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Dominions, zigzags and epimorphisms for partially ordered semigroups

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Abstract. We prove an analogue of Isbell's celebrated zigzag theorem for partially ordered semigroups. This theorem provides a useful description of dominions which we employ to describe absolute closedness of posemigroups and epimorphisms in varieties of absolutely closed posemigroups.

1. Preliminaries

A partially ordered semigroup, briefly *posemigroup*, is a pair $(S, \leq)$ comprising a semigroup S and a partial order $\leq$ (on S) that is *compatible* with the binary operation, i.e. for all $s_1, s_2, t_1, t_2 \in S$, $s_1 \leq t_1$, $s_2 \leq t_2$ implies $s_1 s_2 \leq t_1 t_2$. If S is a monoid we call $(S, \leq)$ a partially ordered monoid, shortly *pomonoid*. A posemigroup *homomorphism* $f : (S, \leq_S) \longrightarrow (T, \leq_T)$ is a monotone semigroup homomorphism. We call f an *order-embedding* if one has $f(s_1) \leq_T f(s_2)$ if and only if $s_1 \leq_S s_2$, $s_1, s_2 \in S$. A surjective order-embedding is called an *order-isomorphism*. *Epimorphisms* of posemigroups are defined in the usual sense of category theory, that is a posemigroup homomorphism $f : (S, \leq_S) \longrightarrow (T, \leq_T)$ is called an epimorphism if $g \circ f = h \circ f$ implies $g = h$ for all posemigroup homomorphisms $g, h : (T, \leq_T) \longrightarrow (U, \leq_U)$. We call $(U, \leq_U)$ a *subposemigroup* of a posemigroup $(S, \leq_S)$ if U is a subsemigroup of S and $\leq_U = \leq_S \cap U \times U$. The corresponding notions for pomonoids are defined analogously.

In the sequel we shall also treat a posemigroup (respectively, pomonoid) $(S, \leq)$ as a semigroup (monoid) by simply disregarding the order. We shall then merely denote it by S. Let $\mathcal{A}$ be a class of posemigroups (pomonoids). Then by $\mathcal{A}'$ we shall denote the class of semigroups (monoids) obtained by disregarding the orders in $\mathcal{A}$ (that is $\mathcal{A}' = \{S : (S, \leq) \in \mathcal{A}\}$).

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A class of posemigroups is called a *variety* of posemigroups, for instance cf. [7], if it is closed under taking products (endowed with componentwise order), homomorphic images and subposemigroups. Varieties of pomonoids may be defined similarly. It is also possible to describe posemigroup (pomonoid) varieties alternatively with the help of inequalities using a Birkhoff type characterization; we refer to [1] for details. Because every term equality in an algebraic variety can be replaced by two (term) inequalities, see [7], in a usual way, a class $\mathcal{A}$ of posemigroups (pomonoids) is a variety if the class $\mathcal{A}'$ is a variety of semigroups (monoids). Also, every variety (whether algebraic or order theoretic) naturally gives rise to a category.

We observe that $f : (S, \leq_S) \longrightarrow (T, \leq_T)$ is necessarily a posemigroup (pomonoid) epimorphism if $f : S \longrightarrow T$ is an epimorphism of semigroups (monoids). One of our aims is to show that the converse of this statement holds in varieties of absolutely closed semigroups. We shall, however, first present two versions of Isbell's zigzag theorem for posemigroups and record some of their consequences.

Before moving to next section, let us recall (for instance from [9]) S-posets and their tensor products. Let $(S, \leq_S)$ be a pomonoid and $(X, \leq_X)$ a poset. Then X is called a *left S-poset*, we shall denote it by ${}_{(S,\leq_S)}(X, \leq_X)$, if X is a left S-act, see [6], such that the left action $S \times X \longrightarrow X$ of S, given by $(s, x) \longmapsto sx$, is monotone, i.e. $(s_1, x_1) \preccurlyeq (s_2, x_2)$ implies $s_1x_1 \leq_X s_2x_2$, where $\preccurlyeq$ is defined componentwise. Right S-posets are defined in a dual manner. A homomorphism of right (left) S-posets is a monotone right (left) S-act homomorphism.

A poset $(A\widehat{\otimes}_S B, \leq)$ is called the *tensor product* (over $(S, \leq_S)$) of a right S-poset $(A, \leq_A)_{(S,\leq_S)}$ and a left S-poset ${}_{(S,\leq_S)}(B, \leq_B)$ if it satisfies the following conditions:

(i) there exists a balanced (i.e. $\alpha(as, b) = \alpha(a, sb)$, $a \in A$, $b \in B$, $s \in S$) monotone map

$$\alpha : (A \times B, \preceq) \longrightarrow (A\widehat{\otimes}_S B, \leq)$$

(where $\preceq$ denotes the componentwise order on $A \times B$) such that

(ii) for any poset $(X, \leq_X)$ admitting a balanced monotone map

$$\beta : (A \times B, \preceq) \longrightarrow (X, \leq_X)$$

there is a unique monotone map $\varphi : (A\widehat{\otimes}_S B, \leq) \longrightarrow (X, \leq_X)$ with $\beta = \varphi \circ \alpha$.

An explicit method of constructing $(A\widehat{\otimes}_S B, \leq)$ may be found, for example, in [9]. We shall henceforth simply write $A\widehat{\otimes}_S B$ instead of $(A\widehat{\otimes}_S B, \leq)$. The image of (a, b) under α will be denoted by $a\widehat{\otimes}b$. On the other hand, given two S-posets $(A, \leq_A)_{(S,\leq_S)}$ and ${}_{(S,\leq_S)}(B, \leq_B)$, we shall denote by $A \otimes_S B$ the (algebraic) tensor product of the S-acts A_S and ${}_S B$ (for details see [6]).

A typical element of $A\otimes_S B$ is denoted by $a\otimes b$. Clearly $(S,\leq)_{(S,\leq)}$, ${}_{(S,\leq)}(S,\leq)$, S_S and ${}_S S$ are special S-posets and S-acts, respectively.

2. Pomonoids

In this section we put together some recent results concerning closure of pomonoids. We begin by recalling dominions.

Definition 1 ([12], Definition 1)**.** Let $(U,\leq_U)$ be a subpomonoid of a pomonoid $(S,\leq_S)$. Then the subpomonoid (of $(S,\leq_S)$)

$$(\widehat{dom}_S(U),\leq)=\{x\in S:\alpha,\beta:(S,\leq_S)\to(T,\leq_T)$$

$$\text{with } \alpha\,|_U=\beta\,|_U \implies \alpha(x)=\beta(x)\}$$

is called the *dominion* of $(U,\leq_U)$ (in $(S,\leq_S)$), where α and β are pomonoid homomorphisms.

Henceforth, we shall simply write $\widehat{dom}_S(U)$ to denote $(\widehat{dom}_S(U),\leq)$. The following zigzag theorem for pomonoids provides a criterion to check if an element $d\in(S,\leq_S)$ lies in $\widehat{dom}_S(U)$.

Theorem 1 ([11], Theorem 3)**.** *Take a subpomonoid $(U,\leq_U)$ of a pomonoid $(S,\leq_S)$. Then $d\in\widehat{dom}_S(U)$ if and only if $d\widehat{\otimes}1=1\widehat{\otimes}d$ in $S\widehat{\otimes}_U S$.*

Given a subpomonoid $(U,\leq_U)$ of a pomonoid $(S,\leq_S)$, one may also consider, while ignoring the orders, the (algebraic) dominion $dom_S(U)$ of U in S; for instance, see [8]. In the unordered scenario we have the following celebrated zigzag theorem, the original formulation of which is due to J. R. Isbell [5].

Theorem 2 ([8], Theorem 2.1)**.** *Let U be a submonoid of a monoid S. Then $d\in dom_S(U)$ if and only if $d\otimes 1=1\otimes d$ in $S\otimes_U S$.*

Recall, for example from [11], that $d\otimes 1=1\otimes d$ in $S\otimes_U S$ implies $d\widehat{\otimes}1=1\widehat{\otimes}d$ in $S\widehat{\otimes}_U S$. We therefore have:

$$U\subseteq dom_S(U)\subseteq\widehat{dom}_S(U)\subseteq S. \tag{1}$$

By analogy with [4], a subpomonoid $(U,\leq_U)$ of $(S,\leq_S)$ will be termed *closed* (in $(S,\leq_S)$) if $\widehat{dom}_S(U)\subseteq U$ (whence indeed $\widehat{dom}_S(U)=U$). We shall call $(U,\leq_U)$ *absolutely closed* if it is closed in all of its pomonoid extensions. One can easily observe that a pomonoid homomorphism $f:(S,\leq_S)\longrightarrow(T,\leq_T)$ is an epimorphism if and only if $\widehat{dom}_T(\operatorname{Im} f)=(T,\leq_T)$.

Theorem 3 (see [12])**.** *Take a subpomonoid $(U,\leq_U)$ of a pomonoid $(S,\leq_S)$. Then $(U,\leq_U)$ is closed in $(S,\leq_S)$ if and only if U is such in S as a monoid.*

3. Posemigroups

In this section we adapt the notions of previous section together with some others to the setting of posemigroups. We define dominions for posemigroups by just replacing pomonoids with posemigroups in Definition 1. We shall use the same notation $\widehat{Dom}_S(U)$ to denote posemigroup dominions.

By a *posemigroup amalgam* $[(U, \leq_U); (S_1, \leq_{S_1}), (S_2, \leq_{S_2}); \psi_1, \psi_2]$, cf. [12], we mean a "span"

$$\psi_i : (U, \leq_U) \longrightarrow (S_i, \leq_i),\ i \in \{1, 2\}, \tag{2}$$

in the category of posemigroups, with ψ_i being order-embeddings. We term (2) a *special posemigroup amalgam* if $(S_1, \leq_1)$ is order-isomorphic to $(S_2, \leq_2)$ via an order-isomorphism ν with $\nu \circ \psi_1 = \psi_2$. An amalgam is said to be *embeddable* (po-embeddable in the sense of [12]) if there exists a posemigroup $(W, \preccurlyeq)$ admitting order-embeddings $\phi_i : (S_i, \leq_i) \longrightarrow (W, \preccurlyeq)$, $i \in \{1, 2\}$, such that

i. $\phi_1 \circ \psi_1 = \phi_2 \circ \psi_2$ and
ii. $\phi_1(s_1) = \phi_2(s_2)$, $s_i \in S_i$, implies that $s_i = \psi_i(u)$ for some $u \in U$.

By relaxing condition (ii) we say that (2) is *weakly embeddable*. In the sequel we shall not make explicit reference to ψ_1 and ψ_2 and shall rather use a shorter list $[(U, \leq_U); (S_1, \leq_{S_1}), (S_2, \leq_{S_2})]$ to denote posemigroup (pomonoid) amalgams.

We first give a zigzag theorem (cf. [8], Theorem 2.1) for posemigroups.

Theorem 4. *Take a subposemigroup $(U, \leq_U)$ of a posemigroup $(S, \leq_S)$. Then an element d of $(S, \leq_S)$ is in $\widehat{Dom}_S(U)$ if and only if*

$$d \widehat{\otimes} 1 = 1 \widehat{\otimes} d \text{ in } S^1 \widehat{\otimes}_{U^1} S^1,$$

where $(U^1, \leq_{U^1})$ and $(S^1, \leq_{S^1})$ are the pomonoids obtained from $(U, \leq_U)$ and $(S, \leq_S)$, respectively, by adjoining an incomparable external identity whether or not they already have one.

Proof. ($\Longrightarrow$) Let **PSgr** denote the category of all posemigroups. Denote by $\mathbf{PSgr}^1$ the category of pomonoids obtained by adjoining an incomparable external identity to every object of **PSgr**, whether or not it has got one (certainly the morphisms in $\mathbf{PSgr}^1$ are required to preserve this identity). Let $1 \neq d \in \widehat{Dom}_S(U)$. Let $f^1, g^1 : (S^1, \leq_{S^1}) \longrightarrow (T^1, \leq_{T^1})$ agree on $(U^1, \leq_{U^1})$ in $\mathbf{PSgr}^1$. Take $f = f^1\big|_{(S, \leq_S)}$, $g = g^1\big|_{(S, \leq_S)}$. Then clearly, in **PSgr**, $f, g : (S, \leq_S) \longrightarrow (T^1, \leq_{T^1})$ are equal on $(U, \leq_U)$. And hence, by assumption, $f(d) = g(d)$. But then $f^1(d) = g^1(d)$. So $d \in \widehat{Dom}_{S^1}(U^1)$ whence $d \widehat{\otimes} 1 = 1 \widehat{\otimes} d$ in $S^1 \widehat{\otimes}_{U^1} S^1$ by Theorem 1.

($\Longleftarrow$) Let $d \widehat{\otimes} 1 = 1 \widehat{\otimes} d$ in $S^1 \widehat{\otimes}_{U^1} S^1$, with $d \neq 1$. Then $d \in \widehat{Dom}_{S^1}(U^1)$. Suppose that $f, g : (S, \leq_S) \longrightarrow (T, \leq_T)$ agree on $(U, \leq_U)$ in **PSgr**. Let

$f', g' : (S^1, \leq_{S^1}) \longrightarrow (T^1, \leq_{T^1})$ be defined by

$$(x \in S \Longrightarrow f'(x) = f(x),\ g'(x) = g(x)),\ f(1) = g(1) = 1.$$

Now clearly $f' \Big|_{(U^1, \leq_{U^1})} = g' \Big|_{(U^1, \leq_{U^1})}$ whence by assumption $f'(d) = g'(d)$. But then $f(d) = g(d)$ and so $d \in \widehat{Dom}_S(U)$ as required. □

One can now reformulate the above theorem in a way that resembles Isbell's original formulation [5].

Theorem 5. *Let U be a subposemigroup of a posemigroup S. Then we have $d \in \widehat{dom}_S(U)$ if and only if $d \in U$ or there exists a system of inequalities*

$$\begin{array}{ll}
d \leq s_1 u_1 & u_1 \leq v_1 t_1 \\
s_1 v_1 \leq s_2 u_2 & u_2 t_1 \leq v_2 t_2 \\
\vdots & \vdots \\
s_{n-1} v_{n-1} \leq u_n & u_n t_{n-1} \leq d \\
v_n \leq s_{n+1} u_{n+1} & d \leq v_n t_n \\
s_{n+1} v_{n+1} \leq s_{n+2} u_{n+2} & u_{n+1} t_n \leq v_{n+1} t_{n+1} \\
\vdots & \vdots \\
s_{n+m} v_{n+m} \leq d & u_{n+m} t_{n+m-1} \leq v_{n+m}
\end{array} \tag{3}$$

with elements $u_1, \ldots, u_{n+m}, v_1, \ldots, v_{n+m} \in U$, $s_1, \ldots, s_{n-1}, s_{n+1}, \ldots, s_{n+m}$, $t_1, \ldots, t_{n+m-1} \in S$.

Proof. ($\Longrightarrow$) Let $d \in \widehat{dom}_S(U)$. Then by the above theorem $d \widehat{\otimes} 1 = 1 \widehat{\otimes} d$ in $S^1 \widehat{\otimes}_{U^1} S^1$. Referring to [9], there exists a system of inequalities,

$$\begin{array}{ll}
d \leq s_1 u_1 & \\
s_1 v_1 \leq s_2 u_2 & u_1 \leq v_1 t_1 \\
\vdots & u_2 t_1 \leq v_2 t_2 \\
s_{n-1} v_{n-1} \leq s'_n u_n & \vdots \\
s'_n v'_n \leq 1 & u_n t_{n-1} \leq v'_n d \\
& \\
1 \leq s'_{n+1} u'_{n+1} & \\
s'_{n+1} v_n \leq s_{n+1} u_{n+1} & u'_{n+1} d \leq v_n t_n \\
\vdots & u_{n+1} t_n \leq v_{n+1} t_{n+1} \\
s_{n+m-1} v_{n+m-1} \leq s_{n+m} u_{n+m} & \vdots \\
s_{n+m} v_{n+m} \leq d & u_{n+m} t_{n+m-1} \leq v_{n+m},
\end{array}$$

where $u_1, \ldots u_{n+m}, v_1, \ldots v_{n+m}, v'_n, u'_{n+1} \in U^1$; $s_1, \ldots, s_{n-1}, s_{n+1}, \ldots, s_{n+m}$, $t_1, \ldots, t_{n+m-1}, s'_n, s'_{n+1} \in S^1$.

Now, because 1 is incomparable, $s'_n v'_n \leq 1$ implies $s'_n v'_n = 1$. But then we have $s'_n = v'_n = 1$, since 1 was adjoined externally. By a similar token we also have $s'_{n+1} = u'_{n+1} = 1$. One can therefore rewrite the above set of inequalities as follows:

$$\begin{array}{ll}
d \le s_1u_1 & u_1 \le v_1t_1 \\
s_1v_1 \le s_2u_2 & u_2t_1 \le v_2t_2 \\
\vdots & \vdots \\
s_{n-1}v_{n-1} \le u_n & u_nt_{n-1} \le d \\
v_n \le s_{n+1}u_{n+1} & d \le v_nt_n \\
s_{n+1}v_{n+1} \le s_{n+2}u_{n+2} & u_{n+1}t_n \le v_{n+1}t_{n+1} \\
\vdots & \vdots \\
s_{n+m}v_{n+m} \le d & u_{n+m}t_{n+m-1} \le v_{n+m}.
\end{array}$$

Employing the argument used by Howie in the unordered context (see [3], p. 272), we next show that all the elements in the above set of inequalities may be assumed to lie in S. We do this by assuming that u_i, v_i, s_i or t_i is not in S for some i and demonstrate that in this case the corresponding inequality can be omitted. It suffices to consider the upper half portion

$$\begin{array}{ll}
d \le s_1u_1 & u_1 \le v_1t_1 \\
s_1v_1 \le s_2u_2 & u_2t_1 \le v_2t_2 \\
\vdots & \vdots \\
s_{n-1}v_{n-1} \le u_n & u_nt_{n-1} \le d.
\end{array} \tag{4}$$

The proof will be accomplished by exhausting all possibilities.
If $u_1 = 1$ then (from $u_1 \le v_1t_1$) $v_1 = t_1 = 1$, and we may write

$$\begin{array}{ll}
d \le s_2u_2 & u_2 \le v_2t_2 \\
s_2v_2 \le s_3u_3 & u_3t_2 \le v_3t_3 \\
\vdots & \vdots \\
s_{n-1}v_{n-1} \le u_n & u_nt_{n-1} \le d.
\end{array}$$

If $v_1 = 1$ then $s_1 \le s_2u_2$, and one can calculate

$$\begin{array}{ll}
d \le s_2(u_2u_1) & u_2u_1 \le v_2t_2 \\
s_2v_2 \le s_3u_3 & u_3t_2 \le v_3t_3 \\
\vdots & \vdots \\
s_{n-1}v_{n-1} \le u_n & u_nt_{n-1} \le d.
\end{array}$$

If $u_i = 1$, $2 \le i \le n-1$, then the set of inequalities

$$\begin{array}{ll}
s_{i-2}v_{i-2} \le s_{i-1}u_{i-1} & u_{i-1}t_{i-2} \le v_{i-1}t_{i-1} \\
s_{i-1}v_{i-1} \le s_iu_i & u_it_{i-1} \le v_it_i \\
s_iv_i \le s_{i+1}u_{i+1} & u_{i+1}t_i \le v_{i+1}t_{i+1}
\end{array}$$

collapses to

$$\begin{array}{ll}
s_{i-2}v_{i-2} \le s_{i-1}u_{i-1} & u_{i-1}t_{i-2} \le v_{i-1}v_it_i \\
s_{i-1}v_{i-1}v_i \le s_{i+1}u_{i+1} & u_{i+1}t_i \le v_{i+1}t_{i+1}.
\end{array}$$

If $v_i = 1$, $2 \leq i \leq n-2$, then the set of inequalities

$$\begin{array}{ll} s_{i-1}v_{i-1} \leq s_i u_i & u_i t_{i-1} \leq v_i t_i \\ s_i v_i \leq s_{i+1}u_{i+1} & u_{i+1}t_i \leq v_{i+1}t_{i+1} \end{array}$$

reduces to

$$s_{i-1}v_{i-1} \leq s_{i+1}u_{i+1}u_i \qquad u_{i+1}u_i t_{i-1} \leq v_{i+1}t_{i+1}.$$

If $u_n = 1$ then (as noted above) $s_{n-1} = v_{n-1} = 1$, and the set of inequalities

$$\begin{array}{ll} s_{n-2}v_{n-2} \leq s_{n-1}u_{n-1} & u_{n-1}t_{n-2} \leq v_{n-1}t_{n-1} \\ s_{n-1}v_{n-1} \leq u_n & u_n t_{n-1} \leq d, \end{array}$$

may be replaced by

$$s_{n-2}v_{n-2} \leq u_{n-1} \qquad u_{n-1}t_{n-2} \leq d.$$

If $v_{n-1} = 1$ then we have $s_{n-1} \leq u_n$ and the set of inequalities

$$\begin{array}{ll} s_{n-2}v_{n-2} \leq s_{n-1}u_{n-1} & u_{n-1}t_{n-2} \leq v_{n-1}t_{n-1} \\ s_{n-1}v_{n-1} \leq u_n & u_n t_{n-1} \leq d \end{array}$$

can be replaced with

$$s_{n-2}v_{n-2} \leq u_n u_{n-1} \qquad u_n u_{n-1}t_{n-2} \leq d.$$

If $s_1 = 1$ then $d = u_1 \in U$ and there is nothing to prove.
If $s_i = 1$, $2 \leq i \leq n-1$, such that $s_j \in S$ for all $j \leq i-1$, then (starting from the top of (4)) one may write

$$d \leq s_1 u_1 \leq s_1 v_1 t_1 \leq s_2 u_2 t_1 \leq \cdots \leq u_i t_{i-1}.$$

On the other hand, (starting from the bottom of (4)) we also have

$$d \geq u_n t_{n-1} \geq s_{n-1}v_{n-1}t_{n-1} \geq s_{n-1}u_{n-1}t_{n-2} \geq \cdots \geq u_i t_{i-1}.$$

Thus $d = u_i t_{i-1}$, and we may shorten the inequalities (4) to

$$\begin{array}{ll} d \leq s_1 u_1 & u_1 \leq v_1 t_1 \\ s_1 v_1 \leq s_2 u_2 & u_2 t_1 \leq v_2 t_2 \\ \vdots & \vdots \\ s_{i-1}v_{i-1} \leq u_i & u_i t_{i-1} = d. \end{array}$$

Similarly, if $t_i = 1$, $1 \leq i \leq n-1$, with $t_j \in S$ for all $j \leq i-1$, then we have

$$d \leq s_1 u_1 \leq s_1 v_1 t_1 \leq s_2 u_2 t_1 \leq \cdots \leq s_i u_i t_{i-1} \leq s_i v_i$$

on one hand, and

$$d \geq u_n t_{n-1} \geq s_{n-1}v_{n-1}t_{n-1} \geq s_{n-1}u_{n-1}t_{n-2} \geq \cdots \geq s_{i+1}u_{i+1} \geq s_i v_i$$

on the other hand. This gives $d = s_i v_i = s_{i+1}u_{i+1}$ and one can shorten (4) to

$$\begin{array}{ll} d \le s_{i+1}u_{i+1} & u_{i+1} \le v_{i+1}t_{i+1} \\ s_{i+1}v_{i+1} \le s_{i+2}u_{i+2} & u_{i+2}t_{i+1} \le v_{i+2}t_{i+2} \\ \vdots & \vdots \\ s_{n-1}v_{n-1} \le u_n & u_n t_{n-1} \le d. \end{array}$$

This completes the proof of the direct part.

($\Longleftarrow$) To prove the converse part, let there exist a set of inequalities (3). Then, by [9], $d\widehat{\otimes}1 = 1\widehat{\otimes}d$ in $S^1\widehat{\otimes}_{U^1}S^1$. So $d \in \widehat{dom}_{S^1}(U^1)$ by Theorem 1. Now, $d \in dom_S(U)$ by the above theorem. $\square$

Proposition 1. *A posemigroup amalgam $[(U, \le_U); (S, \le_S), (T, \le_T)]$ is embeddable (weakly embeddable) if and only if the corresponding pomonoid amalgam $[(U^1, \le_{U^1}); (S^1, \le_{S^1}), (T^1, \le_{T^1})]$ is embeddable (weakly embeddable) in some pomonoid, where $(U^1, \le_{U^1})$, $(S^1, \le_{S^1})$ and $(T^1, \le_{T^1})$ are the pomonoids obtained from $(U, \le_U)$, $(S, \le_S)$ and $(T, \le_T)$ by adjoining an incomparable external identity, whether or not they already have one.*

Proof. Let the posemigroup amalgam $[(U, \le_U); (S, \le_S), (T, \le_T)]$ be embeddable in a posemigroup $(W, \le_W)$. Let $(W^1, \le_{W^1})$ be the pomonoid obtained by externally adjoining an incomparable identity to $(W, \le_W)$. Extend, in a natural way, the order-embeddings

$$\alpha : (S, \le_S) \longrightarrow (W, \le_W),$$
$$\beta : (T, \le_T) \longrightarrow (W, \le_W)$$

to

$$\alpha^1 : (S^1, \le_{S^1}) \longrightarrow (W^1, \le_{W^1}),$$
$$\beta^1 : (T^1, \le_{T^1}) \longrightarrow (W^1, \le_{W^1}).$$

It is now straightforward to see that the pomonoid amalgam $[(U^1, \le_{U^1}); (S^1, \le_{S^1}), (T^1, \le_{T^1})]$ is embeddable in $(W^1, \le_{W^1})$.

To prove the converse, we may assume without loss of generality that $[(U^1, \le_{U^1}); (S^1, \le_{S^1}), (T^1, \le_{T^1})]$ is embeddable in the pushout $(S^1 \circledast_{U^1} T^1, \preccurlyeq)$, see [10]. Because 1 was adjoined externally to $(U, \le_U)$, $(S, \le_S)$ and $(T, \le_T)$, the pair $((S^1 \circledast_{U^1} T^1)\backslash\{1\}, \preccurlyeq)$ is a posemigroup; let us reserve for it the notation $(S \circledast_U T, \preccurlyeq)$. Again, since 1 was adjoined externally to $(U, \le_U)$, $(S, \le_S)$ and $(T, \le_T)$, the posemigroup amalgam $[(U, \le_U); (S, \le_S), (T, \le_T)]$ embeds in $(S \circledast_U T, \preccurlyeq)$ if the order-embeddings from $(S^1, \le_{S^1})$ and $(T^1, \le_{T^1})$ to $(S^1 \circledast_{U^1} T^1, \preccurlyeq)$ are restricted to $(S, \le_S)$ and $(T, \le_T)$. $\square$

We shall henceforth denote $(S \circledast_U T, \preccurlyeq)$ (respectively, $(S^1 \circledast_{U^1} T^1, \preccurlyeq)$) by simply $S \circledast_U T$ $(S^1 \circledast_{U^1} T^1)$.

Corollary 1. *Every special amalgam* $[(U, \leq_U); (S_1, \leq_{S_1}), (S_2, \leq_{S_2})]$ *of posemigroups is weakly embeddable in* $S_1 \circledast_U S_2$.

Proof. Because $[(U^1, \leq_{U^1}); (S_1^1, \leq_{S_1^1}), (S_2^1, \leq_{S_2^1})]$ is weakly embeddable by Remark 1 of [12], it is weakly embeddable in $S^1 \circledast_{U^1} T^1$, see [10]. Thus $[(U, \leq_U); (S_1, \leq_{S_1}), (S_2, \leq_{S_2})]$ is weakly embeddable in $S_1 \circledast_U S_2$. □

Consequently, we also have the following result (its proof, being similar to that of the corresponding result in the unordered context, see for instance [8], is omitted).

Corollary 2. *Let* $(U, \leq_U)$ *be a subposemigroup (subpomonoid) of a posemigroup (monoid)* $(S, \leq_S)$. *Also, let* $(S_1, \leq_1)$ *and* $(S_2, \leq_2)$ *be two disjoint order-isomorphic copies of* $(S, \leq_S)$. *Then*

$$\widehat{dom}_S(U) \cong \widehat{dom}_{S_i}(U) = \pi_i^{-1}(\pi_1(S_1, \leq_1) \cap \pi_2(S_2, \leq_2)),\ i \in \{1, 2\}\,,$$

where $\pi_i : (S_i, \leq_i) \longrightarrow S_1 \circledast_U S_2$ *are the order-embeddings and where* $(U, \leq_U)$ *is identified with its order isomorphic copies in* $(S_1, \leq_1)$ *and* $(S_2, \leq_2)$.

Now observe that a special posemigroup (respectively, pomonoid) amalgam $[(U, \leq_U); (S_1, \leq_{S_1}), (S_2, \leq_{S_2})]$ is embeddable if and only if

$$\pi_i^{-1}(\pi_1(S_1, \leq_1) \cap \pi_2(S_2, \leq_2)) = U \ (i \in \{1, 2\}\,).$$

Therefore we have the following corollary.

Corollary 3. *A subposemigroup (subpomonoid)* $(U, \leq_U)$ *of a posemigroup (pomonoid)* $(S, \leq_S)$ *is closed in* $(S, \leq_S)$ *if and only if the special posemigroup (pomonoid) amalgam* $[(U, \leq_U); (S_1, \leq_{S_1}), (S_2, \leq_{S_2})]$ *is embeddable;* $(S_1, \leq_{S_1})$ *and* $(S_2, \leq_{S_2})$ *are order isomorphic copies of* $(S, \leq_S)$.

Theorem 6. *A subposemigroup* $(U, \leq_U)$ *of a posemigroup* $(S, \leq_S)$ *is closed in* $(S, \leq_S)$ *if and only if* U *is such in* S *as a semigroup.*

Proof. ($\Longrightarrow$) This follows from inclusions (1), which also hold for posemigroups.

($\Longleftarrow$) Suppose U is closed in S as a semigroup. Consider the special posemigroup amalgam $[(U, \leq_U); (S_1, \leq_{S_1}), (S_2, \leq_{S_2})]$, where $(S_1, \leq_{S_1})$ and $(S_2, \leq_{S_2})$ are order-isomorphic copies of $(S, \leq_S)$. Ignoring the orders, note first that $[U; S_1, S_2]$ is embeddable (particularly in $S_1 *_U S_2$, see [3] Theorem 8.2.4) because U is closed in S (cf. [8]). But then $[U^1; S_1^1, S_2^1]$ is embeddable (particularly in $S_1^1 *_{U^1} S_2^1$). So U^1 is closed in S_1^1, by (unordered analogue, see [8], of) the above corollary. Now, $(U^1, \leq_{U^1})$ is closed in $(S_1^1, \leq_{S_1^1})$ by Theorem 3. This means, by the above corollary, that $[(U^1, \leq_{U^1}); (S_1^1, \leq_{S_1^1}), (S_2^1, \leq_{S_2^1})]$ is embeddable (particularly in $S_1^1 \circledast_{U^1} S_2^1$). Finally, repeating the argument used in Proposition 1, $[(U, \leq_U); (S_1, \leq_{S_1}), (S_2, \leq_{S_2})]$ is embeddable in $S_1 \circledast_U S_2$. The proof now follows from the above corollary. □

Corollary 4. *A posemigroup $(U, \leq)$ is absolutely closed if and only if it is such as a semigroup within the class of semigroups underlying the posemigroup extensions of $(U, \leq)$.*

In the unordered scenario Higgins has determined all varieties of absolutely closed semigroups.

Theorem 7 ([2], Theorem 2)**.** *The absolutely closed (algebraic) varieties of semigroups are exactly the varieties consisting entirely of semilattices of groups, or entirely of right groups or entirely of left groups.*

Corollary 5. *The varieties of posemigroups obtained by endowing with compatible orders (the members of) the varieties of above theorem are all absolutely closed.*

Problem 1. Do there exist absolutely closed (order theoretic) varieties of posemigroups other than those of the above corollary?

Proposition 2. *Let $\mathcal{V}'$ be a variety of absolutely closed semigroups. Let $\mathcal{V}$ be the variety of posemigroups obtained by equipping members of $\mathcal{V}'$ with compatible orders. Then a posemigroup homomorphism f is epi in $\mathcal{V}$ if and only if it is such in $\mathcal{V}'$.*

Proof. ($\Longleftarrow$) This part is straightforward.

($\Longrightarrow$) Let $f : U \longrightarrow T$ be non-epi in $\mathcal{V}'$. Then $\operatorname{Im} f$ is absolutely closed in $\mathcal{V}'$ and $dom_T(\operatorname{Im} f) \subsetneq T$. But then by Corollary 4, $\operatorname{Im} f$ is also absolutely closed in $\mathcal{V}$ and thus $\operatorname{Im} f = dom_T(\operatorname{Im} f) = \widehat{dom}_T(\operatorname{Im} f)$. Therefore $\widehat{dom}_T(\operatorname{Im} f) \subsetneq T$. So f is non-epi in $\mathcal{V}$. □

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