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# Equiconvergence of expansions in multiple trigonometric Fourier series and Fourier integral with "$J_k$-lacunary sequences of rectangular partial sums"

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Abstract. We give a review of recent results on equiconvergence of expansions in multiple trigonometric Fourier series and Fourier integral in the case of summation over rectangles.

**1.** Let us consider the $N$-dimensional Euclidean space $\mathbb{R}^N$ whose elements will be denoted as $x = (x_1, \dots, x_N)$, and let $(nx) = n_1x_1 + \cdots + n_Nx_N$, $|x| = (x_1^2 + \cdots + x_N^2)^{1/2}$. Let us introduce the set $\mathbb{R}_0^N = \{(x_1, \dots, x_N) \in \mathbb{R}^N : x_j \geq 0,\ j = 1, \dots, N\}$, and the set $\mathbb{Z}^N \subset \mathbb{R}^N$ of all vectors with integer coordinates. Let $\mathbb{Z}_0^N = \mathbb{R}_0^N \cap \mathbb{Z}^N$.

Let $2\pi$-periodic (in each argument) function $f \in L_1(\mathbb{T}^N)$, where $\mathbb{T}^N = \{x \in \mathbb{R}^N : -\pi \leq x_j < \pi, j = 1, \dots, N\}$, be expanded in multiple trigonometric Fourier series: $f(x) \sim \sum_{k \in \mathbb{Z}^N} c_k e^{i(kx)}$.

For any vector $n = (n_1, \dots, n_N) \in \mathbb{Z}_0^N$ consider the rectangular partial sum of this series

$$S_n(x; f) = \sum_{|k_1| \leq n_1} \cdots \sum_{|k_N| \leq n_N} c_k e^{i(kx)}, \tag{1}$$

the particular case of which is the cubic partial sum $S_{n_0}(x; f)$, when $n_1 = \cdots = n_N = n_0$.

Let function $g \in L_1(\mathbb{R}^N)$ be expanded in multiple Fourier integral:

$$g(x) \sim \int_{\mathbb{R}^N} \widehat{g}(\xi) e^{i(\xi x)} d\xi.$$

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For any vector $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{R}^N_0$ consider the proper Fourier integral

$$J_\alpha(x; g) = \frac{1}{(2\pi)^N} \int\limits_{-\alpha_1}^{\alpha_1} \dots \int\limits_{-\alpha_N}^{\alpha_N} \widehat{g}(\xi) e^{i(\xi x)} d\xi_1 \dots d\xi_N. \tag{2}$$

The particular case of the "rectangular partial sum" (2) is the "cubic partial sum" $J_{\alpha_0}(x; g)$, when $\alpha_1 = \dots = \alpha_N = \alpha_0$.

Suppose that $g(x) = f(x)$ for $x \in \mathbb{T}^N$. Let $R_\alpha(x; f, g)$ denote the following difference:

$$R_\alpha(x; f, g) = R_{\alpha, n}(x; f, g) = S_n(x; f) - J_\alpha(x; g), \tag{3}$$

and let $R_\alpha(x; f)$ denote the difference

$$R_\alpha(x; f) = R_{\alpha, n}(x; f) = S_n(x; f) - J_\alpha(x; g), \text{ if } g(x) = 0 \text{ outside } \mathbb{T}^N. \tag{4}$$

**2.** Consider the differences (3) and (4) under condition that the components of the indexes $n \in \mathbb{Z}^N_0$ and $\alpha \in \mathbb{R}^N_0$ of "partial sums" $S_n(x; f)$ and $J_\alpha(x; g)$ are connected by relations

$$n_j = [\alpha_j], \text{ where } [\alpha_j] \text{ is the integral part of } \alpha_j \in \mathbb{R}^1_0, \; j = 1, \dots, N.$$

In the case $N = 1$ for any function $f \in L_1(\mathbb{T}^1)$ on any segment, which lies entirely inside the interval $(-\pi, \pi)$, the difference $R_\alpha(x; f, g)$ uniformly converges to zero as $\alpha \to \infty$ (see [23, pp. 362–364]) [1]. Thus, in the one-dimensional case the uniform equiconvergence of expansions in trigonometric Fourier series and Fourier integral takes place.

Basing on results by L. Carleson (1966) [16] and R. Hunt (1967) [19], I. L. Bloshanskii in 1975 in the paper [2] proved that for $N = 2$ and $p > 1$ $R_\alpha(x; f, g) \to 0$ as $\alpha \to \infty$ (i.e., $\min\limits_{1 \le j \le N} \alpha_j \to \infty$) almost everywhere (a.e.) on $\mathbb{T}^2$. More precisely (see [2, Theorem 4]), the following theorem was proved.

**Theorem A.** *For any functions $g(x)$ and $f(x)$ such that $g \in L_p(\mathbb{R}^2)$, $f \in L_p(\mathbb{T}^2)$, $p > 1$, and $g(x) = f(x)$ for $x \in \mathbb{T}^2$,*

$$\lim_{\alpha \to \infty} R_\alpha(x; f, g) = 0 \text{ almost everywhere on } \mathbb{T}^2.$$

*Moreover,*

$$\left\| \sup_{\alpha > 0} |R_\alpha(x; f, g)| \right\|_{L_p(\mathbb{T}^2)} \le C(p) \|g\|_{L_p(\mathbb{R}^2)},$$

*where the constant $C(p)$ does not depend on $f$ or $g$.*

In the same paper the essence of the type of convergence of $R_\alpha(x; f)$ and of the conditions $N = 2$, $p > 1$ was determined. Namely, there were constructed continuous functions $f_1 \in \mathbb{C}(\mathbb{T}^2)$, such that $\overline{\lim\limits_{\alpha \to \infty}} |R_\alpha(0; f_1)| = +\infty$, and

[1] Moreover, this result is true for any function $g(x)$ such that $\frac{|g(x)|}{1+|x|} \in L_1(\mathbb{R}^1)$.

$f_2 \in \mathbb{C}(\mathbb{T}^N)$, $N > 2$, such that $\overline{\lim\limits_{\alpha\to\infty}} |R_\alpha(x; f_2)| = +\infty$ everywhere inside $\mathbb{T}^N$ (see [2, Theorem 7]). As for the class $L_1$, there was presented an example of a function $f_3$ such that $\overline{\lim\limits_{\alpha\to\infty}} |R_\alpha(x; f_3)| = +\infty$ in each point $x \in \mathbb{T}^N$, $N \geq 2$ (see [2, Theorem 6]).

Further investigations of the problems of equiconvergence have been conducted in two directions. As for the first one, there arose a question concerning the possibility to construct counterexamples (for $p = 1$, $N \geq 2$) in the case of summation over cubes. This is connected with the fact that, during the construction of counterexamples in [2], the rectangular method of summation of Fourier series was taken into consideration, which, in its turn, gave an opportunity to make the components $\alpha_j$ varied. So, in the case of summation over cubes the construction of such counterexamples appeared to be an essentially more complicated process.

Thus, in 1976 in the paper [3] there was constructed a summable function $f \in L_1(\mathbb{T}^2)$ such that $\overline{\lim\limits_{\alpha_0\to\infty}} |R_{\alpha_0}(x; f)| = +\infty$ for almost all $x \in \mathbb{T}^2$. The latter estimate is true due to the "different speed of divergence" of the double Fourier series of function $f$ and of the double Fourier integral of function $g$, $g(x) = f(x)$ for $x \in \mathbb{T}^2$ ($g(x) = 0$ outside $\mathbb{T}^2$) over the same subsequences $\{\alpha_0(k, x)\}$, $\alpha_0(k, x) \in \mathbb{R}^1$, $k = 1, 2, \ldots$. Later, in 1990 in the paper [6] there were constructed two summable functions $f \in L_1(\mathbb{T}^N)$ and $g \in L_1(\mathbb{R}^N)$, $N \geq 2$, coinciding on $\mathbb{T}^N$ and such that the multiple Fourier series of function $f$ unboundedly diverges a.e. on $\mathbb{T}^N$ over some subsequences, while, at the same time, the multiple Fourier integral of function $g$ converges a.e. over the same subsequences.

Further, taking into account that, beginning from the three-dimensional case (as it was proved in [2]), equiconvergence a.e. (of the expansions under consideration) is not true even for continuous functions, the problem to find the "classes of equiconvergence" for $N \geq 3$ became the second direction of investigation.

Thus, in 1978 I. L. Bloshanskii (see [4]) established that for functions $f \in H^\omega(\mathbb{T}^3)$, where

$$H^\omega(\mathbb{T}^N) = \left\{ f \in \mathbb{C}(\mathbb{T}^N) : \omega(\delta, f) = \sup_{\substack{|x-y|<\delta, \\ x,y\in\mathbb{T}^N}} |f(x) - f(y)| = O(\omega(\delta)) \right\},$$

$\omega(\delta) = o(\omega_0(\delta))$ as $\delta \to +0$, and $\omega_0(\delta) = (\log \frac{1}{\delta} \log\log\log \frac{1}{\delta})^{-1}$, [2]

$$R_\alpha(x; f) \to 0 \quad \text{as} \quad \alpha \to \infty \quad \text{a.e. on} \quad \mathbb{T}^3.$$

[2] The class of functions $H^\omega(\mathbb{T}^2)$, $\omega(\delta) = o(\omega_0(\delta))$ as $\delta \to +0$, for the first time appeared in the paper [21] by K. I. Oskolkov, where the convergence a.e. of double Fourier series (summed over rectangles) was proved for functions in this class.

However, in the class $H^{\omega_2}(\mathbb{T}^3)$, defined by the modulus of continuity $\omega_2(\delta) = \lambda(\delta)\cdot\omega_1(\delta)$ (where $\omega_1(\delta) = \left(\log\frac{1}{\delta}\right)^{-1}$, and an arbitrary function $\lambda(\delta)$ satisfies (as $\delta \to +0$) two conditions: $\lambda(\delta)$ monotonically tends to $+\infty$ and $\lambda(\delta)\cdot\left(\log\frac{1}{\delta}\right)^{-1}$ tends to $+0$), equiconvergence a.e. is not true (proof of this fact is based on the estimates from the paper [1] by M. Bahbuh and E. M. Nikishin, see [4] [3]).

Further, in 1996 in the paper [12] I. L. Bloshanskii, O. K. Ivanova and T. Yu. Roslova proved that for functions $f \in L(\log^+ L)^2(\mathbb{T}^2)$ $\lim\limits_{\alpha\to\infty} R_\alpha(x; f) = 0$ a.e. on $\mathbb{T}^2$.

**3.** The results obtained in the framework of the second direction of investigation posed a question concerning the validity of equiconvergence a.e. (of expansions under consideration) with additional conditions on functions $f$ and $g$, in particular, in the classes $L_p$, $p > 1$, for $N \geq 3$, and with additional restrictions on the vector $\alpha$.

Let us note that the possibility to obtain new results in the case of additional restrictions on the vector $\alpha$ is connected with the following results on the convergence of multiple trigonometric Fourier series.

It is well known that in the classes $L_p$, $p > 1$, some subsequences of partial sums of multiple Fourier series have better properties of convergence a.e. in comparison with the whole sequence $S_n(x; f)$. It is true, for example, for those subsequences whose components of vector $n$ are elements of (single) lacunary sequences. [4]

Thus, in 1971 in the paper [22] P. Sjolin proved that if $f \in L_p(\mathbb{T}^2)$, $p > 1$, and $\{n_1^{(\lambda_1)}\}$, $n_1^{(\lambda_1)} \in \mathbb{Z}_0^1, \lambda_1 = 1, 2, \ldots$, is a lacunary sequence, then

$$\lim_{\lambda_1,\, n_2\to\infty} S_{n_1^{(\lambda_1)},\, n_2}(x; f) = f(x) \quad \text{a.e. on} \quad \mathbb{T}^2.$$

In 1977 in the paper [20] M. Kojima extended this result and proved that if $f \in L_p(\mathbb{T}^N)$, $p > 1$, $N \geq 2$, and $\{n_j^{(\lambda_j)}\}$, $n_j^{(\lambda_j)} \in \mathbb{Z}_0^1, \lambda_j = 1, 2, \ldots, j = 1, \ldots, N-1$, are lacunary sequences, then

$$\lim_{\lambda_1,\ldots,\lambda_{N-1},\, n_N\to\infty} S_{n_1^{(\lambda_1)},\ldots,n_{N-1}^{(\lambda_{N-1})},\, n_N}(x; f) = f(x) \quad \text{a.e. on} \quad \mathbb{T}^N.$$

Further, the analogous tendency was found during the research of *generalized localization almost everywhere* (GL) and *weak generalized localization*

[3] In [1] there was constructed a function from the class $H^{\omega_1}(\mathbb{T}^2)$ such that its rectangular partial sums of double Fourier series diverge in each point of the square $[-\pi+\varepsilon, \pi-\varepsilon]^2$, $\varepsilon > 0$.

[4] The sequence $\{n^{(\lambda)}\}, n^{(\lambda)} \in \mathbb{Z}_0^1$, is a lacunary sequence if $n^{(1)} = 1$ and $\frac{n^{(\lambda+1)}}{n^{(\lambda)}} \geq q > 1, \lambda = 1, 2, \ldots$.

*almost everywhere* (WGL) [5] for multiple trigonometric Fourier series of functions in $L_p(\mathbb{T}^N)$, $p>1$, $N\geq 3$.

Thus, I. L. Bloshanskii and O. V. Lifantseva showed that, as opposed to the case when all components of the index $n$ of the rectangular partial sum $S_n(x;f)$ of multiple trigonometric Fourier series are “free” (see, in particular, [5]), in the case when $S_n(x;f)$ have index $n$, in which some components are elements of (single) lacunary sequences, there takes place an “enlargement” of the set, on which the series with such “lacunary sequences of partial sums” converges, with the simultaneous “decrease” of the set on which the equivalence of function $f$ to zero is necessary, see [14], [15] or [7] (for $N=3$ the results concern GL, for $N>3$ they concern WGL).

Note that neither the result by M. Kojima, nor the result by I. L. Bloshanskii and O. V. Lifantseva cannot be “essentially strengthened”. In both cases (with different degree of complexity) the authors used Ch. Fefferman’s function [6] from [17] in order to construct counterexamples.

**4.** Further, let $\{n^{(\lambda)}\}$, $n^{(\lambda)}\in\mathbb{Z}^1_0$, $\lambda=1,2,\ldots$, be some lacunary sequence and let $\varrho$ be some constant. In [9] I. L. Bloshanskii and D. A. Grafov introduced the following notions.

**Definition 1.** We will call a sequence $\{\alpha^{(\lambda)}\}$, $\alpha^{(\lambda)}\in\mathbb{R}^1_0$, $\lambda=1,2,\ldots$, a *real lacunary sequence* if $[\alpha^{(\lambda)}]=n^{(\lambda)}$, $\lambda=1,2,\ldots$ (here $[\xi]$ is the integral part of $\xi\in\mathbb{R}^1$), and a *generalized real lacunary sequence* if

$$|\,\alpha^{(\lambda)}-n^{(\lambda)}\,|\leq\varrho,\quad \lambda=1,2,\ldots. \tag{5}$$

First of all, in connection with this definition the question arises: how the difference $R_\alpha(x;f,g)=S_n(x;f)-J_\alpha(x;g)$ (see (3) and (4)) for $N\geq 2$ behaves when components $n_j$ of vector $n\in\mathbb{Z}^N_0$ and components $\alpha_j$ of vector $\alpha\in\mathbb{R}^N_0$ are connected by the relation

$$|\,\alpha_j-n_j\,|\leq\varrho,\quad j=1,\ldots,N, \tag{5$'$}$$

where $\varrho$ is some constant which does not depend on $n$ or $\alpha$.

The following result generalizes Theorem A.

**Theorem 1** (see [9])**.** *For any $\alpha=(\alpha_1,\alpha_2)$, $\alpha\in\mathbb{R}^2_0$, satisfying condition* (5$'$)*, and for any functions $g(x)$ and $f(x)$ such that $g\in L_p(\mathbb{R}^2)$, $f\in L_p(\mathbb{T}^2)$, $p>1$, and $g(x)=f(x)$ for $x\in\mathbb{T}^2$,*

$$\lim_{\alpha_1,\,\alpha_2\to\infty} R_{\alpha_1,\alpha_2}(x;f,g)=0 \ \textit{almost everywhere on } \mathbb{T}^2.$$

[5] These notions mean that for multiple Fourier series of function $f$, which equals zero on the set $\mathfrak{A}$, $\mu\mathfrak{A}>0$ ($\mu$ is the $N$-dimensional Lebesgue measure), the question of convergence a.e. is investigated either on the whole set $\mathfrak{A}$ (GL), or on some of its subsets $\mathfrak{A}_1\subset\mathfrak{A}$, $\mu\mathfrak{A}_1>0$ (WGL).

[6] The continuous function whose double trigonometric Fourier series (summed over rectangles) diverges unboundedly everywhere inside $\mathbb{T}^2$.

**Remark 1.** The analogous result is true as well in the one-dimensional case for any functions $g \in L_1(\mathbb{R}^1)$ and $f \in L_1(\mathbb{T}^1)$ such that $g(x) = f(x)$ for $x \in \mathbb{T}^1$.

It is obvious that when one of the components of vector $\alpha \in \mathbb{R}^2_0$ in the difference $R_\alpha(x; f, g)$ is a real lacunary sequence or a generalized real lacunary sequence, then Theorem 1 is all the more true.

Further, let us denote $RS_{n+m}(x; f) := S_{n+m}(x; f) - S_n(x; f)$, $n$, $m \in \mathbb{Z}^N_0$. The following result is equivalent to Theorem 1.

**Theorem 1$'$** (see [8]). *For any bounded sequence $\{m(n)\}$, $m(n) \in \mathbb{Z}^2_0$, $n \in \mathbb{Z}^2_0$, and for any function $f \in L_p(\mathbb{T}^2)$, $p > 1$,*

$$\lim_{n\to\infty} RS_{n+m(n)}(x; f) = 0 \ \text{ almost everywhere on } \mathbb{T}^2.$$ [7]

This result shows that the "almost" Cauchy property is true for the double Fourier series of functions in $L_p$, $p > 1$.

**Corollary of Theorems 1 and 1$'$.** *For any two bounded functions $\beta_1(\alpha)$, $\beta_2(\alpha) \in \mathbb{R}^1_0$, $\alpha \in \mathbb{R}^2_0$, and for any function $g \in L_p(\mathbb{R}^2)$, $p > 1$,*

$$\lim_{\alpha\to\infty} (J_{\alpha+\beta(\alpha)}(x; g) - J_\alpha(x; g)) = 0 \ \text{ almost everywhere on } \mathbb{T}^2,$$

*where $\beta(\alpha) = (\beta_1(\alpha), \beta_2(\alpha)) \in \mathbb{R}^2_0$.*

Theorems 1 and 1$'$ pose a question about the behavior of the differences $RS_{n+m}(x; f)$ and $R_\alpha(x; f)$ for $N \geq 3$. It turned out that, beginning from the three-dimensional case, these differences are not equivalent. More precisely, the following results are true. Consider a result about the difference $RS_{n+m}(x; f)$.

**Theorem 2** (see [8]). *For any bounded sequence $\{m(n)\}$, $m(n) \in \mathbb{Z}^2_0$, $n = (n_1, n_2) \in \mathbb{Z}^2_0$, for any lacunary sequences $\{n_j^{(\lambda_j)}\}$, $n_j^{(\lambda_j)} \in \mathbb{Z}^1_0$, $\lambda_j = 1, 2, \ldots$, $j = 3, \ldots, N$, and for any function $f \in L_p(\mathbb{T}^N)$, $p > 1$, $N \geq 3$,*

$$\lim_{n_1, n_2, \lambda_3, \ldots, \lambda_N\to\infty} RS_{n_1+m_1(n),\, n_2+m_2(n),\, n_3^{(\lambda_3)}, \ldots, n_N^{(\lambda_N)}}(x; f) = 0$$

*almost everywhere on $\mathbb{T}^N$.*

In its turn, for the difference $R_\alpha(x; f)$ the following holds.

**Theorem 3** (see [9], [11]). *There exists a function $f \in \mathbb{C}(\mathbb{T}^N)$, $N \geq 3$, such that for any $N-2$ sequences $\{\alpha_j^{(\nu_j)}\}$, $\alpha_j^{(\nu_j)} \in \mathbb{R}^1_0$, $\alpha_j^{(\nu_j)} \to \infty$ as $\nu_j \to \infty$, $j = 3, \ldots, N$,* [8]

$$\overline{\lim_{\alpha_1, \alpha_2, \nu_3, \ldots, \nu_N\to\infty}} |R_{\alpha_1, \alpha_2, \alpha_3^{(\nu_3)}, \ldots, \alpha_N^{(\nu_N)}}(x; f)| = +\infty \ \text{ everywhere inside } \mathbb{T}^N.$$

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[7] Note that this estimate is true for the divergent a.e. Fourier series as well.

[8] In particular, each of the sequences $\{\alpha_j^{(\nu_j)}\}$ can be a lacunary sequence.

**5.** Note that Theorem 3, in view of problems of equiconvergence of expansions in multiple Fourier series and integral (which are investigated in the present paper), shows that as soon as two components of the vector $\alpha$ become "free" (i.e., in particular, they are not elements of any lacunary sequences), the class $\mathbb{C}(\mathbb{T}^N)$, $N \geq 3$, does not remain the "class of equiconvergence a.e." of the indicated expansions. Nevertheless, the class $L_p(\mathbb{T}^N)$, $p > 1$, without additional conditions on functions $g$ and $f$ remains the "class of equiconvergence a.e." in the case $N \geq 3$, the same as for $N = 2$, if only one component of the vector $\alpha$ is "free". Let us illustrate what was said with the following result.

In order to formulate this result (as well as the next ones) in a shorter way let us introduce the following notations. Let $M = \{1, 2, \dots, N\}$, $J_k = \{j_1, \dots, j_k\} \subset M$, $j_1 < \cdots < j_k$, $1 \leq k \leq N$, and let $\lambda = \lambda(J_k) = (\lambda_{j_1}, \dots, \lambda_{j_k}) \in \mathbb{Z}_0^k$, $j_s \in J_k$, $s = 1, \dots, k$. By the symbol $\alpha^{(\lambda)} = \alpha^{(\lambda)}[J_k] = (\alpha_1, \dots, \alpha_N) \in \mathbb{R}_0^N$ we denote the $N$-dimensional vector whose components $\alpha_j$, $j \in J_k$, are elements of some (single) generalized real lacunary sequences. So, consequently we will call the sequences $S_{n^{(\lambda)}[J_k]}(x; f)$, $J_{\alpha^{(\lambda)}[J_k]}(x; g)$ and $R_{\alpha^{(\lambda)}[J_k]}(x; f, g)$ the "$J_k$-lacunary sequences of rectangular partial sums" of Fourier series, of Fourier integral and of the difference (of the indicated expansions), respectively.

The following result is true.

**Theorem 4** (see [9]). *For $J_{N-1} \subset M$ and for functions $g(x)$ and $f(x)$ such that $g \in L_p(\mathbb{R}^N)$, $f \in L_p(\mathbb{T}^N)$, $p > 1$, $N \geq 3$, and $g(x) = f(x)$ for $x \in \mathbb{T}^N$,*

$$\lim_{\substack{\lambda_j \to \infty, j \in J_{N-1}, \\ \alpha_j \to \infty, j \in M \setminus J_{N-1}}} R_{\alpha^{(\lambda)}[J_{N-1}]}(x; f, g) = 0 \quad \textit{almost everywhere on} \quad \mathbb{T}^N.$$

*Moreover,*

$$\left\| \sup_{\substack{\lambda_j > 0,\, j \in J_{N-1}, \\ \alpha_j > 0,\, j \in M \setminus J_{N-1}}} |R_{\alpha^{(\lambda)}[J_{N-1}]}(x; f, g)| \right\|_{L_p(\mathbb{T}^N)} \leq C(p) \|g\|_{L_p(\mathbb{R}^N)},$$

*where the constant $C(p)$ does not depend on functions $f$ or $g$.*

The assertion of Theorem 4 (for $N \geq 3$) appeared to be equivalent in some sense to the assertion of Theorem A (for $N = 2$). Namely, the presence of $N - 1$ lacunary components in the $N$-dimensional vector $\alpha$ in the difference $R_\alpha(x; f, g)$ (Theorem 4) "replaces" the presence of one free component in the two-dimensional vector $(\alpha_1, \alpha_2)$ in the difference $R_{\alpha_1, \alpha_2}(x; f, g)$ (Theorem A). Thus, the question stands, as before, concerning the validity of equiconvergence a.e. (of the considered expansions) for $N \geq 3$ either in more "narrow classes" than $\mathbb{C}(\mathbb{T}^N)$, or in the classes $L_p$, $p > 1$, with additional conditions on functions $f$ and $g$, but already in the case when two or more components of the vector $\alpha$ are free (see the assertions of Theorems 3 and 4).

Some advance in the first of the above-mentioned directions of investigation is the following result. Denote

$$H^{\omega^*}(\mathbb{T}^3) = \Big\{ f \in \mathbb{C}(\mathbb{T}^3) :$$

$$\omega^*(\delta, f) = \sup_{\substack{(x_2-y_2)^2+(x_3-y_3)^2<\delta^2,\\ x_j,\, y_j \in \mathbb{T}^1,\, j=1,2,3}} |f(x_1,x_2,x_3) - f(x_1,y_2,y_3)| = O(\omega(\delta)) \Big\},$$

here $\omega(\delta) = o(\omega_0(\delta))$ as $\delta \to +0$ (it is obvious that $H^{\omega}(\mathbb{T}^3) \subset H^{\omega^*}(\mathbb{T}^3)$).

**Theorem 5** (see [18])**.** *For $J_1 = \{1\}$ and for any function $f \in H^{\omega^*}(\mathbb{T}^3)$,*

$$\lim_{\substack{\lambda_j \to \infty,\, j \in J_1,\\ \alpha_j \to \infty,\, j \in M \setminus J_1}} R_{\alpha^{(\lambda)}[J_1]}(x; f) = 0 \quad \textit{almost everywhere on} \quad \mathbb{T}^3.$$

Concerning advance in the second direction (to determine additional conditions on functions $f$ and $g$ in the classes $L_p$, $p > 1$, in order to obtain equiconvergence of expansions), in the present paper we consider (as a kind of these additional conditions) the equivalence to zero of function $f$ on sets of definite form.

Denote $\mathbb{R}[J_k] = \{x \in \mathbb{R}^N : x_j = 0 \text{ as } j \in M \setminus J_k\}$, $\mathbb{T}[M \setminus J_k] = \{x \in \mathbb{R}[M \setminus J_k] : -\pi \le x_j < \pi \text{ as } j \in M \setminus J_k\}$, let $\Omega \subset \mathbb{T}^N$ be an arbitrary (nonempty) open set, and let $\Omega[J_2] = pr_{(J_2)}\{\Omega\}$ be an orthogonal projection of the set $\Omega$ onto the plane $\mathbb{R}[J_2]$, $J_2 \subset M$. Set $W[J_2] = \Omega[J_2] \times \mathbb{T}[M \setminus J_2]$, $J_2 \subset M$.

Let us fix an arbitrary sample $J_k$ from $M$, $1 \le k \le N-2$, and define the following sets

$$W(J_k) = \bigcup_{J_2 \subset M \setminus J_k} W[J_2] \quad \text{and} \quad W^0(J_k) = \bigcap_{J_2 \subset M \setminus J_k} W[J_2]. \tag{6}$$

In [13] I. L. Bloshanskii and O. V. Lifantseva introduced the following notion.

**Definition 2.** Let $\mathfrak{A} \subset \mathbb{T}^N$, $J_k \subset M$, $1 \le k \le N-2$, $N \ge 3$.

1. We will say that the set $\mathfrak{A}$ *possesses the property* $\mathbb{B}_2^{(J_k)}$, if there exists a set $W = W(J_k)$ of type (6) such that $\mu(W \setminus \mathfrak{A}) = 0$, moreover, the property $\mathbb{B}_2^{(J_k)}$ is *the property* $\mathbb{B}_2^{(J_k)}(W^0)$, if $W = W(W^0, J_k)$.

2. We will call the property $\mathbb{B}_2^{(J_k)}(W^0)$ of the set $\mathfrak{A}$ the *maximal property* $\mathbb{B}_2^{(J_k)}$ of the set $\mathfrak{A}$, if for any set $\widetilde{W}^0 = \widetilde{W}^0(J_k)$ of type (6) such that $\mu(\widetilde{W}^0 \setminus W^0) > 0$, the set $\mathfrak{A}$ does not possess the property $\mathbb{B}_2^{(J_k)}(\widetilde{W}^0)$.

Further, let an arbitrary measurable set $\mathfrak{A} \subset \mathbb{T}^N$, $0 < \mu\mathfrak{A} < (2\pi)^N$, $N \geq 3$, satisfy the following conditions on the boundary:

$$\mu(\mathfrak{B} \setminus \overline{int\mathfrak{B}}) = 0; \tag{7}$$

$$\mu_2 Fr\, pr_{(J_2)}\{int\mathfrak{B}\} = 0, \quad J_2 \subset M \setminus J_k, \tag{8}$$

where $\mathfrak{B} = \mathbb{T}^N \setminus \mathfrak{A}$, $J_k \subset M$, $1 \leq k \leq N-2$ ($\mu_2$ is the measure on the plane, $intP$ is the set of interior points, $\overline{P}$ is the closure, $FrP$ is the boundary of the set $P$).

**Theorem 6** (see [10]). *Let $\mathfrak{A}$ be an arbitrary measurable set, $\mathfrak{A} \subset \mathbb{T}^N$, $N \geq 3$, $0 < \mu\mathfrak{A} < (2\pi)^N$, and let $J_k \subset M$, $1 \leq k \leq N-2$.*

1. *If there exists a set $W^0 = W^0(J_k)$ of type* (6) *such that the set $\mathfrak{A}$ possesses the property $\mathbb{B}_2^{(J_k)}(W^0)$, then for any function $f \in L_p(\mathbb{T}^N)$, $p > 1$, $f(x) = 0$ on $\mathfrak{A}$,*

$$\lim_{\substack{\lambda_j \to \infty,\, j \in J_k, \\ \alpha_j \to \infty,\, j \in M \setminus J_k}} R_{\alpha^{(\lambda)}[J_k]}(x; f) = 0 \ \text{almost everywhere on } W^0.$$

*In addition, let the set $\mathfrak{A}$ satisfy conditions* (7), (8). *Then the following holds.*

2. *If the property $\mathbb{B}_2^{(J_k)}(W^0)$ of the set $\mathfrak{A}$ is the maximal property $\mathbb{B}_2^{(J_k)}$, then there exists a function $f_1 \in L_\infty(\mathbb{T}^N)$ such that $f_1(x) = 0$ on $\mathfrak{A}$ and for any $k$ sequences of real numbers $\{\alpha_j^{(\lambda_j)}\}$, $j \in J_k$, $\alpha_j^{(\lambda_j)} \to \infty$ as $\lambda_j \to \infty$, the following estimate is true:*

$$\overline{\lim_{\substack{\lambda_j \to \infty,\, j \in J_k, \\ \alpha_j \to \infty,\, j \in M \setminus J_k}}} |R_{\alpha^{(\lambda)}[J_k]}(x; f_1)| = +\infty \ \text{almost everywhere on } \mathbb{T}^N \setminus W^0.$$

3. *In particular, if the set $\mathfrak{A}$ does not possess the property $\mathbb{B}_2^{(J_k)}$ at all, then there exists a function $f_2 \in L_\infty(\mathbb{T}^N)$ such that $f_2(x) = 0$ on $\mathfrak{A}$ and for any $k$ sequences of real numbers $\{\alpha_j^{(\lambda_j)}\}$, $j \in J_k$, $\alpha_j^{(\lambda_j)} \to \infty$ as $\lambda_j \to \infty$, the following estimate is true:*

$$\overline{\lim_{\substack{\lambda_j \to \infty,\, j \in J_k, \\ \alpha_j \to \infty,\, j \in M \setminus J_k}}} |R_{\alpha^{(\lambda)}[J_k]}(x; f_2)| = +\infty \ \text{almost everywhere on } \mathbb{T}^N.$$

Theorem 6 shows that for any $k$, $1 \leq k \leq N-2$, the validity or invalidity of equiconvergence of expansions in multiple Fourier series and Fourier integral (in the case of $J_k$-lacunary sequence of the difference) in the classes $L_p$, $p > 1$, on the set $\mathfrak{A} \subset \mathbb{T}^N$ are defined by the structure and geometry of the set $\mathfrak{A}$, which, in their turn, are described by the property $\mathbb{B}_2^{(J_k)}$, where the quantity $k$ is the number of "lacunary components" of the vector $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{R}^N$ (the index of $R_\alpha(x; f)$).

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The following theorem shows that the first part of Theorem 6 cannot be strengthened by not requiring the equivalence to zero of the function $g(x)$ outside $\mathbb{T}^N$ (see (4)).

**Theorem 7** (see [9], [11]). *There exists a function $g(x)$, $g \in \mathbb{C}(\mathbb{R}^3)$, $g(x) = 0$ for $x \in \mathbb{T}^3$, such that for any sequence $\{\alpha_3^{(\nu_3)}\}$, $\alpha_3^{(\nu_3)} \in \mathbb{R}^1_0$, $\alpha_3^{(\nu_3)} \to \infty$ as $\nu_3 \to \infty$,*

$$\overline{\lim_{\alpha_1,\,\alpha_2,\,\nu_3 \to \infty}} |R_{\alpha_1,\,\alpha_2,\,\alpha_3^{(\nu_3)}}(x;0,g)| = +\infty \ \ \textit{everywhere inside} \quad \mathbb{T}^3.$$

As a corollary of Theorem 7 the following result is true.

**Corollary of Theorem 7.** *For any $N \geq 3$ there exist functions $g(x)$ and $f(x)$, $g \in \mathbb{C}(\mathbb{R}^N)$, $f \in \mathbb{C}(\mathbb{T}^N)$, $g(x) = f(x)$ for $x \in \mathbb{T}^N$, such that for any $N-2$ sequences $\{\alpha_j^{(\nu_j)}\}$, $\alpha_j^{(\nu_j)} \in \mathbb{R}^1_0$, $\alpha_j^{(\nu_j)} \to \infty$ as $\nu_j \to \infty$, $j = 3, \ldots, N$,*

$$\lim_{n\to\infty} S_n(x;f) = f(x) \ \ \textit{in each point} \ \ \mathbb{T}^N,$$

*but*

$$\overline{\lim_{\alpha_1,\,\alpha_2,\,\nu_3,\,\ldots,\,\nu_N \to \infty}} |J_{\alpha_1,\,\alpha_2,\,\alpha_3^{(\nu_3)},\ldots,\alpha_N^{(\nu_N)}}(x;g)| = +\infty \ \ \textit{everywhere inside} \ \ \mathbb{T}^N.$$

Note that in Theorem 7 the sequence $\{\alpha_3^{(\nu_3)}\}$ can be lacunary; and in its corollary each sequence $\{\alpha_j^{(\nu_j)}\}$, $j = 3, \ldots, N$, can be a lacunary sequence.

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