function means that the function belongs to the real Hardy space $H^1(\mathbb{R})$.

Let g_0 be a non-negative monotone decreasing function on $(0,\infty)$ such that

$$\int_0^{\infty} g_0(t)\,dt < \infty,$$

and let $f(t) = g_0(t)$ on $(0,\infty)$, and $f(-t) = -f(t)$. The Paley–Wiener theorem then states that $f \in H^1(\mathbb{R})$. The proof in [21, Chapter IV, 6.2] goes along the following lines. For $-\infty < k < \infty$, let

$$a_k(t) = \frac{1}{2^{k+2} g_0(2^k)} g_0(|t|) \operatorname{sign} t$$

when $2^k \le |t| < 2^{k+1}$ and zero otherwise. Obviously, each a_k is an atom. To see that the absolute value of the function is less than the reciprocal of the length of the support interval $[-2^{k+1}, 2^{k+1}]$, just the monotonicity is used. Taking $\lambda_k = 2^{k+2} g_0(2^k)$ and observing that $\sum\limits_{k=-\infty}^{\infty} \lambda_k \le 8 \int_0^\infty g_0(t)\,dt$, we see that the series $\sum\limits_{k=-\infty}^{\infty} \lambda_k a_k(t)$ converges to $f(t)$ except at the origin. Since thus we have an atomic decomposition of f, it belongs to $H^1(\mathbb{R})$.

We can immediately extend the result by taking g to be weak monotone. To define the latter notion, we will assume a function to be defined on $(0, \infty)$, of locally of bounded variation, and vanishing at infinity.

Definition 3. We say that a non-negative function f defined on $(0, \infty)$, is *weak monotone*, written WM, if

$$f(t) \le C f(x) \quad \text{for any} \quad t \in [x, 2x]. \tag{16}$$

Using in the above proof $g_0(t) \le C g_0(2^k)$ provided $g_0 \in WM$ instead of $g_0(t) \le g_0(2^k)$ for monotone g_0, we immediately arrive at the following result more general than Theorem 6.1 in [18].

Theorem 4. *Let g_0 be integrable on $(0, \infty)$ and $g_0 \in WM$. Then $f \in H^1(\mathbb{R})$.*

This shows that for the integrability of the Hilbert transform smoothness conditions are frequently not of crucial importance; certain regularity of the functions rather works.

5. A Hardy–Littlewood theorem

The following result is due to Hardy and Littlewood (see [6] or, e.g., [25, Volume I, Chapter VII, (8.6)]).

If a (periodic) function f and its conjugate $\widetilde{f}$ are both of bounded variation, their Fourier series converge absolutely.

In [16], we generalize the Hardy–Littlewood theorem to functions on the real axis, hence the absolute convergence of the Fourier series should be replaced by the integrability of the Fourier transform.

Since a function f of bounded variation may be not integrable, its Hilbert transform, a usual substitute for the conjugate function, may not exist. One has to use the modified Hilbert transform (see, e.g., [5])

$$\widetilde{f}(x) = \text{(P.V.)}\, \frac{1}{\pi} \int_{\mathbb{R}} f(t) \left\{ \frac{1}{x-t} + \frac{t}{1+t^2} \right\} dt$$

well adjusted for bounded functions. As a singular integral, it behaves like the usual Hilbert transform; the additional term in the integral makes it to be well defined near infinity.

Theorem 5. *Let f be a function of bounded variation which vanish at infinity: $\lim\limits_{|t|\to\infty} f(t) = 0$. If its conjugate $\widetilde{f}$ is also of bounded variation, then the Fourier transforms of the both functions are integrable on $\mathbb{R}$.*

It is interesting that the initial Hardy–Littlewood theorem is a partial case of Theorem 5, where the function is taken to be with compact support.

The proof of Theorem 5 contains two main ingredients: one of them is

Lemma 1. *Under the assumptions of the theorem, we have at almost every x*

$$\frac{d}{dx}\widetilde{f}(x) = \mathcal{H}f'(x).$$

More precisely, "almost everywhere" is specified as at the Lebesgue points of the integrable function f'. This lemma is a direct analog of the various known results for the Hilbert transform of a function from the spaces different from the space of functions of bounded variation; see, e.g., [20, 3.3.1, Theorem 1] or [11, 4.8]. Since the assumptions are different, we use different arguments while interchanging limits.

The second ingredient (and one of the main tools in other issues) is the well-known extension of Hardy's inequality:

$$\int_{\mathbb{R}} \frac{|\widehat{g}(x)|}{|x|}\, dx \ll \|g\|_{H^1(\mathbb{R})}.$$

This inequality shows that for the Fourier transform of a function from the Hardy space we have more than just the Riemann–Lebesgue lemma for an integrable function and its Fourier transform. It also inspires the following observation.

Proposition 1. *Let $\int_{\mathbb{R}} \frac{|\widehat{g}(x)|}{|x|}\, dx$ be finite. Then g is the derivative of a function of bounded variation f which is locally absolutely continuous, $\lim\limits_{|t|\to\infty} f(t) = 0$, and its Fourier transform is integrable.*

This motivates a special study of the Fourier transform of a function of bounded variation, where obtained estimates lead, in particular, to a refinement of Hardy's inequality in certain cases.

6. The Fourier transform of a function of bounded variation

The study of the finiteness of the integral in Proposition 1 brings up a question of existence (or non-existence) of the widest space for the integrability of the Fourier transform of a function of bounded variation.

In fact, such a space does exist (see [14]) and has in essence been introduced (for different purposes) by Johnson and Warner in [9] as

$$Q = \{g : g \in L^1(\mathbb{R}), \int_{\mathbb{R}} \frac{|\widehat{g}(x)|}{|x|}\,dx < \infty\}.$$

With the obvious norm $\|g\|_{L^1(\mathbb{R})} + \int_{\mathbb{R}} \frac{|\widehat{g}(x)|}{|x|}\,dx$ it is a Banach space and ideal in $L^1(\mathbb{R})$.

The space Q_o of the odd functions from Q is of special importance:

$$Q_o = \{g : g \in L^1(\mathbb{R}), g(-t) = -g(t), \int_0^\infty \frac{|\widehat{g_s}(x)|}{x}\,dx < \infty\};$$

such functions naturally have mean zero.

An even counterpart of Q_o is

$$Q_e = \{g : g \in L^1(\mathbb{R}), g(-t) = g(t), \int_0^\infty \frac{|\widehat{g_c}(x)|}{x}\,dx < \infty\}.$$

This makes sense only if $\int_0^\infty g(t)\,dt = 0$.

Theorem 6. *Let $f : \mathbb{R}_+ \to \mathbb{C}$ be locally absolutely continuous, of bounded variation, and $\lim\limits_{t\to\infty} f(t) = 0$.*

a) *Then the cosine Fourier transform of f is Lebesgue integrable on $\mathbb{R}_+$ if and only if $f' \in Q$.*

b) *The sine Fourier transform of f is Lebesgue integrable on $\mathbb{R}_+$ if and only if $f' \in Q$.*

In fact, in a) and b) the conditions are given in terms of different subspaces of Q, namely Q_o and Q_e, respectively.

Proof of Theorem 6. The assumptions of the theorem give a possibility to integrate by parts. First,

$$\widehat{f_c}(x) = -\frac{1}{x}\int_0^\infty f'(t)\sin xt\,dt = -\frac{1}{x}\widehat{f'_s}(x),$$

which immediately proves a).

Further,

$$\widehat{f_s}(x) = \frac{f(0)}{x} + \frac{1}{x}\int_0^\infty f'(t)\cos xt\,dt = \frac{f(0)}{x} + \frac{1}{x}\widehat{f'_c}(x).$$

Integrating both sides over $\mathbb{R}_+$ would complete the proof if we show that $f(0) = 0$. Indeed, if $\widehat{f_s}$ is integrable, then the function f extended from $\mathbb{R}_+$ to the whole $\mathbb{R}$ as an odd function, should be continuous. On the other hand, in this case $f' \in Q$ implies $\int_0^\infty f'(t)\,dt = 0$, since otherwise the integral in the definition of Q diverges at 0. Therefore,

$$\begin{aligned}\int_0^\infty f'(t)\,dt &= \lim_{\delta\to 0+}\lim_{A\to\infty}\int_\delta^A f'(t)\,dt \\ &= \lim_{\delta\to 0+}\lim_{A\to\infty}[f(A)-f(\delta)] = -f(0).\end{aligned}$$

This completes the proof. □

The fact that finding the widest space of integrability turned out to be a rather simple task does not play down the importance of this result. But what is more essential is that it gives rise to various interesting problems, first of all to the one of establishing convenient intermediate spaces between Q and the spaces known in this topic from previous work (see, e.g., [13]).

7. Concluding remarks

What knits the considered issues together? On the one hand, they are somewhat unbalanced, one may get an impression that they are miscellaneous. On the other hand, numerous connections in definitions, tools, etc. prompt that there must be something in common, maybe beyond visual considerations. As an attempt to formulate this, at least, in a heuristic way, one may suppose that the real Hardy space (and possibly those close to it) is a natural environment for many functions if the integrability of the Fourier transform is desirable for them. First of all, the derivatives of functions of bounded variation should belong to such spaces.

The above considerations are purely one-dimensional. However, multidimensional versions are either in work or one may anticipate how they look like. However, in any case the multidimensional picture will necessarily be much more complicated. The presence of various types of variation in many dimensions or/and of several types of Hardy spaces is already a serious obstacle to getting the results similar to those in dimension one. And there are even more obstacles. On the other hand, these open a wide field of action and will hopefully lead to a variety of interesting approaches and results.

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