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Traces of operators and their history

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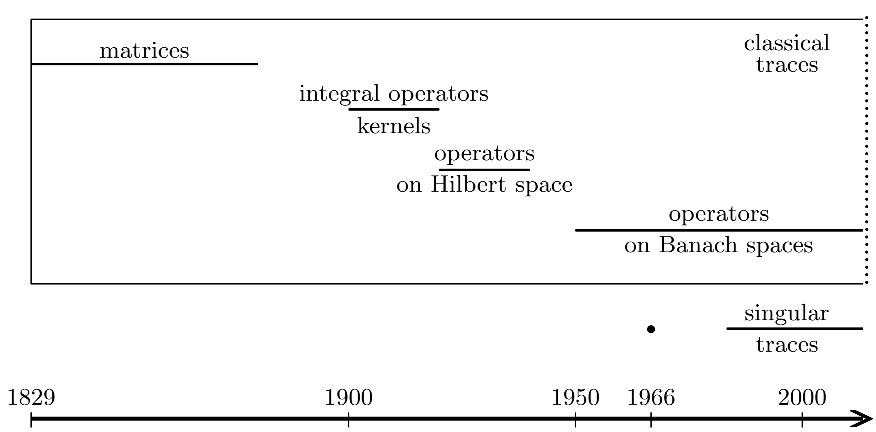
Abstract. As well known, the trace of an $n \times n$-matrix is defined to be the sum of all entries of the main diagonal. Extending this concept to the infinite-dimensional setting does not always work, since non-converging infinite series may occur. So one had to identify those operators that possess something like a trace. In a first step, integral operators generated from continuous kernels were treated. Then the case of operators on the infinite-dimensional separable Hilbert space followed. The situation in Banach spaces turned out to be more complicated, since the missing approximation property causes a lot of trouble. To overcome those difficulties, we present an axiomatic approach in which operator ideals play a dominant rule. The considerations include also singular traces that, by definition, vanish on all finite rank operators.

1. Chronology

The following diagram illustrates the history of trace theory:



Note that the theory of singular traces begins at a singular point.

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2. Preliminaries

The symbol $\mathbb{K}$ stands for the real field $\mathbb{R}$ as well as for the complex field $\mathbb{C}$. If not otherwise specified, we consider both cases simultaneously. Whenever eigenvalues occur, the complex setting is absolutely necessary.

Throughout, $\mathfrak{L}(X,Y)$ denotes the Banach space of all (bounded linear) operators from the Banach space X into the Banach space Y. If $\mathfrak{A}(X,Y)$ is any subset of $\mathfrak{L}(X,Y)$, then we write $\mathfrak{A}(X)$ instead of $\mathfrak{A}(X,X)$. In particular, $\mathfrak{L}(X)$ is the ring of all operators on X.

3. The trace of matrices

In 1829, Cauchy [4] introduced the *characteristic polynomial*

$$P_n(\lambda) := \det(\lambda I_n - S_n) = \lambda^n + d_1\lambda^{n-1} + \cdots + d_{n-1}\lambda + d_n$$

of an $n\times n$-matrix $S_n = (\sigma_{hk})$, where $I_n = (\delta_{hk})$ stands for the unit matrix and λ is any complex number. The explicit form of the coefficients was determined by Jacobi [17, p. 15]. In particular, it turned out that

$$-d_1 = \operatorname{trace}(S_n) := \sum_{k=1}^{n} \sigma_{kk}.$$

Finally, Borchardt [1] obtained the *trace formula*

$$\operatorname{trace}(S_n) = \sum_{k=1}^{n} \lambda_k,$$

in which $\lambda_1, \dots, \lambda_n$ denote the roots of $P_n(\lambda)$, eigenvalues of S_n, counted according to their multiplicities.

4. The origin of the name "trace"

In 1882, Dedekind [8] invented the name *trace*, when he studied number fields of degree n. Such a field Ω is viewed as an n-dimensional linear space over $\mathbb{Q}$, the field of rational numbers. Let $\omega_1, \omega_2 \dots, \omega_n$ be any basis. Then, for every element $\theta \in \Omega$, there exist coefficients $e_{h,k} \in \mathbb{Q}$ such that

$$\begin{aligned}
\theta\omega_1 &= e_{1,1}\omega_1 + e_{2,1}\omega_2 + \cdots + e_{n,1}\omega_n \\
\theta\omega_2 &= e_{1,2}\omega_1 + e_{2,2}\omega_2 + \cdots + e_{n,2}\omega_n \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\theta\omega_n &= e_{1,n}\omega_1 + e_{2,n}\omega_2 + \cdots + e_{n,n}\omega_n.
\end{aligned}$$

Besides $\theta^{(1)} = \theta$, the characteristic polynomial $P_n(\lambda) := \det\big(\lambda\delta_{h,k} - e_{h,k}\big)$ has $n-1$ further zeros $\theta^{(2)}, \dots, \theta^{(n)}$, the so-called conjugates (*conjugirte Zahlen*).

Dedekind wrote: *Unter der* Spur *der Zahl* θ *verstehen wir die Summe aller mit ihr conjugirten Zahlen; wir bezeichnen diese offenbar rationale Zahl mit* $S(\theta)$*; dann ist* $S(\theta) = \theta^{(1)} + \theta^{(2)} + \cdots + \theta^{(n)} = e_{1,1} + e_{2,2} + \cdots + e_{n,n}$.

This means that every $\theta \in \Omega$ *leaves a trace* $S(\theta)$ in the rational field $\mathbb{Q}$.

5. The trace of kernels

In 1912, the Romanian mathematician Lalesco published one of the first books about integral equations, in which he referred to

$$\operatorname{trace}(K) := \int_0^1 K(z,z)\,dz$$

as the *"trace du noyau"*; see [22, p. 30, footnote (^{1})].

To simplify the presentation, we consider only *continuous* kernels K defined on the unit square $[0,1]\times[0,1]$. Working with bounded measurable kernels would yield complications, since we integrate over diagonals, which are null sets.

From the modern point of view, every continuous kernel generates a compact operator

$$K_{\mathrm{op}} : f(t) \mapsto g(s) = \int_0^1 K(s,t)f(t)\,dt$$

on $L_p[0,1]$ with $1 \le p \le \infty$ and $C[0,1]$, whose eigenvalues do not depend on the underlying classical Banach space.

The famous Mercer theorem [26, p. 446] says that, for positive definite symmetric real kernels, we have

$$\int_0^1 K(z,z)\,dz = \sum_{n=1}^{\infty} \lambda_n(K),$$

where $\big(\lambda_n(K)\big)$ is the sequence of eigenvalues, counted according to their multiplicities.

Unfortunately, such a *trace formula* does not hold for arbitrary continuous kernels. This is mainly due to the fact that the eigenvalue sequence need not be absolutely summable. We know from Schur's inequality [42, p. 506] that

$$\sum_{n=1}^{\infty} |\lambda_n(K)|^2 \le \int_0^1 \int_0^1 |K(s,t)|^2\,dsdt.$$

However, trigonometric series constructed by Carleman [3] show that the convergence of the left-hand side with exponent 2 is the best possible result.

One may look at those kernels that have no eigenvalue $\lambda \ne 0$. According to Lalesco [22, p. 32], this happens if and only if the traces of the iterated kernels K_n,

$$K_{n+1}(s,t) := \int_0^1 K_n(s,z)K(z,t)\,dz \quad \text{and} \quad K_1(s,t) := K(s,t),$$

vanish for $n \ge 3$; see also [43, p. 312]. Kaucký [19] added $\operatorname{trace}(K_2) = 0$. Finally, Fenyö [11] "proved" even $\operatorname{trace}(K) = 0$. This result was included also in [12, p. 105]:

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Ein stetiger Kern besitzt genau dann keine von Null verschiedenen Eigenwerte, falls alle seine Spuren verschwinden.

If this result were true, it would yield an affirmative answer to the *approximation problem in Banach spaces*; see Sections 8 and 15.

6. The trace of operators on the Hilbert space

Throughout, H denotes the *infinite-dimensional separable Hilbert space*, which can be identified with ℓ_2.

In his famous book *"Mathematische Grundlagen der Quantenmechanik"* [27] von Neumann defined the *trace* of a positive operator $A : H \to H$ by

$$\operatorname{trace}(A) := \sum_{n=1}^{\infty} (Ae_n|e_n).$$

This quantity does not depend on the special choice of the orthonormal basis (e_n). In view of the assumption, $(Ax|x) \geq 0$ for all $x \in H$, the right-hand infinite series converges to a non-negative real number or ∞. In what follows, we consider only operators with a *finite* trace.

In a next step, Schatten and von Neumann [39, Definitions 2.1 and 2.3] introduced the *trace class operators*, which form an ideal in the ring $\mathfrak{L}(H)$.

We refer to $S \in \mathfrak{L}(H)$ as a *Hilbert–Schmidt operator* if

$$\sum_{n=1}^{\infty} \|Se_n\|^2 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |(Se_n|e_m)|^2$$

is finite for some/every orthonormal basis (e_n).

The *trace class* consists of all products ST of Hilbert–Schmidt operators. Then

$$\operatorname{trace}(ST) := \sum_{n=1}^{\infty} (STe_n|e_n)$$

is well-defined, since the right-hand series converges absolutely, and its value does not depend on the special choice of the orthonormal basis (e_n).

Finally, using von Neumann's work about finite matrices [28], Schatten and von Neumann [40, Theorem 3.2, Lemma 4.3, and Remark 4.1] presented the most natural definition of trace class operators, which is based on the *Schmidt representation*:

Every compact operator $S : H \to H$ can be written in the form

$$S = \sum_{n=1}^{\infty} \sigma_n u_n \otimes v_n, \qquad u_n \otimes v_n : x \mapsto (x|u_n)v_n, \tag{$\mathfrak{S}$}$$

where (u_n), (v_n) are orthonormal sequences in H, and (σ_n) is a non-negative zero sequence; see [41, p. 465].

By easy manipulations, we may arrange that $\sigma_1 \geq \sigma_2 \geq \cdots \geq 0$. Then these coefficients are uniquely determined, and one refers to $s_n(S) := \sigma_n$ as the n-th *singular number*, in short, *s-number*, of S; see [41, pp. 461–462].

Let $0 < p < \infty$. The main observation says that

$$\mathfrak{S}_p(H) := \Big\{ S \in \mathfrak{L}(H) : \sum_{n=1}^{\infty} s_n(S)^p < \infty \Big\}$$

is an ideal in the ring $\mathfrak{L}(H)$. In particular,

$\mathfrak{S}_2(H) = \{\text{Hilbert–Schmidt operators}\}$ and $\mathfrak{S}_1(H) = \{\text{trace class operators}\}$.

For every $S \in \mathfrak{S}_1(H)$, the expression

$$\operatorname{trace}(S) := \sum_{n=1}^{\infty} \sigma_n (v_n | u_n)$$

does not depend on the special choice of the Schmidt representation $(\mathfrak{S})$.

As remarked in [40, p. 575, Added in proof], Schatten and von Neumann proved their results without knowing Calkin's paper [2], which is the starting point of the theory of operator ideals on the Hilbert space. On the other hand, Calkin wrote in [2, Introduction and p. 842] that *"the author is indebted to J. v. Neumann"*. Unfortunately, e-mail was not available at the mid-1940s.

7. The trace of finite rank operators on Banach spaces

Let X and Y be Banach spaces. Denote their dual spaces by X^* and Y^*, respectively.

An operator $F : X \to Y$ is said to have *finite rank* if the range

$$\mathcal{M}(F) := \{ Fx : x \in X \}$$

is finite-dimensional. We denote the set of these operators by $\mathfrak{F}(X, Y)$.

An operator has finite rank if and only if there exists a finite representation

$$F = \sum_{k=1}^{n} u_k^* \otimes v_k, \quad u_k^* \otimes v_k : x \mapsto \langle x, u_k^* \rangle v_k, \tag{$\mathfrak{F}$}$$

where $u_1^*, \ldots, u_n^* \in X^*$ and $v_1, \ldots, v_n \in Y$. Note that the smallest number of summands that can be achieved is equal to $\operatorname{rank}(F) := \dim \big[\mathcal{M}(F)\big]$.

The concept of a trace makes sense only for square matrices. Similarly, in the setting of operators, we must assume that the operators act on one and the same Banach space.

For every finite rank operator $F : X \to X$, we let

$$\operatorname{trace}(F) := \sum_{k=1}^{n} \langle v_k, u_k^* \rangle.$$

Algebraic manipulations show that the right-hand expression does not depend on the special choice of the finite representation $(\mathfrak{F})$.

8. The trace of nuclear operators on Banach spaces

Following Grothendieck [15, Chapter I, p. 80], an operator $S : X \to Y$ is said to be *nuclear* if there exists a representation

$$S = \sum_{k=1}^{\infty} u_k^* \otimes v_k, \quad u_k^* \otimes v_k : x \mapsto \langle x, u_k^* \rangle v_k, \tag{$\mathfrak{N}$}$$

where $u_1^*, u_2^*, \ldots \in X^*$ and $v_1, v_2, \ldots \in Y$ such that

$$\sum_{k=1}^{\infty} \|u_k^*\| \|v_k\| < \infty.$$

The set of these operators is denoted by $\mathfrak{N}(X, Y)$.

Since $S : H \to H$ is nuclear if and only if $S \in \mathfrak{S}_1(H)$, nuclear operators on Banach spaces were, for a while, referred to as *trace class operators* [38] or *opérateurs à trace* [14]. Ironically, there are trace class operators without a trace.

Now we state the **Basic Problem:**
Do there exist nuclear operators $S : X \to X$ for which the expression

$$\operatorname{trace}(S) := \sum_{k=1}^{\infty} \langle v_k, u_k^* \rangle$$

depends on the special choice of the nuclear representation $(\mathfrak{N})$*?*

According to [15, Chapter I, p. 165], the trace of all nuclear operators on a fixed Banach space X is well-defined if and only if X has the *approximation property*:

For every compact subset K of X and every $\varepsilon > 0$ there exists a finite rank operator F on X such that

$$\|x - Fx\| \le \varepsilon \quad \text{whenever} \quad x \in K.$$

In 1972, Enflo [10] discovered a counterexample, which became a turning point in the history of Banach space theory.

An operator theoretical approach to Enflo's construction is based on Grothendieck's observation that the following statements, *which turned out to be false*, are equivalent; see [15, Chapter I, pp. 170–171]:

(1) Every Banach space has the approximation property.

(2) If $A = (\alpha_{hk})$ is any infinite matrix such that

$$\sum_{h=1}^{\infty} \sup_{1 \le k < \infty} |\alpha_{hk}| < \infty \quad \text{and} \quad \sum_{i=1}^{\infty} \alpha_{hi} \alpha_{ik} = 0 \quad \text{for} \quad h, k = 1, 2, \ldots\, ,$$

then

$$\sum_{i=1}^{\infty} \alpha_{ii} = 0.$$

(3) If K is any continuous kernel on $[0,1]\times[0,1]$ such that

$$\int_0^1 K(s,z)K(z,t)\,dz = 0 \quad \text{for} \quad s,t\in[0,1],$$

then

$$\int_0^1 K(z,z)\,dz = 0.$$

This means that the approximation problem has been reduced to problems of classical analysis.

We stress that **(3)** is equivalent to Problem 153 posed by Mazur (prize: a live goose, November 6, 1936) in *The Scottish Book*; see also [15, Chapter I, p. 171, (f) and (g)], [24, p. 36], and [34, p. 285]:

Is the following true?

(4) For every continuous kernel K on $[0,1]\times[0,1]$ and $\varepsilon>0$, there exist points $s_1,\dots,s_n$ and $t_1,\dots,t_n$ in $[0,1]$ as well as constants $c_1,\dots,c_n$ such that

$$\Big|K(s,t)-\sum_{i=1}^{n} c_i K(s,t_i)K(s_i,t)\Big| \le \varepsilon \quad \text{if } 0\le s,t\le 1.$$

This estimate says that K can be approximated by a very special kind of degenerated kernels.

9. p-Nuclear operators

Let $0<p\le 1$. Following Grothendieck [15, Chapter II, p. 3], an operator $S: X\to Y$ is said to be *de puissance p.ème sommable*, now *p-nuclear*, if there exists a representation

$$S=\sum_{k=1}^{\infty} u_k^*\otimes v_k, \quad u_k^*\otimes v_k : x\mapsto \langle x,u_k^*\rangle v_k, \tag{$\mathfrak{N}_p$}$$

where $u_1^*,u_2^*,\ldots\in X^*$ and $v_1,v_2,\ldots\in Y$ such that

$$\sum_{k=1}^{\infty}\|u_k^*\|^p\|v_k\|^p<\infty.$$

The set of these operators is denoted by $\mathfrak{N}_p(X,Y)$. We have $\mathfrak{N}_p(H)=\mathfrak{S}_p(H)$.

In the case when $0<p\le 2/3$, every p-nuclear operator $S: X\to X$ has a well-defined trace; see [15, Chapter II, pp. 18–19]. Davie [7, p. 265] showed that $p=2/3$ is indeed the point at which the behavior of $\mathfrak{N}_p$ turns from the good to the bad.

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10. Operators with p-summable approximation numbers

The *n-th approximation number* of an operator $S : X \to Y$ is defined by

$$a_n(S) := \inf\big\{\|S - F\| : \operatorname{rank}(F) < n\big\}.$$

For $0<p<\infty$, we let

$$\mathfrak{C}_p^{\mathrm{app}}(X,Y) := \Big\{S \in \mathfrak{L}(X,Y) : \sum_{n=1}^{\infty} a_n(S)^p < \infty\Big\}.$$

In the case of compact operators on the Hilbert space, the approximation numbers coincide with Schmidt's s-numbers; see [13, pp. 28–29] and [41, pp. 461–462]. Therefore $\mathfrak{C}_p^{\mathrm{app}}(H)=\mathfrak{S}_p(H)$.

As shown in [29, p. 437], a trace can be assigned to every operator $S\in\mathfrak{C}_1^{\mathrm{app}}(X)$. Its construction will be described in Section 15.

11. Products of 2-summing operators

An operator $S : X \to Y$ is called 2-*summing* if there exists a constant $c\geq 0$ such that

$$\Big(\sum_{k=1}^{n} \|Sx_k\|^2\Big)^{1/2} \leq c \sup_{\|x^*\|\leq 1} \Big(\sum_{k=1}^{n} |\langle x_k, x^*\rangle|^2\Big)^{1/2}$$

for all finite families of elements $x_1, \ldots, x_n \in X$. We denote the set of these operators by $\mathfrak{P}_2(X,Y)$.

Recall that, according to Schatten and von Neumann, the trace class on H consists of all products ST of Hilbert–Schmidt operators. Therefore, in view of $\mathfrak{P}_2(H)=\mathfrak{S}_2(H)$, it seems reasonable to study the set

$$\mathfrak{P}_2\circ\mathfrak{P}_2(X,Y) := \Big\{ST : T\in\mathfrak{P}_2(X,M),\ S\in\mathfrak{P}_2(M,Y),\ M \text{ a Banach space}\Big\}.$$

Since

$$\mathfrak{P}_2\circ\mathfrak{P}_2(X,Y) := \Big\{ST : T\in\mathfrak{N}(X,H),\ \ S\in\mathfrak{L}(H,Y)\Big\},$$

we know from [16] that $\mathfrak{P}_2\circ\mathfrak{P}_2(X)$ supports a trace.

12. Axiomatic theory of traces

So far, we described several classes of operators on Banach spaces for which there exists a trace. Thus the question arises *"What are the characteristic properties of a trace and what is its appropriate domain of definition?"* In contrast to the Hilbert space setting, it makes less sense to fix a specific Banach space. Hence we should adopt the position of the theory of categories.

Recall that $\mathfrak{L}(X,Y)$ denotes the collection of all (bounded linear) operators from the Banach space X into the Banach space Y. Write

$$\mathfrak{L} := \bigcup_{X,Y} \mathfrak{L}(X,Y).$$

Suppose that, for every couple (X,Y), we have selected a subset $\mathfrak{A}(X,Y)$ of $\mathfrak{L}(X,Y)$. According to [30, 1.1.1],

$$\mathfrak{A} := \bigcup_{X,Y} \mathfrak{A}(X,Y)$$

is called an *operator ideal* if $S,T \in \mathfrak{A}(X,Y)$, $A \in \mathfrak{L}(X_0,X)$, $B \in \mathfrak{A}(Y,Y_0)$, $u^* \in X^*$, and $v \in Y$ imply

(O$_1$) $S+T \in \mathfrak{A}(X,Y)$,

(O$_2$) $BSA \in \mathfrak{A}(X_0,Y_0)$,

(O$_3$) $u^* \otimes v \in \mathfrak{A}(X,Y)$.

Obviously, $\mathfrak{A}(X) := \mathfrak{A}(X,X)$ is an ideal in the ring $\mathfrak{L}(X) := \mathfrak{L}(X,X)$.

A *trace* on an operator ideal $\mathfrak{A}$ is a scalar-valued function τ defined on all components $\mathfrak{A}(X)$ for which the following holds:

(T$_1$) $\tau(\alpha S + \beta T) = \alpha\tau(S) + \beta\tau(T)$ for $S,T \in \mathfrak{A}(X)$ and $\alpha,\beta \in \mathbb{K}$,

(T$_2$) $\tau(AS) = \tau(SA)$ whenever

$$\begin{tikzcd} X \arrow[rr, "AS\in\mathfrak{A}"] \arrow[dr, "S\in\mathfrak{A}"'] & & X \arrow[dr, "S\in\mathfrak{A}"] & \\ & Y \arrow[ur, "A\in\mathfrak{L}"'] \arrow[rr, "SA\in\mathfrak{A}"'] & & Y \end{tikzcd} .$$

In the case when τ is defined only on a specific component $\mathfrak{A}(X)$, condition **(T$_2$)** means some kind of commutativity.

To reduce the huge amount of traces that may exist on some operator ideals it is natural to require continuity with respect to a suitable topology. The best way to obtain such topologies is via *quasi-norms* $\|\cdot|\mathfrak{A}\|$. We assume that the following holds for $S,T \in \mathfrak{A}(X,Y)$, $A \in \mathfrak{L}(X_0,X)$, $B \in \mathfrak{A}(Y,Y_0)$, $u^* \in X^*$, $v \in Y$, and some constant $c \geq 1$:

(Q$_1$) $\|S+T|\mathfrak{A}\| \leq c\,\big[\|S|\mathfrak{A}\| + \|T|\mathfrak{A}\|\big]$,

(Q$_2$) $\|BSA|\mathfrak{A}\| \leq \|B\|\|S|\mathfrak{A}\|\|A\|$,

(Q$_3$) $\|u^* \otimes v|\mathfrak{A}\| = \|u^*\|\|v\|$.

This leads to the concept of a *quasi-Banach operator ideal.* More information about continuous traces can be found in [32, Chapter 4] and [34, Section 6.5].

13. The spectral trace

A trace τ on an operator ideal $\mathfrak{A}$ is called *spectral* if

$$\tau(S) = \sum_{n=1}^{\infty} \lambda_n(S) \quad \text{for all} \quad S \in \mathfrak{A}(X). \qquad (\textit{trace formula})$$

This includes the assumption that, from the spectral point of view, all operators $S \in \mathfrak{A}(X)$ behave like compact operators; see [30, 26.5.1], [31, 3.2.12], and [34, 5.2.3.2]. Then we can define the *eigenvalue sequence* $\big(\lambda_n(S)\big)$ in which every eigenvalue $\lambda \neq 0$ is counted according to its *finite* multiplicity. We further assume that

$$\sum_{n=1}^{\infty} |\lambda_n(S)| < \infty.$$

For trace class operators on the Hilbert space the latter property was shown by Weyl [44], and Lidskiĭ [23] proved the trace formula by using methods from the theory of entire functions. This result was certainly known to Grothendieck [15, Chapter II, pp. 18–19] when he discovered that the operator ideal $\mathfrak{N}_{2/3}$ supports the spectral trace. Further examples, namely $\mathfrak{C}_1^{\text{app}}$ and $\mathfrak{P}_2 \circ \mathfrak{P}_2$, were exhibited by König [20, p. 164], [21, p. 259].

According to Enflo's counterexample, we find a continuous kernel K on $[0,1] \times [0,1]$ such that

$$\int_0^1 K(s,z)K(z,t)\,dz = 0 \quad \text{for} \quad s,t \in [0,1] \quad \text{and} \quad \int_0^1 K(z,z)\,dz = 1.$$

The associated operator K_{op} turns out to be nuclear on $C[0,1]$, which has the approximation property. Hence

$$\text{trace}(K_{\text{op}}) = \int_0^1 K(z,z)\,dz = 1$$

is well-defined. On the other hand, it follows from $K_{\text{op}}^2 = O$ that there exists no eigenvalue $\lambda \neq 0$. This shows that, even on *nice* Banach spaces, the nuclear trace fails to be spectral.

14. Classical and singular traces

Until the late 1980s, only *classical* traces were viewed as useful; see [31], [32, 4.2.1], and [34, 6.5.1]. Those traces are supposed to satisfy the additional condition that $\tau(u^* \otimes v) = \langle v, u^* \rangle$ for $u^* \in X^*$ and $v \in X$ or just $\tau(I_{\mathbb{K}}) = 1$, where $I_{\mathbb{K}}$ is the identity map of the 1-dimensional Banach space $\mathbb{K}$. In other words, $\tau(F)$ coincides with trace(F) for all operators $F \in \mathfrak{F}(X)$.

The question *"Whether a classical trace is uniquely determined by its axiomatic properties"* led to the concept of a *singular* trace: $\tau(u^* \otimes v) = 0$ for $u^* \in X^*$ and $v \in Y$. Already in 1966 and motivated by the theory of $C^\star$-algebras, Dixmier [9] had constructed an example, in the Hilbert space setting. About 20 years later a miracle happened, when the teacher explained this pathological monster to his best pupil. As soon as Connes understood (after approximately thirty seconds), he said: *"C'est ce qu'il me faut – This is what I need";* see [25, pp. 217–218]. The outcome can be found in [5, 6].

15. Construction of traces

A linear space $\mathfrak{z}(\mathbb{N}_0)$ of bounded scalar sequences $a = (\alpha_h)$ indexed by $\mathbb{N}_0 := \{0, 1, 2, \dots\}$ is called a *shift-monotone sequence ideal* if the following conditions are satisfied:

(M$_1$) $\mathfrak{z}(\mathbb{N}_0)$ is invariant under the shift operators

$$S_- : (\alpha_0, \alpha_1, \alpha_2, \dots) \mapsto (\alpha_1, \alpha_2, \alpha_3, \dots),$$

and

$$S_+ : (\alpha_0, \alpha_1, \alpha_2, \dots) \mapsto (\ 0\ , \alpha_0, \alpha_1, \dots).$$

(M$_2$) If $(\beta_h) \in \mathfrak{z}(\mathbb{N}_0)$ and $|\alpha_h| \le |\beta_h|$, then $(\alpha_h) \in \mathfrak{z}(\mathbb{N}_0)$.

(M$_3$) For every sequence $(\alpha_h) \in \mathfrak{z}(\mathbb{N}_0)$ there exists a sequence $(\beta_h) \in \mathfrak{z}(\mathbb{N}_0)$ such that $|\alpha_h| \le \beta_h$ and $\beta_0 \ge \beta_1 \ge \beta_2 \ge \dots \ge 0$.

We refer to $\mathfrak{z}(\mathbb{N}_0)$ as *strictly shift-monotone* if **(M$_3$)** is replaced by

(M$_4$) $(\alpha_h) \in \mathfrak{z}(\mathbb{N}_0)$ implies $\Big(\sum_{h=k}^{\infty} |\alpha_h| \Big) \in \mathfrak{z}(\mathbb{N}_0)$.

With every shift-monotone sequence ideal $\mathfrak{z}(\mathbb{N}_0)$ we associate the operator ideal

$$\mathfrak{D}_{\mathfrak{z}}^{\text{app}} := \big\{ S \in \mathfrak{L} : \big(a_{2^h}(S) \big) \in \mathfrak{z}(\mathbb{N}_0) \big\}.$$

In the case when $\mathfrak{z}(\mathbb{N}_0)$ is strictly shift-monotone, an operator $S : X \to Y$ belongs to $\mathfrak{D}_{\mathfrak{z}}^{\text{app}}$ if and only if it admits a representation

$$S = \sum_{h=1}^{\infty} F_h \quad \text{such that} \quad \operatorname{rank}(F_h) \le 2^h \quad \text{and} \quad \big(\|F_h\| \big) \in \mathfrak{z}(\mathbb{N}_0). \tag{$\mathfrak{z}$}$$

Suppose that λ is a $\frac{1}{2} S_+$-invariant linear form on $\mathfrak{z}(\mathbb{N}_0)$,

$$\lambda(\tfrac{1}{2} S_+ a) = \lambda(a) \quad \text{for all} \quad a \in \mathfrak{z}(\mathbb{N}_0).$$

Then

$$\tau(S) := \lambda\big(\tfrac{1}{2^h} \operatorname{trace}(F_h) \big)$$

defines a trace on $\mathfrak{D}_{\mathfrak{z}}^{\text{app}}$; see [33, p. 35]. Indeed, it follows from

$$|\operatorname{trace}(F)| \le n \|F\| \quad \text{whenever} \quad \operatorname{rank}(F) \le n$$

and $\big(\|F_h\| \big) \in \mathfrak{z}(\mathbb{N}_0)$ that $\big(\frac{1}{2^h} \operatorname{trace}(F_h) \big) \in \mathfrak{z}(\mathbb{N}_0)$. So $\lambda\big(\frac{1}{2^h} \operatorname{trace}(F_h) \big)$ makes sense, and we can show that this expression does not depend on the special choice of the representation ($\mathfrak{z}$).

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The linear space of all $\frac{1}{2}S_+$-invariant linear forms on a shift-monotone sequence ideal is studied in [37]. Its dimension can at most be $2^{2^{\aleph_0}}$. In all known cases we have either 0 or 1 or $2^{2^{\aleph_0}}$.

An example of a strictly shift-monotone sequence ideal is given by

$$\mathfrak{l}_1[2^{-h}](\mathbb{N}_0) := \Big\{(\alpha_h) : \sum_{h=0}^{\infty} 2^h|\alpha_h| < \infty\Big\}.$$

The associated operator ideal $\mathfrak{D}^{\mathrm{app}}_{\mathfrak{l}_1[2^{-h}]}$ coincides with $\mathfrak{C}^{\mathrm{app}}_1$, and the canonical $\frac{1}{2}S_+$-invariant linear form

$$\lambda(a) := \sum_{h=0}^{\infty} 2^h\alpha_h \quad \text{for all} \quad a = (\alpha_h) \in \mathfrak{l}_1[2^{-h}](\mathbb{N}_0)$$

generates the spectral trace.

The preceding construction, which works in the setting of Banach spaces, yields traces only on *specific* operator ideals. So our knowledge is still rudimentary. However, if the considerations are restricted to operators on the infinite-dimensional separable Hilbert space, then the situation becomes perfect. Based on previous work of Figiel (unpublished), Kalton [18], et al., it was shown in [36] that there exist one-to-one correspondences between *all* operator ideals $\mathfrak{A}(H)$ and *all* shift-monotone sequence ideals $\mathfrak{z}(\mathbb{N}_0)$ as well as between *all* traces on $\mathfrak{A}(H)$ and *all* $\frac{1}{2}S_+$-invariant linear forms on the associated $\mathfrak{z}(\mathbb{N}_0)$.

A trace on $\mathfrak{A}(H)$ can be characterized as a linear form that is invariant under the non-commutative group of unitary operators, which looks like a rather involved property. On the other hand, the corresponding linear form on $\mathfrak{z}(\mathbb{N}_0)$ needs to be invariant just under the single operator $\frac{1}{2}S_+$. In this way, we have reached a drastic simplification of trace theory.

Dedication

This paper is dedicated to Erhard Schmidt, who was born on January 1, 1876 (Julian calendar), in Tartu (in those days named Dorpat); see [35]. From 1893 to 1899, he studied at the University of Tartu. After two years at the University of Berlin, Schmidt went to Göttingen. Under the supervision of Hilbert, he wrote his famous thesis [41]. The Ph.D. was awarded to him on June 29, 1905. Schmidt spent the main part of his life in Berlin.

Being a fresh mathematician, I was appointed from 1958 to 1965 at the *Forschungsinstitut für Mathematik der Deutschen Akademie der Wissenschaften*, founded in 1946 on the initiative of Schmidt. Unfortunately, I did not have the privilege to meet him personally, since he passed away on December 6, 1959. Nevertheless, reading Schmidt's papers influenced my mathematical work and style considerably; I was able to extend some of his results about operators on the Hilbert space to the setting of Banach spaces.

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