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$\mathcal{I}_\lambda$-statistically convergent sequences in topological groups

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Abstract. Let $2^{\mathbb{N}}$ be the family of all subsets of $\mathbb{N}$. Using an ideal $\mathcal{I} \subset 2^{\mathbb{N}}$, Savaş and Das in 2011 defined $\mathcal{I}_\lambda$-statistical convergence of real sequences as a generalization of λ-statistical convergence introduced in 2000 by Mursaleen. In this paper we define $\mathcal{I}_\lambda$-statistical convergence for sequences in topological groups and present some inclusion theorems.

1. Introduction

The idea of convergence of a real sequence was extended to statistical convergence by Fast [6] (see also Schoenberg [19]) as follows.

A sequence (x_k) of real numbers is said to be statistically convergent to L if, for arbitrary $\epsilon > 0$, the set $K(\epsilon) = \{k \in \mathbb{N} \colon |x_k - L| \geq \epsilon\}$ has natural density zero, i.e.,

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} \chi_{K(\varepsilon)}(k) = 0,$$

where $\chi_{K(\varepsilon)}$ denotes the characteristic function of $K(\varepsilon)$.

Statistical convergence turned out to be one of the most active areas of research in summability theory after the works of Fridy [7] and Šalát [11]. Di Maio and Kočinac [5] introduced the concept of statistical convergence in topological spaces and statistical Cauchy condition in uniform spaces, and established the topological nature of this convergence. Albayrak and

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Pehlivan [1] studied this notion in locally solid Riesz spaces. Recently, Savaş [17] introduced the generalized double statistical convergence in locally solid Riesz spaces.

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers such that

$$\lambda_1 = 1,\ \lambda_{n+1} \leq \lambda_n + 1 \text{ and } \lambda_n \to \infty \text{ as } n \to \infty.$$

The collection of all such sequences λ will be denoted by Δ.

In [10], a new type of convergence called λ-statistical convergence was introduced. A sequence (x_k) of real numbers is said to be λ-statistically convergent to L if for any $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \Big| \big\{ k \in I_n \colon\ |x_k - L| \geq \epsilon \big\} \Big| = 0,$$

where $I_n = [n - \lambda_n + 1, n]$ and $|A|$ denotes the cardinality of $A \subset \mathbb{N}$. In [10] the relation between λ-statistical convergence and statistical convergence was established among other things. Savaş [15] studied λ-statistical convergence in random 2-normed spaces.

Let $2^{\mathbb{N}}$ be the family of all subsets of $\mathbb{N}$. Recall that a family $I \subset 2^{\mathbb{N}}$ is said to be an ideal if the following conditions hold:

(a) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$,
(b) $A \in \mathcal{I},\ B \subset A$ implies $B \in \mathcal{I}$.

An ideal $\mathcal{I}$ is called proper if $\mathbb{N} \notin \mathcal{I}$, and it is called admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$. For example, the family $\mathcal{I}_{fin}$ of all finite subsets of $\mathbb{N}$ is a proper admissible ideal.

Throughout, $\mathcal{I}$ will stand for a proper admissible ideal.

In [8], Kostyrko et al. introduced the concept of $\mathcal{I}$-convergence of sequences in a metric space and studied some properties of such convergence. Note that $\mathcal{I}$-convergence is an interesting generalization of statistical convergence. A sequence (x_k) of elements of $\mathbb{R}$ is said to be $\mathcal{I}$-convergent to $L \in \mathbb{R}$ if for each $\epsilon > 0$,

$$\big\{ k \in \mathbb{N} \colon\ |x_k - L| \geq \epsilon \big\} \in \mathcal{I}.$$

Furthermore, Savaş and Das [18] defined and studied $\mathcal{I}$-statistical convergence and $\mathcal{I}_\lambda$-statistical convergence. A real sequence (x_k) is said to be

$\mathcal{I}_\lambda$-statistically convergent to L if for any $\epsilon > 0$ and $\delta > 0$,

$$\left\{n \in \mathbb{N} \colon \frac{1}{\lambda_n}\left|\left\{k \in I_n \colon |x_k - L| \geq \epsilon\right\}\right| \geq \delta\right\} \in \mathcal{I}.$$

More investigations in this direction and applications of ideals can be found in [3, 4, 9, 12, 13, 14, 16].

By X we will denote a Hausdorff topological abelian group, written additively, which satisfies the first axiom of countability. In [2], an X-valued sequence (x_k) is called statistically convergent to an element $L \in X$ if for each neighbourhood U of 0,

$$\lim_{n\to\infty} \frac{1}{n}\left|\left\{k \leq n \colon x_k - L \notin U\right\}\right| = 0.$$

The purpose of this paper is to define $\mathcal{I}_\lambda$-statistical convergence of sequences in topological groups and to give some important inclusion theorems.

2. Main results

We start with the definitions of $\mathcal{I}$-statistical convergence and $\mathcal{I}_\lambda$-statistical convergence in topological groups.

Definition 2.1. A sequence (x_k) in X is said to be $\mathcal{I}$-statistically convergent to L if for each neighbourhood U of 0 and each $\delta > 0$,

$$\left\{n \in \mathbb{N} \colon \frac{1}{n}\left|\left\{k \leq n \colon x_k - L \notin U\right\}\right| \geq \delta\right\} \in \mathcal{I}.$$

In this case we write $x_k \to L(S^{\mathcal{I}})$. The class of all $\mathcal{I}$-statistically convergent sequences will be denoted by $S^{\mathcal{I}}(X)$.

Definition 2.2. A sequence (x_k) in X is said to be $\mathcal{I}_\lambda$-statistically convergent to L if for any neighbourhood U of 0 and any $\delta > 0$,

$$\left\{n \in \mathbb{N} \colon \frac{1}{\lambda_n}\left|\left\{k \in I_n \colon x_k - L \notin U\right\}\right| \geq \delta\right\} \in \mathcal{I}.$$

In this case we write $x_k \to L(S_\lambda^{\mathcal{I}})$ and denote by $S_\lambda^{\mathcal{I}}(X)$ the set of all $\mathcal{I}_\lambda$-statistically convergent sequences in X.

It is obvious that every $\mathcal{I}_\lambda$- statistically convergent sequence has only one limit, that is, if a sequence is $\mathcal{I}_\lambda$-statistically convergent to L_1 and L_2 then $L_1 = L_2$.

Remark 2.3. For $\mathcal{I} = \mathcal{I}_{fin}$, $\mathcal{I}$-statistical convergence becomes statistical convergence in topological groups which is studied by Çakalli [2], and $\mathcal{I}_\lambda$-statistical convergence defines the λ-statistical convergence in topological groups. If $\lambda_n = n$, then $\mathcal{I}_\lambda$-statistical convergence reduces to $\mathcal{I}$-statistical convergence.

We now prove our main theorems.

Theorem 2.4. *If $\lambda \in \triangle$ with $\liminf_{n\to\infty} \frac{\lambda_n}{n} > 0$, then $S^{\mathcal{I}}(X) \subset S_\lambda^{\mathcal{I}}(X)$.*

Proof. Let us take any neighbourhood U of 0. Then

$$\begin{aligned}\frac{1}{n}\Big|\big\{k \le n\colon\ x_k - L \notin U\big\}\Big| &\ge \frac{1}{n}\Big|\big\{k \in I_n\colon\ x_k - L \notin U\big\}\Big| \\ &= \frac{\lambda_n}{n}\frac{1}{\lambda_n}\Big|\big\{k \in I_n\colon\ x_k - L \notin U\big\}\Big|.\end{aligned}$$

If $\liminf_{n\to\infty} \frac{\lambda_n}{n} = a$, then the set $\big\{n \in N\colon\ \frac{\lambda_n}{n} < \frac{a}{2}\big\}$ is finite. Thus, for $\delta > 0$ and any neighbourhood U of 0,

$$\begin{aligned}&\left\{n \in N\colon\ \frac{1}{\lambda_n}\Big|\big\{k \in I_n\colon\ x_k - L \notin U\big\}\Big| \ge \delta\right\} \\ &\quad\subset \left\{n \in N\colon\ \frac{1}{n}\Big|\big\{k \le n\colon\ x_k - L \notin U\big\}\Big| \ge \frac{a}{2}\,\delta\right\} \cup \left\{n \in N\colon\ \frac{\lambda_n}{n} < \frac{a}{2}\right\}.\end{aligned}$$

So, if $x_k \to L(S^{\mathcal{I}})$, then the set on the right hand side belongs to I. This completes the proof. □

Theorem 2.5. *Let $\lambda \in \triangle$ be such that $\lim_n \frac{\lambda_n}{n} = 1$. Then $S_\lambda^{\mathcal{I}}(X) \subset S^{\mathcal{I}}(X)$.*

Proof. Let $\delta > 0$ be given. Since $\lim_n \frac{\lambda_n}{n} = 1$, we can choose $m \in N$ such that $\frac{n-\lambda_n+1}{n} < \frac{\delta}{2}$ for all $n \ge m$. Let us take any neighbourhood U of 0. Now observe that

$$\begin{aligned}\frac{1}{n}\Big|\big\{k \le n\colon\ x_k - L \notin U\big\}\Big| &= \frac{1}{n}\Big|\big\{k < n - \lambda_n + 1\colon\ x_k - L \notin U\big\}\Big| \\ &\qquad + \frac{1}{n}\Big|\big\{k \in I_n\colon\ x_k - L \notin U\big\}\Big| \\ &< \frac{n - \lambda_n + 1}{n} + \frac{1}{n}\Big|\big\{k \in I_n\colon\ x_k - L \notin U\big\}\Big| \\ &< \frac{\delta}{2} + \frac{1}{\lambda_n}\Big|\big\{k \in I_n\colon\ x_k - L \notin U\big\}\Big|,\end{aligned}$$

for all $n \geq m$. Hence for $\delta > 0$ and any neighbourhood U of 0,

$$\left\{n \in N\colon \frac{1}{n}\Big|\big\{k \leq n\colon x_k - L \notin U\big\}\Big| \geq \delta\right\}$$
$$\subset \left\{n \in N\colon \frac{1}{n}\Big|\big\{k \in I_n\colon x_k - L \notin U\big\}\Big| \geq \frac{\delta}{2}\right\} \cup \{1, \ldots, m\}.$$

If $x_k \to L(S_\lambda^{\mathcal{I}})$, then the set on the right hand side belongs to $\mathcal{I}$ and so the set on the left hand side also belongs to $\mathcal{I}$. This shows that (x_k) is $\mathcal{I}$-statistically convergent to L. □

Remark 2.6. We do not know whether the condition in Theorem 2.5 is necessary and leave it as an open problem.

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