

## Norm approximation property

ANATOLIJ PLICHKO

ABSTRACT. We introduce and study a general approximation property which takes origin in Numerical Analysis.

Let  $\mathcal{F}(X)$  be the set of all finite rank bounded linear operators in a Banach space  $X$ . A Banach space  $X$  has the *approximation property* (AP) if for every  $\varepsilon > 0$  and every compact subset  $K \subset X$  there is  $T \in \mathcal{F}(X)$  such that for all  $x \in K$ ,

$$\|Tx - x\| \leq \varepsilon.$$

We say that a Banach space  $X$  has the *norm approximation property* (norm AP) if there is  $\lambda \geq 1$  such that for every  $\varepsilon > 0$  and every finite-dimensional subspace  $E \subset X$  there is  $T \in \mathcal{F}(X)$  with  $\|T\| \leq \lambda$  and such that for all  $x \in E$ ,

$$(1 - \varepsilon)\|x\| \leq \|Tx\| \leq (1 + \varepsilon)\|x\|. \quad (1)$$

Of course, one may consider the norm approximation property for different Banach spaces  $X$  and  $Y$ . This property is a kind of “finite representability” of  $X$  in  $Y$ . The norm approximation property takes origin in Numerical Analysis, see, e.g., Vainikko [6], Heinrich [1], Plichko [5].

**Proposition 1.** *A Banach space  $X$  has the norm AP provided there is a sequence  $T_n \in \mathcal{F}(X)$  such that for all  $x \in X$ ,*

$$\|T_n x\| \rightarrow \|x\| \text{ as } n \rightarrow \infty. \quad (2)$$

*Proof.* Indeed, by the Uniform Boundedness Principle, in this Proposition the operators  $(T_n)$  are automatically uniformly bounded. Moreover, (2) implies that this convergence is uniform on the unit ball of every finite-dimensional subspace  $E \subset X$ .  $\square$

---

Received December 22, 2013.

2010 *Mathematics Subject Classification.* Primary 46B28; Secondary 46B07.

*Key words and phrases.* Approximation property, finite representability.

<http://dx.doi.org/10.12097/ACUTM.2014.18.07>

For separable Banach spaces the converse statement is also valid. If  $X$  has the norm AP, then there exists a sequence  $T_n \in \mathcal{F}(X)$  with the property (2).

Now we present an example of a Banach space without the norm AP.

Denote by  $\mathcal{L}(X, Y)$  the space of all bounded linear operators from a Banach space  $X$  to a Banach space  $Y$ . Let us recall that an operator  $T \in \mathcal{L}(X, Y)$  is *2-absolutely summing* if there is a constant  $C$  such that, for all finite subsets  $(x_i)$  in  $X$ , we have

$$\left( \sum \|Tx_i\|^2 \right)^{1/2} \leq C \sup_{f \in B_{X^*}} \left( \sum |f(x_i)|^2 \right)^{1/2},$$

where  $B_{X^*}$  is the dual unit ball. We denote by  $\pi_2(T)$  the smallest constant  $C$  satisfying the previous inequality. This  $\pi_2(T)$  is a norm on the space of all 2-absolutely summing operators and  $\pi_2(T) \geq \|T\|$ . Moreover, if  $T \in \mathcal{L}(X, Y)$  is 2-absolutely summing, and for subspaces  $X' \subset X$  and  $Y' \subset Y$  we have  $T(X') \subset Y'$ , then the restriction  $T' = T|_{X'} \in \mathcal{L}(X', Y')$  is again 2-absolutely summing and  $\pi_2(T') \leq \pi_2(T)$ , see, e.g., Pisier [4, p. 9].

**Lemma 2.** *Let  $T \in \mathcal{L}(X, Y)$  and  $E$  be a subspace of  $X$ ,  $\dim E = n$ , for which (1) is satisfied. Then there exists a constant  $a > 0$ , depending on  $\varepsilon$  only, such that*

$$\pi_2(T|_E) \geq a\sqrt{n}.$$

*Proof.* Let  $Id$  be the identity operator on  $X$ . As is well known, there exists an absolute constant  $b > 0$  such that  $\pi_2(Id|_E) \geq b\sqrt{n}$  (see, e.g., Pisier [3, p. 201] or [4, p. 145]). Since  $Id|_E = (T|_E)(T|_E)^{-1}$  and  $\pi_2(Id|_E) \leq \pi_2(T|_E)\|(T|_E)^{-1}\|$  (see, e.g., Pisier [4, p. 9]), the Lemma is proved.  $\square$

Pisier [3, p. 201], [4, p. 145] constructed a Banach space  $\mathbf{X}$  for which there is a constant  $c > 0$  such that for every  $T \in \mathcal{F}(\mathbf{X})$  we have

$$\|T\| \geq c\pi_2(T). \quad (3)$$

**Proposition 3.** *Every Banach space  $X$  for which (3) is satisfied fails the norm AP.*

*Proof.* Indeed, let  $E \subset X$ ,  $\dim E = n$ , and let an operator  $T \in \mathcal{F}(X)$  satisfy condition (1). Then, by (3) and Lemma 2,

$$\|T\| \geq c\pi_2(T) \geq c\pi_2(T|_E) \geq ca\sqrt{n}.$$

Therefore, there exists no  $\lambda$  with  $\|T\| \leq \lambda$  for the operators  $T \in \mathcal{F}(X)$  satisfying (1).  $\square$

**Proposition 4.** *Let  $X$  be a subspace of a Banach space  $Y$  which has the norm AP and is finitely representable in  $X$ . Then  $X$  has the norm AP.*

*Proof.* Take a finite-dimensional subspace  $E \subset X$ ,  $\varepsilon > 0$ , and let  $I : X \rightarrow Y$  be the identity embedding. Since  $Y$  has the norm AP (with a constant  $\lambda$ ), there is  $T \in \mathcal{F}(Y)$  with  $\|T\| \leq \lambda$  such that

$$(1 - \varepsilon)\|x\| \leq \|Tx\| \leq (1 + \varepsilon)\|x\|$$

for every  $x \in I(E)$ . Since  $Y$  is finitely representable in  $X$ , there exists an operator  $S : T(Y) \rightarrow X$  with  $\|S\| < 1 + \varepsilon$  and  $\|S^{-1}\| < 1 + \varepsilon$ . Put  $U = STI$ . Then  $U \in \mathcal{F}(X)$ ,  $\|U\| < \lambda(1 + \varepsilon)$ ,  $\|Ux\| < (1 + \varepsilon)^2\|x\|$  and  $\|Ux\| > \frac{1 - \varepsilon}{1 + \varepsilon}\|x\|$  for every  $x \in E$ . Hence,  $X$  has the norm AP.  $\square$

**Corollary 5.** *Every Banach space  $X$ , which contains  $\ell_\infty^n$  uniformly, has the norm AP.*

*Proof.* Indeed, each Banach space  $X$  is a subspace of  $Y = \ell_\infty(\Gamma)$ , for a suitable set  $\Gamma$ , which of course has the norm AP. By the assumption of Corollary,  $Y$  is finitely representable in  $X$  and one can apply the previous Proposition.  $\square$

**Corollary 6.** *Let  $Y = (\sum_n \ell_\infty^n)_2$  and  $X$  be an arbitrary Banach space. Then the space  $Z = (X \oplus Y)_{\ell_2}$  has the norm AP. So, each Banach space is isometric to a 1-complemented subspace of a Banach space with the norm AP. Each reflexive Banach space is isometric to a 1-complemented subspace of a reflexive Banach space with the norm AP.*

**Corollary 7.** *Every subspace  $X$  of the space  $\ell_p$ ,  $1 \leq p < \infty$ , or  $c_0$  has the norm AP. So, there exists a superreflexive separable Banach space with the norm AP but without the AP.*

*Proof.* Indeed, each (infinite-dimensional) subspace  $X$  of  $\ell_p$  or  $c_0$  contains a subspace,  $(1 + \varepsilon)$ -isometric to  $\ell_p$  or  $c_0$  [2, Proposition 2.a.2], and one can apply the previous Proposition. It is well known that there exists a subspace  $X$  of  $\ell_p$ ,  $p \neq 2$ , (or  $c_0$ ) without the AP [2, p. 90].  $\square$

**Acknowledgements.** The author expresses his thanks to G. Godefroy and W. B. Johnson for helpful discussions.

## References

- [1] S. Heinrich, *A problem in discretization theory related to the approximation property of Banach spaces*, Math. Nachr. **111** (1983), 147–152.
- [2] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces*, Vol. I, *Ergebn. Math. Grenzgeb.* **92**, Springer, Berlin, 1977.
- [3] G. Pisier, *Counterexamples to a conjecture of Grothendieck*, Acta Math. **151** (1983), 181–208.
- [4] G. Pisier, *Factorization of Linear Operators and Geometry of Banach Spaces*, Amer. Math. Soc., Providence, RI, 1986.

- [5] A. M. Plichko, *Totality of a set of weak\*  $\mathfrak{B}$ -approximable functionals*, Dopovidi Akad. Nauk Ukrain. RSR Ser. A, No. 2 (1983), 17–20. (Ukrainian)
- [6] G. Vainikko, *Funktionalanalysis der Diskretisierungsmethoden*, Teubner-Texte zur Mathematik. B.G. Teubner Verlag, Leipzig, 1976.

DEPARTMENT OF MATHEMATICS, CRACOW UNIVERSITY OF TECHNOLOGY, CRACOW,  
POLAND

*E-mail address:* `aplichko@pk.edu.pl`