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After the Bishop–Phelps theorem

"Give me a support point and I'll move the world" (Archimedes)

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Abstract. We give an expository background to the Bishop–Phelps–Bollobás theorem and explore some progress and questions in recently developed areas of related research.

1. Introduction

There has been a lot of recent work related to the Bishop–Phelps–Bollobás theorem, and it is our goal to give a very brief survey describing the background to this theorem as well as some related new developments. The paper will concentrate on three topics: (i) The Bishop–Phelps theorem, (ii) Some extensions of the Bishop–Phelps theorem in the years immediately following the appearance of this result, and (iii) Recent work and problems in this area. We will limit ourselves to the Bishop–Phelps theorem itself, at the expense of omitting completely other approaches to the same problem (such as those of Bronsted–Rockafeller and Ekeland). Readers are encouraged to consult a much more complete survey (as of 2006) by Acosta [1].

Throughout, X will be a Banach space which is (usually) over $\mathbb{R}$, with open unit ball B_X. Given another Banach space Y and $n \in \mathbb{N}$, $\mathcal{L}(^nX, Y)$ will be the space of all n-linear, continuous mappings $X \times \cdots \times X \to Y$, endowed with the natural norm $A \in \mathcal{L}(^nX, Y) \rightsquigarrow \|A\| = \sup\{\|A(x_1, \ldots, x_n)\| \colon x_1, \ldots, x_n \in \overline{B}_X\}$. The continuous linear operators $X \to Y$ are denoted $\mathcal{L}(X, Y)$ (instead of $\mathcal{L}(^1X, Y)$). We agree to write $\mathcal{L}(^nX)$ for $\mathcal{L}(^nX, \mathbb{K})$.

There may be some who believe that this area has its origins before the Common Era (see, e.g., Figure 1). However, there is little controversy that

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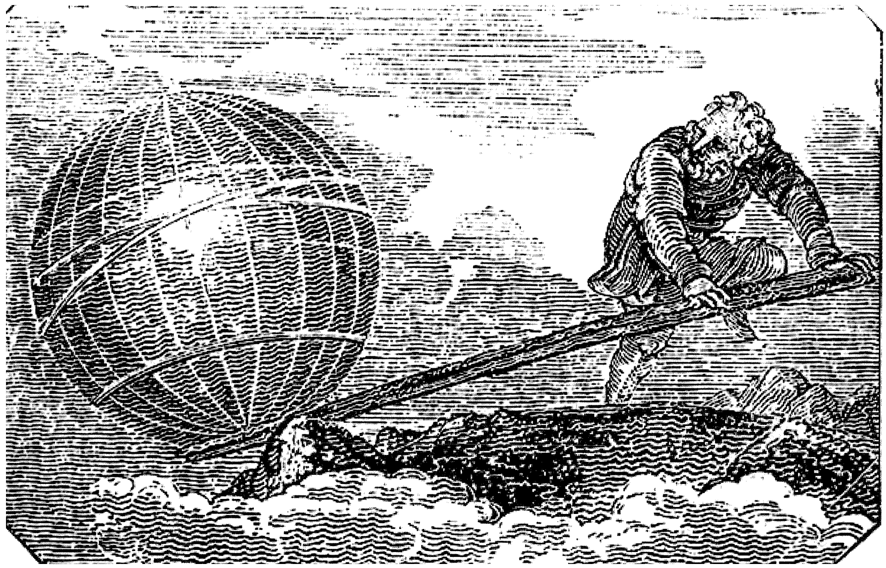

FIGURE 1. Archimedes finding a support point

the modern origin of norm-attaining operators began with Victor Klee's work over fifty years ago.

Definition 1.1 (**V. Klee**, 1958)**.** [20] Let C be a convex subset of a real locally convex space E. A point x in the boundary of C is a *support point* if there exists a continuous linear form $\varphi \in E^*$ such that $|\varphi(x)| = \sup|\varphi(C)|$ (and then φ is a *support functional*).

Klee asked if every closed bounded convex subset C of a Banach space X has at least one support point. Around the same time, R. C. James proved that a Banach space X is *reflexive* (that is, the natural inclusion $X \hookrightarrow X^{**}$ is surjective) if and only if for all $\varphi \in X^*$, there is $x \in \overline{B_X}$ such that $\varphi(x) = \|\varphi\|$. In other words, reflexive spaces X are exactly those with the property that every norm one point of $\overline{B_X}$ is a support point for $\overline{B_X}$ and every norm one functional in X^* is a support functional. (This result was first published in 1957 for separable X [17] and then, six years later, for arbitrary Banach spaces [18].) It is important to emphasize that the question of the existence of support points on closed bounded convex subsets C of X is clearly norm-dependent. What is not quite so evident is that this question is also heavily dependent on the underlying scalar field of the Banach space [27]. Now, in 1977, Bourgain [11] proved the remarkable result that if a complex Banach space X has the Radon–Nikodým property, then the Bishop–Phelps theorem holds for all such sets $C \subset X$. On the other hand, in 2000, the second author showed that, in general, in a *complex* Banach space, there are closed bounded convex sets with no support points [22].

This shows that in general Banach spaces, the complex version of Klee's question has a negative answer. In particular, it means that the Bishop–Phelps theorem cannot be extended to a general complex Banach space.

Let us expand on this somewhat. A Banach space X is said to have the *attainable approximation property* (AAP) if the set of support functionals for any closed bounded convex subset of X is norm-dense in X^*. Since the counterexample of [22] appeared to be very pathological, there was hope that the Bishop–Phelps theorem (for all closed bounded convex subsets of a complex Banach space) would hold for "most" Banach spaces. Unfortunately, in [23], the second author proved that the reality is not so nice. In order to understand the statement of the following result, recall that if $\mathcal{R}$ is a subalgebra of $\mathcal{L}(H, H)$ (with H being a Hilbert space) which is closed in the ultraweak topology, then $\mathcal{R}$ is said to be a *dual algebra.* It has a natural predual $\mathcal{R}_*$ which is the quotient space of the space of trace-class operators $tr(H)$ by the annihilator $\mathcal{R}^\top$ of $\mathcal{R}$ in $tr(H)$. A Banach algebra $\mathcal{R}$ is *uniform* if the spectral radius of any operator in $\mathcal{R}$ coincides with its norm. Such an algebra is always commutative.

Theorem 1.2. [23] *Let $\mathcal{R}$ be a dual uniform algebra. If $\mathcal{R}_*$ has the AAP, then the algebra $\mathcal{R}$ is self-adjoint.*

In other words, the Bishop–Phelps theorem fails in a space which is a predual for any uniform non-self-adjoint dual operator algebra.

The following important technical result is interesting in its own right.

Theorem 1.3. [23] *With the above notation, let $A \in \mathcal{R}$ be a support functional for a closed, bounded, and convex set. Then, viewed as an operator, A has an eigenvalue λ such that $\|A\| = |\lambda|$ and such that the orthogonal projection P_λ onto the corresponding eigenspace L_λ belongs to $\mathcal{R}$.*

2. Bishop–Phelps theorem. Contributions of Bollobás and Lindenstrauss

We begin with the statement of the Bishop–Phelps theorem, which dates from 1961. From it, we will proceed in a non-chronological manner to the 1970 Bollobás improvement of this theorem. Then we will describe the work of Lindenstrauss on a vector-valued version of the Bishop–Phelps theorem (1963).

Theorem (E. Bishop and R. Phelps). [9] *Let X be a Banach space, let $\varphi \in X^*$ and let $\epsilon > 0$ be arbitrary. Then there exists an element of the dual, $\theta \in X^*$, $\|\theta\| = \|\varphi\|$, with the following two properties:*

- *the functional θ attains its norm, i.e., there exists $y \in X, \|y\| = 1$, such that $\theta(y) = \|\theta\|$;*

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- $\|\varphi - \theta\| < \epsilon$.

It is perhaps worth remarking that the paper of Bishop and Phelps is $1\frac{1}{2}$ pages long. It is slightly longer than what amounts to the following significant observation that Bollobás made nine years later and that is known as the **Bishop–Phelps–Bollobás** theorem.

Theorem (B. Bollobás). [10] *Let X be a Banach space and let $\varphi \in X^*$. Let $\epsilon > 0$ be arbitrary and let $x \in \overline{B_X}$ be such that $|\varphi(x)| \geqslant \|\varphi\| - \epsilon^2/4$. Then there exist $\theta \in X^*$ and $y \in X$, $\|y\| = 1$, with the following three properties:*

- *θ attains its norm at the point y;*
- $\|\varphi - \theta\| < \epsilon$;
- $\|x - y\| < \epsilon$.

In short, not only can every continuous linear functional φ be approximated by a norm-attaining one θ, but also the vector at which φ almost attains its norm can be approximated by a vector at which θ does attain the norm.

We now backtrack to 1963, to the following result of Lindenstrauss, which was motivated by a question raised in [9].

Theorem (J. Lindenstrauss). [21] *Let X and Y be Banach spaces and let $T \in \mathcal{L}(X, Y)$ be a continuous linear operator from X to Y. Let $\epsilon > 0$. Then there exists $S \in \mathcal{L}(X, Y)$ with the following properties:*

- $\|S - T\| < \epsilon$;
- *the double transpose $S^{**} \in \mathcal{L}(X^{**}, Y^{**})$ attains its norm.*

The following remarks may help to put Lindenstrauss' theorem into context. First, if X is reflexive, the Lindenstrauss' theorem gives us nothing more than the exact analogy of the Bishop–Phelps theorem, extended to Banach space-valued operators: For all $T \in \mathcal{L}(X, Y)$ and for all $\epsilon > 0$, there is an operator $S \in \mathcal{L}(X, Y)$ such that $\|S - T\| < \epsilon$ and S attains its norm. On the other hand, for non-reflexive spaces, the result is in some sense the best possible. Namely, as the following example (from [21]) shows, if X is non-reflexive, then the exact analogy alluded to above may be simply false. Indeed, let $Y \backsimeq c_0$ be such that Y is strictly convex. (see, e.g., [19], p. 33). Suppose that $T : c_0 \to Y$ is an isomorphism. Then T cannot be approximated by a norm-attaining $S \in \mathcal{L}(c_0, Y)$. To see this, suppose that $S \in \mathcal{L}(c_0, Y)$ attains its norm at $x = (x_n) \in c_0$, $\|x\|_{c_0} = 1$. Without loss of generality, we may take $\|S\| = 1$. For some $N \in \mathbb{N}$, one has $|x_n| \leqslant \frac{1}{2}$ whenever $n \geqslant N$. For all such n, one has $\|x \pm \frac{1}{2}\, e_n\|_{c_0} = 1$. Now,

$$\begin{aligned} 1 = \|S\| = \|Sx\| &= \Big\|\frac{1}{2}\Big(S\big(x + \frac{1}{2}\, e_n\big) + S\big(x - \frac{1}{2}\, e_n\big)\Big)\Big\| \\ &\leqslant \Big\|\frac{1}{2}\, S\big(x + \frac{1}{2}\, e_n\big)\Big\| + \Big\|\frac{1}{2}\, S\big(x - \frac{1}{2}\, e_n\big)\Big\| \leqslant \frac{1}{2} + \frac{1}{2} = 1. \end{aligned}$$

It follows that $\left\| S\bigl(x \pm \frac{1}{2}\, e_n\bigr) \right\| = 1$. Now, since Y is strictly convex and since

$$Sx = \frac{1}{2}\, S\bigl(x + \frac{1}{2}\, e_n\bigr) + \frac{1}{2}\, S\bigl(x - \frac{1}{2}\, e_n\bigr),$$

it must be that $Sx = S\bigl(x \pm \frac{1}{2}\, e_n\bigr)$ for all $n \geqslant N$. Consequently, $Se_n = 0$ for all such n, which means that S is a finite rank operator. But the only operators that can be approximated by finite rank operators are compact ones, and clearly the isomorphism T is not compact. Hence, T is not approximable by norm-attaining operators.

3. Recent research directions and some questions

Roughly speaking, we will focus here on two general areas: **(i)** *Non-linear extensions of the Bishop–Phelps theorem*, and **(ii)** *Extensions of the Bishop–Phelps–Bollobás theorem.*

(i) ***Non-linear extensions of the Bishop–Phelps theorem.*** Our interest here will be centered around continuous multilinear forms. For a Banach space X, let $A \in \mathcal{L}(^n X)$ be a continuous n-linear scalar-valued form. Also, let $\mathcal{P}(^n X)$ be the space of continuous n-homogeneous polynomials $P\colon\ X \to \mathbb{K}$, where by a *continuous n-homogeneous polynomial P* we mean that for some (necessarily unique) symmetric $A \in \mathcal{L}(^n X)$, one has $P(x) = A(x, \ldots, x)$ for all $x \in X$.

Let $\mathcal{L}_{NA}(^n X)$ be the subset of $\mathcal{L}(^n X)$ consisting of all those A whose norm is attained. The first basic question is: Is there a Bishop–Phelps theorem for n-linear forms? That is: *Given $A \in \mathcal{L}(^n X)$ and $\epsilon > 0$, is there a $C \in \mathcal{L}_{NA}(^n X)$ such that $\|A - C\| < \epsilon$?* As we shall see, the answer is both yes and no, depending on particular geometric properities of X. For instance in 1994, C. Finet, E. Werner, and the first author proved the following.

Theorem 3.1. [7] *Suppose that X has the Radon–Nikodým property or that X has Schachermayer's property α (see, e.g.,* [28]*). Then for every $n \in \mathbb{N}$ and every element $A \in \mathcal{L}(^n X)$, one has $A \in \overline{\mathcal{L}_{NA}(^n X)}$.*

We mention parenthetically here that in 1977, Bourgain [11] proved that X has the Radon–Nikodým property if and only if for every equivalent norm on X, the set $\mathcal{L}_{NA}(X, Y)$ is dense in $\mathcal{L}(X, Y)$ for every Banach space Y.

On the other hand, in 1996, M.D. Acosta, F. Aguirre, and R. Payá proved the first negative result in this direction. Namely, they showed that for certain "strange" Banach spaces like Gowers space G, one has $\overline{\mathcal{L}_{NA}(G, G^*)} \subsetneqq \mathcal{L}((G, G^*)$. Since then, similar negative results have been obtained for more "pleasant" Banach spaces. For instance, Y.S. Choi (1997) showed the following.

Example 3.2. (1) [12] For $X = L_1[0,1]$, the set of norm-attaining bilinear forms is not dense in the space of all bounded bilinear forms on X.

In 2009, Y. S. Choi and H. G. Song showed the following.

(2) [15] There is a simple example of a continuous bilinear form $A\colon \ell_1 \times \ell_1 \to \mathbb{R}$ which cannot be approximated by norm-attaining bilinear forms.

The following remark is worth mentioning, since it shows the delicate relation between bilinear forms $X \times X \to \mathbb{K}$ and linear operators $X \to X^*$. In some very special situations, we can employ the techniques used by Lindenstrauss in [21] to transfer results about approximation of linear operators to approximation of bilinear forms (see, e.g., [8]). To be specific, for $A\colon X \times X \to \mathbb{K}$, let $T_A\colon X \to X^*$, $(T_A x_1)(x_2) \equiv A(x_1, x_2)$. This association, $A \rightsquigarrow T_A$, is isometric. However, the isometry does not preserve norm-attaining functions. To see this, let $A\colon \ell_1 \times \ell_1 \to \mathbb{R}$ be defined by

$$A(x,y) = x_1 \sum_{j=1}^{\infty} \frac{j}{j+1} y_j.$$

Then $\|A\| = 1$, although $A \notin \mathcal{L}_{NA}({}^2\ell_1)$. On the other hand, $T_A\colon \ell_1 \to \ell_\infty$ is such that $1 = \|T_A\| = \|T_A(e_1)\|$. In short, the isometry between $\mathcal{L}(\ell_1, \ell_\infty)$ and $\mathcal{L}({}^2\ell_1)$ does not preserve norm-attaining operators.

In the same direction, we note that Finet and Payá [16] have shown that, in contrast to the aforementioned result of [12], every operator in $\mathcal{L}(L_1(\mu), L_1(\mu)^*)$ can be approximated by norm-attaining operators. Now, a characterization is known of spaces Y such that every $T \in \mathcal{L}(\ell_1, Y)$ can be approximated by norm-attaining operators [3]. In contrast to the aforementioned result [15] about bilinear forms on ℓ_1, every $Y = L_1(\mu)$ for σ-finite μ has the above property (as do $\mathcal{C}(K)$, finite dimensional, and uniformly convex spaces Y).

On the other hand, we now show that a version of Lindenstrauss' results holds for multilinear mappings $X \times \cdots \times X \to Y$ and polynomials $P\colon X \to Y$. Roughly speaking, our interest will be in showing the denseness of functions of a certain type, whose "bitranspose" (in a sense to be made precise below) attains the norm, in all functions of that type. Let us now be more specific. Every continuous bilinear mapping $A\colon X \times X \to Y$ has a continuous bilinear extension $\tilde{A}\colon X^{**} \times X^{**} \to Y^{**}$ defined as follows. Fix $z, w \in X^{**}$, and using Goldstine's theorem find bounded nets $(x_\alpha)_\alpha$ and $(y_\beta)_\beta$ in X converging weak-star to z and w, respectively. Following Arens [5], one can then define

$$\tilde{A}(z,w) \equiv \lim_\alpha \lim_\beta A(x_\alpha, y_\beta).$$

(It turns out that the choice of order in these iterated limits is important – in general, $\lim_\alpha \lim_\beta A(x_\alpha, y_\beta) \neq \lim_\beta \lim_\alpha A(x_\alpha, y_\beta)$. However, this will be immaterial for us.) In [8], by extending the techniques in [21], the authors proved that for any Banach spaces X and Y, the space $\mathcal{L}(^2X \times Y)$ of continuous bilinear forms on $X \times Y$ contains as a dense subspace $\{A \in \mathcal{L}(^2X \times Y)\colon\ \tilde{A} \in \mathcal{L}(^2X^{**} \times Y^{**})$ attains the norm$\}$. In this direction, perhaps the best result is due to M. D. Acosta, D. García, and M. Maestre, who extended [8] as follows.

Theorem 3.3. [4] *Let $n \geqslant 2$ and let $X_1, \ldots, X_n$ and F be Banach spaces. The set of continuous n-linear mappings from $X_1 \times \cdots \times X_n$ to F whose Arens extensions to $X_1^{**} \times \cdots \times X_n^{**}$ attain their norms is dense in the space of all n-linear mappings from $X_1 \times \cdots \times X_n$ to F.*

To conclude this part, we briefly mention the two related but more difficult problems of approximation of *symmetric* multilinear mappings and also approximation of n-homogeneous polynomials by norm-attaining functions of the same type. For polynomials, Choi and Kim [13] have shown the following.

Theorem 3.4. *If X has the Radon-Nikodým property, then for every Y, the norm-attaining polynomials in $\mathcal{P}(^nX, Y)$ are dense in $\mathcal{P}(^nX, Y)$.*

It should be noted that in [2], the authors showed that the norm-attaining 2-homogeneous polynomials on Gowers' space are not dense in $\mathcal{P}(^2G, \mathbb{R})$. Thus, some hypothesis (such as the Radon–Nikodým property) is required for the above theorem to hold. In (ii) below, we will briefly revisit this topic, mentioning a recent result of Choi and Kim.

Much less is known concerning approximation using symmetric functions, and in fact this problem appears reasonably difficult. Partial results have been obtained by Payá and Saleh [26], who showed that if X satisfies a certain property α (which is reviewed in [26]) with a sufficiently small constant $\rho > 0$, then $\mathcal{L}_{NA}(^2X)$ is dense in $\mathcal{L}(^2X)$.

(ii) ***Extensions of the Bishop–Phelps–Bollobás theorem.*** The primary objective here is to obtain a type of Bishop–Phelps–Bollobás theorem for operators. For this, we first need the following definition, which comes from [3].

Definition 3.5. A pair of Banach spaces (X, Y) has the *Bishop–Phelps–Bollobás property for operators (BPBp)* if for all $\epsilon > 0$, there are $\eta(\epsilon) > 0$ and $\beta(\epsilon) > 0$ with $\lim_{\epsilon \to 0^+} \beta(\epsilon) = 0$, such that for all $T \in \mathcal{L}(X, Y), \|T\| = 1$, if $x_0 \in X$ is a norm one vector such that $\|T(x_0)\| > 1 - \eta(\epsilon)$, then there exist a norm one point $u_0 \in X$ and a norm one operator $S \in \mathcal{L}(X, Y)$ that satisfy the following conditions:

$$\|S(u_0)\| = 1, \|u_0 - x_0\| < \beta(\epsilon), \text{ and } \|S - T\| < \epsilon.$$

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Of course, that $(X, \mathbb{K})$ has the BPBp follows from the aforementioned Bishop–Phelps–Bollobás theorem. However, there are other pairs (X, Y) that have the BPBp. For this, we need the following definition of *approximate hyperplane series property*, AHSP [3].

Definition 3.6. A Banach space Y is said to have *property AHSP* if for every $\varepsilon > 0$ there exist $\gamma(\varepsilon) > 0$ and $\eta(\varepsilon) > 0$ with $\lim_{\varepsilon \to 0^+} \gamma(\varepsilon) = 0$ such that for every sequence $(y_k)_{k=1}^{\infty}$ in the unit sphere S_Y of Y) and every series $\sum_{k=1}^{\infty} \alpha_k$ of non-negative reals that sums to 1 such that

$$\Big\| \sum_{k=1}^{\infty} \alpha_k y_k \Big\| > 1 - \eta(\varepsilon),$$

there exist a subset $A \subset \mathbb{N}$ and a subset $\{z_k \colon k \in A\} \subset S_Y$ satisfying

(1) $\sum_{k \in A} \alpha_k > 1 - \gamma(\varepsilon)$;
(2) $\|z_k - y_k\| < \varepsilon$ for all $k \in A$;
(3) $y^*(z_k) = 1$ for a certain $y^* \in S_{Y^*}$ and all $k \in A$.

The principal reason for interest in the awkwardly named AHSP is the following result, which comes from [3].

Theorem 3.7. *The pair (ℓ_1, Y) has the BPBp if and only if Y has the AHSP.*

Despite its "ungainly" definition, it is surprisingly difficult to find spaces Y that do not have the AHSP. Indeed, every finite dimensional space, $C(K)$, $L_1(\mu)$ for σ-finite μ, and uniformly convex space has AHSP. A typical example of a space failing AHSP is a strictly convex but not uniformly convex Banach space. Recent work in this area includes the following result of Choi and Kim [14].

Theorem 3.8. *If μ is σ-finite, then $(L_1(\mu), Y)$ has the BPBp implies that Y has the AHSP. Moreover, if Y has the Radon–Nikodým property, then the converse holds.*

We conclude (ii) with a brief overview of recent work on two properties that were originally discussed by Lindenstrauss. We first recall properties A and B [21].

Definition 3.9. A Banach space X is said to have *property A* if for all Banach spaces Y, every $T \in \mathcal{L}(X, Y)$ can be approximated by norm-attaining operators in $\mathcal{L}(X, Y)$. The corresponding property for a range space is the following: Y has *property B* means that for all X, every $T \in \mathcal{L}(X, Y)$ can be approximated by norm-attaining operators in $\mathcal{L}(X, Y)$.

So, by Lindenstrauss' theorem, every reflexive space X has property A. Schachermayer [28] has shown that ℓ_1 has property A. Concerning property

B, of course $Y = \mathbb{R}$ has it, since this is just the Bishop–Phelps theorem. Partington [25] has shown that every Banach space Y can be renormed to have B. On the other hand, it is unknown if $\mathbb{R}^2$ with the Euclidean norm has property B.

We now present a Bishop–Phelps–Bollobás version of properties A and B [6].

Definition 3.10. A Banach space X is said to be a *universal BPB domain space* if for all Y, the pair $(X; Y)$ has the BPBp. Similarly, Y is said to be a *universal BPB range space* if for all X, the pair $(X; Y)$ has the BPBp.

It is clear that if a space is a universal BPB domain (respectively, range) space, then it has property A (respectively, property B). Given that there are Banach spaces Y without AHSP, the aforementioned result of [3] implies that ℓ_1 is not a universal BPB domain space. The situation concerning universal BPB range spaces is somewhat more delicate, since the basic examples of spaces with Lindenstrauss property B are actually universal BPB range spaces [3]. In [6], an example is given of a space that has Lindenstrauss' property B but fails to be a universal BPB range space.

To conclude this note, we state a very new and rather exciting result of M. Martín, and we offer some (apparently) open problems.

Theorem 3.11 (M. Martín)**.** [24] *There are Banach spaces X and Y and a compact operator $T \in \mathcal{L}(X, Y)$ that cannot be approximated by norm-attaining operators.*

Problem 1. For arbitrary X and Y, consider the set of norm-attaining operators in $\mathcal{L}(X, Y)$. Does this set always contain an infinite dimensional vector space? Does it contain a 2-dimensional vector space? This question seems to be open even for $Y = \mathbb{R}$.

Finally, in connection with theorems 1.2 and 1.3, it is natural to ask the following questions.

Problem 2. Does the Banach space $L_1[0, 1]$ have the AAP?

Problem 3. Is the AAP of a predual space equivalent to the self-adjointness for dual uniform subalgebras of $\mathcal{L}(H, H)$?

Added in Proof. The authors are grateful to Dr. Martin Rmoutil who, in April 2014, informed them that ideas from a manuscript "Banach spaces with no proximinal subspaces of codimension 2", arXiv:1307.7958, by C. J. Read can be applied to show that there is an equivalent norm $|||\cdot|||$ on c_0 such that the set of norm attaining functionals on $(c_0, |||\cdot|||)$ does not even contain a 2-dimensional subspace. Thus, the answer to Problem 1 is "no", in general.

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References

[1] M. D. Acosta, *Denseness of norm attaining mappings*, RACSAM. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. **100** (2006), 9–30.

[2] M. D. Acosta, F. J. Aguirre, and R. Payá, *There is no bilinear Bishop–Phelps theorem,* Israel J. Math. **93** (1996), 221–227.

[3] M. D. Acosta, R. Aron, D. García, and M. Maestre, *The Bishop–Phelps–Bollobás theorem for operators*, J. Funct. Anal. **254** (2008), 2780–2799.

[4] M. D. Acosta, D. García, and M. Maestre, *A multilinear Lindenstrauss theorem*, J. Funct. Anal. **235** (2006), 122–136.

[5] R. Arens, *The adjoint of a bilinear operation*, Proc. Amer. Math. Soc. **2** (1951), 839–848.

[6] R. Aron, Y. S. Choi, H. J. Lee, S. G. Kim, and Y. S. Choi, *Bishop–Phelps–Bollobás version of Lindenstrauss properties A and B*, preprint.

[7] R. Aron, C. Finet, and E. Werner, *Some remarks on norm-attaining n-linear forms*, Function spaces (Edwardsville, IL, 1994), 19–28, Lecture Notes in Pure and Applied Mathematics **172**, Dekker, New York, 1995.

[8] R. Aron, D. García, and M. Maestre, *On norm attaining polynomials*, Publ. Res. Inst. Math. Sci. **39** (2003), 165–172.

[9] E. Bishop and R. R. Phelps, *A proof that every Banach space is subreflexive*, Bull. Amer. Math. Soc. **67** (1961), 97–98.

[10] B. Bollobás, *An extension to the theorem of Bishop and Phelps*, Bull. London Math. Soc. **2** (1970), 181-182.

[11] J. Bourgain, *On dentability and the Bishop–Phelps property*, Israel J.Math. **28** (1977), 265–271.

[12] Y. S. Choi, *Norm attaining bilinear forms on* $L^1[0,1]$, J. Math. Anal. Appl. **211** (1997), 295–300.

[13] Y. S. Choi and S. G. Kim, *Norm or numerical radius attaining multilinear mappings and polynomials*, J. London Math. Soc. (2) **54** (1996), 135–147.

[14] Y. S. Choi and S. K. Kim, *The Bishop–Phelps–Bollobás theorem for operators from* $L_1(\mu)$ *to Banach spaces with the Radon–Nikodým property*, J. Funct. Anal. **261** (2011), 1446–1456.

[15] Y. S. Choi and H. G. Song, *The Bishop–Phelps–Bollobás theorem fails for bilinear forms on* $\ell_1 \times \ell_1$, J. Math. Anal. Appl. **360** (2009), 752–753.

[16] C. Finet and R. Payá, *Norm attaining operators from* L_1 *into* L_∞, Israel J. Math. **108** (1998), 139–143.

[17] R. C. James, *Reflexivity and the supremum of linear functionals*, Ann. of Math. (2) **66** (1957), 159–169.

[18] R. C. James, *Characterizations of reflexivity*, Studia Math. **23** (1963/1964), 205–216.

[19] W. B. Johnson and J. Lindenstrauss, *Basic concepts in the geometry of Banach spaces*, Handbook of the Geometry of Banach Spaces, Vol. I, 1–84, North-Holland, Amsterdam, 2001.
[20] V. L. Klee, Jr. *Extremal structure of convex sets. II*, Math. Z. **69** (1958), 90–104.
[21] J. Lindenstrauss, *On operators which attain their norm*, Israel J. Math. **1** (1963), 139–148.
[22] V. Lomonosov, *A counterexample to the Bishop–Phelps theorem in complex spaces*, Israel J. Math. **115** (2000), 25–28.
[23] V. Lomonosov, *The Bishop–Phelps theorem fails for uniform non-selfadjoint dual operator algebras*, J. Funct. Anal. **185** (2001), 214–219.
[24] M. Martín, *Norm-attaining, non-compact operators*, preprint.
[25] J. R. Partington, *Norm attaining operators*, Israel J. Math. **43** (1982), 273–276.
[26] R. Payá and Y. Saleh, *New sufficient conditions for the denseness of norm attaining multilinear forms*, Bull. London Math. Soc. **34** (2002), 212–218.
[27] R. R. Phelps, *The Bishop–Phelps theorem in complex spaces: an open problem*, Function spaces (Edwardsville, IL, 1990), 337–340, Lecture Notes in Pure and Applied Mathematics **136**, Dekker, New York, 1992.
[28] W. Schachermayer, *Norm attaining operators and renormings of Banach spaces,* Israel J. Math. **44** (1983), 201–212.

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