ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 18, Number 1, June 2014
Available online at http://acutm.math.ut.ee

The Hermitian part of a Rickart involution ring, I

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Abstract. Rickart *-rings may be considered as a certain abstraction of the rings $\mathcal{B}(H)$ of bounded linear operators of a Hilbert space H. In 2006, S. Gudder introduced and studied a certain ordering (called the logical order) of self-adjoint Hilbert space operators; the set $\mathcal{S}(H)$ of these operators, which is a partial ring, may be called the Hermitian part of $\mathcal{B}(H)$. The new order has been further investigated also by other authors. In this first part of the paper, an abstract analogue of the logical order is studied on certain partial rings that approximate the Hermitian part of general *-rings; the special case of Rickart *-rings is postponed to the next part.

1. Introduction

1. Evidently, the set $S(R) := \{x \in R\colon\ x^* = x\}$ of symmetric elements of an involution ring $(R, +, \cdot, 0, ^*)$ is closed under addition, and contains the product of its elements x, y if and only if $xy = yx$. In particular, $(S(R), +, 0)$ is a subgroup of the additive group of R. $S(R)$ contains also all projections (idempotent symmetric elements) of R. By the *Hermitian part* of R we shall mean the partial ring $(S(R), +, \cdot, 0)$ equipped with the inherited addition and (partial) multiplication.

Our interest in Hermitian parts is determined by the paper [9] and its successors [13, 2]. In the standard Hilbert space formalism for quantum mechanics, an appropriate Hilbert space H is associated with a physical system, and observables of the system are identified with bounded self-adjoint operators on H. In [9], Gudder introduced on the set $\mathcal{S}(H)$ of all such operators a certain ordering called by him the *logical order*, which may be characterized

Received December 29, 2013.
2010 *Mathematics Subject Classification.* 06A06, 06F25, 13J25, 16W10, 47L30.
Key words and phrases. Hermitian part, involution ring, logical order, partial ring, projection, self-adjoint operator.
http://dx.doi.org/10.12097/ACUTM.2014.18.10

by

$$A \preceq B :\equiv A|\overline{\mathrm{ran}}A = B|\overline{\mathrm{ran}}A.$$

He explained its physical meaning ($A \preceq B$ if and only if the event that A has a value in any Borel subset Δ of reals not containing 0 implies the event that B has a value in Δ), and investigated general properties of the new order. His results were extended and improved by Pulmannová and Vinceková in [13] and the present author in [2]. In particular, existence and descriptions of lattice operations in $\mathcal{S}(H)$ were studied in these three papers. We list some facts about the order structure of $\mathcal{S}(H)$ stated there:

(1) $\preceq$ agrees with the natural ordering of orthogonal projection (i.e., idempotent self-adjoint) operators,
(2) $\mathcal{S}(H)$ is a lower semilattice under $\preceq$, with O (the zero operator) as the least element and without the greatest element,
(3) every initial segment $[O, A]$ of $\mathcal{S}(H)$ is an orthomodular lattice embeddable into an initial segment of the lattice of projection operators,
(4) the segment $[O, I]$, where I is the unit operator, coincides with the lattice of projection operators,
(5) $\preceq$ is the natural ordering of a naturally defined generalized orthoalgebra on $\mathcal{S}(H)$,
(6) every subset of $\mathcal{S}(H)$ bounded from the above has the least upper bound, and every non-empty subset has the greatest lower bound.

To obtain these, in fact, lattice-theoretic results, several topological properties of Hilbert spaces, as well as some facts of spectral theory of operators and of weak operator topology were used along with purely algebraic reasoning ((2) and (6) depended also on completeness of the lattice of projections in $\mathcal{S}(H)$).

However, the logical order (sometimes called also the Gudder order in the literature) itself can be characterized algebraically. It turns out [9] that

$$A \preceq B \text{ if and only if } A^2 = AB$$

(where multiplication means the composition of operators). This observation rises an interest in the following problem, which has motivated the present work:

> Which results on $\preceq$ from [2, 9, 13] (more generally, which properties of the logical order) can be derived algebraically, and what algebraic properties of bounded self-adjoint operators are necessary for this purpose?

2. Let $\mathcal{B}(H) := (\mathcal{B}(H), +, \cdot, ^*, O, I)$ be the involution ring (*-ring) of all bounded linear operators of a Hilbert space H, where *, as usual, stands for the operation associating with every operator A its adjoint, $\cdot$ is the composition, O is the zero operator, and I is the unit operator. Its Hermitian part, $\mathcal{S}(H)$, is a kind of commutative partial ring. As the orthogonal projections play an essential role in describing the order structure of $\mathcal{S}(H)$, we consider also an additional less familiar operation that assigns to any operator A the projection onto its kernel (null space) $\ker A$. Denote this operation by $'$; then A'' takes A into the projection operation P_A onto the subspace $\overline{\mathrm{ran}}A$ (the closure of $\mathrm{ran}\, A$), and $A' = I - A''$. The ring $\mathcal{B}(H)$ equipped with this operation is an example of a *Rickart *-ring*; of course this operation acts also on $\mathcal{S}(H)$. Such partial Rickart rings form our starting point.

Let R be a Rickart *-ring. Suppose that an order relation $\preceq$ (the "logical order") on its Hermitian part $S(R)$ is given. There are several ways for investigating the structure of $S(R)$:

(1) We may work in the full Rickart *-ring R, and relativize the results to $S(R)$. This way, which repeats abstractly the actual approach of [9, 13, 2], seems to be the more easy and more fruitful one. It was partially realized in [4], where properties of the so-called star order on a Rickart *-ring were studied (the logical order on $S(R)$ is the restriction of the star order of R). As a by-product, the abstract analogues of some results from the mentioned original papers were demonstrated in [4].

(2) On the contrary, we may fix some basic properties of $S(R)$ as a partial Rickart ring (commutativity among these) and work solely in a partial ring S possessing these properties. This is the approach which we follow in the present paper. The main difficulty of this approach is the fact that many standard constructions and arguments become more complicated or cannot be realized at all just because of the partiality of multiplication, and the main challenge is to find an appropriate axiom system for partial Rickart rings and to demonstrate that it is sufficiently strong. A more distant goal of independent interest could be a reasonable representation theorem for these partial rings.

(3) Like [7], we could treat $S(R)$ abstractly as an Abelian group $(G, +, 0)$ embedded in an appropriate enveloping (total) ring R and to work in R. In fact, the group G is not supposed there to be a part of any *-ring; instead, the relations between G and R are subject to certain axioms. Moreover, G is assumed to be ordered; however, the order corresponds to the usual ordering of Hilbert space operators rather than to their logical order. (The paper [8] presents a simplified variant of this scheme.) The logical order on G in such algebraic structures is investigated in [11].

3. We are not going, in this work, to reconsider all results of [9, 13, 2], and restrict ourselves only to a few basic ones. Actually, our aim is to clear up which (arithmetical) properties of a partial ring are involved with various properties of the logical order on it. In the present first part of the paper we deal with partial rings arising from general involution rings; the more special case of partial Rickart rings will be discussed in the next part. Section 2 contains the necessary information on some ordered structures related to orthomodular posets. A version of abstract partial rings approximating the Hermitian parts of involution rings is introduced in Section 3. We also describe here the order structure of the set of idempotents of such partial rings (in particular, idempotents of a unital partial ring form a partial Boolean algebra of certain type) and show that a reduced partial ring supports some generalized orthoalgebra (these algebras are known well in quantum logic; see [5]). The logical order on a reduced partial ring is the subject of the last section. The main results of this section concern the existence of certain joins and meets on S.

2. Preliminaries

A poset with the least element 0 and the greatest element 1 is *orthocomplemented* (in short, an *orthoposet*) if it is equipped with a unary operation $^\perp$ such that

$$x^{\perp\perp} = x, \quad \text{if } x \leq y, \text{ then } y^\perp \leq x^\perp, \quad x \vee x^\perp = 1, \quad x \wedge x^\perp = 0,$$

where $u \vee v$ means the least upper bound (join), and $u \wedge v$, the greatest lower bound (meet) of x and y. Then $x^\perp$ is the *orthocomplement* of x, and $0^\perp = 1$, $1^\perp = 0$. The de Morgan laws hold in an orthoposet M in the following form: if one side in the identities $(x \vee y)^\perp = x^\perp \wedge y^\perp$ and $(x \wedge y)^\perp = x^\perp \vee y^\perp$ exists, then the other also exists, and both are equal. The *induced orthogonality* on M is the relation $\perp$ defined by $x \perp y$ if and only if $y \leq x^\perp$; it has the properties

($\perp_1$): if $x \perp y$, then $y \perp x$,
($\perp_2$): if $x \leq y$ and $y \perp z$, then $x \perp z$,
($\perp_3$): $0 \perp x$,
($\perp_4$): if $x \perp y, z$ and $y \vee z$ exists, then $x \perp y \vee z$,
($\perp_5$): if $x \perp x$, then $x = 0$.

An orthoposet M is *orthomodular*, if its induced orthogonality satisfies conditions

($\perp_6$): if $x \perp y$, then $x \vee y$ exists,
($\perp_7$): if $x \leq y$, then $y = x \vee z$ for some z with $x \perp z$.

In such a poset, also

($\perp_8$): if $x \perp y, z$ and $y \leq x \vee z$, then $y \leq z$.

An *orthomodular lattice* is a lattice-ordered orthomodular poset. A poset equiped with an arbitrary relation $\perp$ such that ($\perp_1$)–($\perp_3$) and ($\perp_6$)–($\perp_8$) hold true is called *quasi-orthomodular* in [3]. Notice that ($\perp_5$) is a particular case of ($\perp_8$).

We shall need some more information from [3]. A *sectionally orthocomplemented poset*, or a *sectional orthoposet*, is a poset with 0, in which every initial section $[0, p]$ is orthocomplemented; we denote the "local" orthocomplementation in this section by ${}^{\perp}_{p}$, and the corresponding induced orthogonality on $[0, p]$, by $x \perp_p y$. Therefore, a sectional orthoposet with the greatest element is orthocomplemented. A *sectionally orthomodular* poset is defined similarly. The *induced orthogonality* $\perp$ on a sectional orthoposet is defined to be the union of all local orthogonalities $\perp_p$. It satisfies $(\perp_1)$–$(\perp_3)$ and $(\perp_5)$, but not necessary $(\perp_4)$.

A sectional orthoposet is said to be *relatively orthocomplemented* (in short, a *relative orthoposet*) if (i) any pair of elements $x, y \leq p$ has the join whenever $x \perp_p y$, and (ii) if $x \leq p \leq q$, then $x^{\perp}_{p} = p \wedge x^{\perp}_{q}$; the latter condition may even be weakened to (ii') if $x \leq p \leq q$, then $x^{\perp}_{p} \leq x^{\perp}_{q}$. Moreover, all sections of a relative orthoposet are actually orthomodular, so that such a poset could also be called *relatively orthomodular*.

Proposition 2.1 ([3, Theorem 5.5]). *A poset A with the least element* 0*, supplied with a binary relation $\perp$, is quasi-orthomodular if and only if it is relatively orthocomplemented and $\perp$ is its induced orthogonality.*

Generalized and *weak generalized* orthomodular posets were introduced by Mayet-Ippolito in [12]; they may be shortly characterized as relative orthoposets satisfying respectively the condition $(\perp_4)$ and its weakening

$(\perp_9)$: if $x \perp y, z$ and $y \perp z$, then $x \perp y \vee z$.

Every orthomodular poset is a generalized orthomodular poset; conversely, a bounded weak generalized orthomodular poset is orthomodular. The set of all bounded self-adjoint operators in a Hilbert space is an example of a weak generalized orthomodular poset with respect to the logical order and the usual orthogonality of operators defined by $A \perp B$ if and only if $AB = 0$ ([13], Theorem 4.3). It is well known that projection operators in a Hilbert space form even a (complete) orthomodular lattice.

3. Multiplicatively partial commutative rings

1. Since we shall deal in this paper only with partial rings that are commutative, we omit the attribute "commutative" in the subsequent definition.

Definition 3.1. A system $(S, +, \cdot, 0)$ is a *(multiplicatively) partial ring* if

- $(S, +, 0)$ is an Abelian group,
- $(S, \cdot, 0)$ is a partial commutative semigroup with zero, i.e.,
  - (a) $\cdot$ is a partial binary operation (we usually write uv for $u \cdot v$, and uDv, to mean that uv is defined),
  - (b) if xDy, then yDx and $yx = xy$,
  - (c) if xDy, $xyDz$ and yDz, then $xDyz$ and $x \cdot yz = xy \cdot z$,

(d) $0Dx$ and $0x = 0$,

- S is distributive in the following sense:
  (e) if any two of the conditions $xD(y+z)$, xDy and xDz are fulfilled, then the third one is also fulfilled and $x(y+z) = xy + xz$,
- in addition,
  (f) xDx,
  (g) if xDy, xDz and yDz, then $xDyz$.

A partial commutative ring S is said to be *unital* if it contains an element 1 such that

- $(S, \cdot, 1)$ becomes a partial monoid, i.e.,
  (h) $1Dx$ and $1x = x$.

In particular, every subset $C(x) := \{y\colon xDy\}$ of S, which may be called the *commutant of* x, is a partial subring of S: besides x itself, it contains 0 and also sums and existing products of its elements. The proof of the following proposition is a routine work.

Proposition 3.2. *The following holds in any partial ring:*

(a) *if yDz, $xDyz$ and xDy, then $xyDz$ and $xy \cdot z = x \cdot yz$,*
(b) *if xDy, then $-xDy$ and $(-x)y = -(xy)$,*
(c) *if any two of the conditions $xD(y-z)$, xDy and xDz are fulfilled, then the third one is also fulfilled and $x(y-z) = xy - xz$,*
(d) *if xDy, then $xDxy$, x^2Dy and $x \cdot xy = x^2y$,*
(e) *if xDy, then x^mDy^n for all natural numbers m, n.*

We shall usually apply the properties of the relation D and of ring operations listed in Definition 3.1 and this proposition without explicit references.

Theorem 3.3. *The Hermitian part of any involution ring R is a partial ring.*

Proof. We shall demonstrate only the items (c), (e) and (g) of the above definition. Assume that $x, y, z \in S(R)$. Recall that xDy if and only if $xy \in S(R)$ if and only if $xy = yx$ in R.

(c) Suppose that xDy, $xyDz$, yDz. Then $xy \cdot z \in S(R)$ and, in R, $xy \cdot z = z \cdot xy = z \cdot yx = zy \cdot x = yz \cdot x$. So, $yz \cdot x \in S(R)$ and $yz \cdot x = x \cdot yz$.

(e) Suppose that $xD(y+z)$ and xDy. Then in R, $yx + zx = (y+z)x = x(y+z) = xy + xz = yx + xz$, from where $xz = zx \in S(R)$ and xDz.

Suppose that xDy and xDz. Then, in R, $x(y+z) = xy + xz = yx + zx = (y+z)x$, i.e., $xDy + z$.

(g) Suppose that xDy, xDz, yDz. Then in R, $x \cdot yz = x \cdot zy = xz \cdot y = zx \cdot y = z \cdot xy = z \cdot yx = zy \cdot x = yz \cdot x$, i.e., $xDyz$. □

A subset A of a partial ring S is said to be *compatible* if aDb for all $a, b \in A$. The empty set, any one-element set and the subset $\{0, 1\}$ are trivial examples of compatible sets. The standard argument based on Zorn's lemma shows that every compatible set is included in a maximal one. A maximal compatible set is a total subring of S; clearly, it is a maximal subring, and every maximal subring arises in this way. Let us call the maximal compatible subsets *blocks* of S.

For the rest of the paper, we assume that S is a fixed partial ring.

2. We denote by P the set of all idempotent elements, or *projections* of S, and let the letters e, f, g stand for arbitrary elements of P. Similarly to ordinary commutative rings, P can naturally be ordered: $e \leq f :\equiv eDf$ and $ef = e$ (equivalently, $fe = e$); 0 is the least, and 1 (if S is unital) is the greatest projection. As observed in [12, Proposition 1], the set of all idempotent elements of a ring provides an example of a weak generalized orthomodular poset. We noticed already in the Introduction that all projections of a *-ring R belong to $S(R)$. In a unital involution ring, projections form an orthomodular poset, which is a lattice if the ring is Rickart [10]. An old result by Foster [6] says that idempotents of a unital commutative ring even form a Boolean algebra. We are now going to show how these facts are reflected in an arbitrary partial ring. (Of course, the last-mentioned result concerns every block of S.)

Let the usual symbols $\vee$ and $\wedge$ stand for the lattice operations in P, which normally are partial. The well-known descriptions of these operations in total rings can be transferred to S as in (a) and (b) in the subsequent proposition.

Proposition 3.4. *Suppose that eDf. Then*

(a) *$e \wedge f$ exists, and $e \wedge f = ef$,*
(b) *$e \vee f$ exists, and $e \vee f = e + f - ef$,*
(c) *if $e, f \in C(a)$ for some $a \in S$, then also $e \wedge f, e \vee f \in C(a)$,*
(d) *if $e, f \in [0, g]$ for some $g \in P$, then also $e \wedge f, e \vee f \in [0, g]$.*

If also eDg and fDg, then $(e \vee f)Dg$, $(e \wedge g)D(f \wedge g)$, and

(e) *$(e \vee f) \wedge g = (e \wedge g) \vee (f \wedge g)$.*

Proof. If eDf, then $ef \in P$ and $eDef$, $fDef$.

(a) Clearly, $ef \leq e, f$, and if $g \leq e, f$, then $g = ge$ and $g = gf = ge \cdot f = g \cdot ef$, i.e., $g \leq ef$. Hence, ef is the greatest lower bound of e and f.

(b) Clearly, $g := e + f - ef$ is an upper bound of e and f. If h is any other upper bound, then $ef \leq h$, $e, f, ef, g \in C(h)$ and $gh = eh + fh - ef \cdot h = e + f - ef = g$, i.e., $g \leq h$. Therefore, g is the least upper bound of a and b.

(c) and (d) follow from (a) and (b) by virtue of properties of D.

(e) Under the two additional assumptions, also $ef, eg, fg \in C(g)$ and $geDgf$. Then the equality to be proved reduces to $(e + f - ef)g = eg + fg - eg \cdot fg$, where both sides exist. □

Let us say that a subset A of P is a *D-partial lattice* if $e \wedge f, e \vee f \in A$ whenever $e, f \in A$ and eDf. Proposition 3.4 shows that such a lattice is in a sense distributive. In particular, P itself, every commutant $C(a)$, and every segment $[0, g]$ are distributive D-partial lattices.

Just as in commutative rings, elements x, y of S are said to be *orthogonal*, in symbols, $x \perp y$, if xDy and $xy = 0$. We shall sometimes write $x \oplus y$ for $x + y$ in the case when $x \perp y$. Evidently, $e + f \in P$ if and only if $e \perp f$, and then $e \vee f = e + f = e \oplus f$. Also, $e \perp f$ if and only if eDf and $e \wedge f = 0$.

We leave to the reader the axiom checking, necessary to ensure that the poset P is quasi-orthomodular with respect to $\perp$ and fulfills also $(\perp_9)$. Proposition 2.1 now leads to the following conclusion.

Proposition 3.5. *The projections in S form a weak generalized orthomodular poset. In every initial segment $[0, g]$ of P, $e_g^{\perp} = g - e$ and $e \perp f$ if and only if $f \leq e_g^{\perp}$. In particular, if S is unital, then it is an orthomodular poset and $e^{\perp} = 1 - e$.*

It now follows that each segment $[0, g]$ of P, being a distributive and orthomodular D-partial lattice, could be regarded as a *D-partial Boolean algebra*. It is worth to note that if elements e and f of $[0, g]$ are compatible i.e., eDf, then $e, f, ef, e_g^{\perp} \in C(g)$, $(e \wedge f)D(e_g^{\perp} \wedge f)$ and $f = (e \wedge f) \vee (e_g^{\perp} \wedge f)$. Therefore, e and f commute as elements of an orthomodular poset $[0, g]$.

Finally, if S is unital, then $C(a)$ is a suborthoposet of P and, hence, also a D-Boolean algebra.

3. A total ring usually is said to be reduced if it has no nilpotent elements other than 0. As is known well, this is the case if and only if the following condition holds:

$$\text{if } x^2 = 0, \text{ then } x = 0. \tag{1}$$

Since only one variable is involved in (1), this connection can be established in a standard way also for partial rings (see Proposition 3.2(a,d,e)). However, some useful consequences of (1) (for instance, if $x^2y = 0$, then $xy = 0$) cannot be derived in a partial ring. For this reason, we formally extend the notion of reducibility.

Definition 3.6. Let us call a partial ring *reduced* if it satisfies the conditions (1) and

$$\text{if } x^2Dy \text{ and } x^2y = 0, \text{ then } xDy. \tag{2}$$

It is easily seen that, in such a partial ring S,

$$\text{if } x^2Dy \text{ and } x^2y = 0, \text{ then } xDy \text{ and } xy = 0. \tag{3}$$

Indeed, if $x^2y = 0$, then $0 = y \cdot x^2y = yx \cdot xy = (xy)^2$ and, by (1), $xy = 0$. Therefore, if S is total, then (3) follows from (1), and if S is unital, then (1) is a particular case of (3). We note that a similar implication “if $A^2B = 0$, then

$AB = 0$" for self-adjoint Hilbert space operators A and B was demonstrated in [9, proof of Theorem 4.2] using a property of scalar products.

Involution in a $*$-ring R is said to be *proper* if it satisfies the so-called **-cancellation rule*

$$\text{if } x^*x = 0, \text{ then } x = 0.$$

For example, $\mathcal{B}(H)$ has a proper involution.

Theorem 3.7. *If a ring R has a proper involution, then the Hermitian part of R is reduced.*

Proof. If $x \in S(R)$ and $x^2 = 0$, then, in R, $x^*x = 0$ and $x = 0$. Further, if $x, y \in S(R)$ and $x^2 y = 0$, then, in R, $0 = yx^2y = (xy)^*(xy)$ and $xy = 0 \in S(R)$, i.e., xDy. $\square$

Most of the basic properties of the orthogonality relation and operation $\oplus$ are collected in the subsequent theorem, the prototype of the first part of which is Theorem 4.2 in [9] proved for self-adjoint Hilbert space operators. See [5, 9, 13] for more on generalized orthoalgebras.

Theorem 3.8. *If the partial ring S is reduced, then the partial algebra $(S, \oplus, 0)$ is a generalized orthoalgebra, i.e., for all appropriate x, y, z it satisfies the conditions*

(a) *if $x \perp y$, then $y \perp x$ and $x \oplus y = y \oplus x$,*
(b) *if $x \perp y$ and $(x \oplus y) \perp z$, then $y \perp z$, $x \perp (y \oplus z)$ and $(x \oplus y) \oplus z = x \oplus (y \oplus z)$,*
(c) *$x \perp 0$ and $x \oplus 0 = x$,*
(d) *if $x \perp y$, $x \perp z$ and $x \oplus y = x \oplus z$, then $y = z$,*
(e) *if $x \oplus x$ is defined (i.e., $x \perp x$), then $x = 0$.*

Moreover,

(f) *if $x \perp y, z$ and $y \perp z$, then $x \perp (y \oplus z)$.*

Proof. (b) Suppose that $x \perp y$ and $(x + y) \perp z$. Then xDy, $xD(x + y)$, $(x+y)Dz$ and $x0 = x \cdot (x+y)z = x(x+y) \cdot z = x^2 z$. Now $x \perp z$ by (3), and likewise $y \perp z$. It immediately follows that $x \perp (y \oplus z)$ and $(x \oplus y) \oplus z = x \oplus (y \oplus z)$.

(e) If $x \oplus y = 0$, then xDy, $xy = 0$ and $0 = x(x + y) = x^2$. By (1), $x = 0$.

(f) If $xy = xz = yz = 0$, then $xD(y + z)$ and $x(y + z) = xy + xz = 0$. $\square$

Notice that none of the properties (1)–(3) of S is needed to prove (a), (c) and (d). As for (b), cf. the remark following the proof of (3) above.

4. The logical order on S

Let us consider a relation $\preceq$ on a partial ring S defined by

$$x \preceq y :\equiv xDy \text{ and } x^2 = xy \text{ (or, equivalently, } x^2 = yx\text{)}. \tag{4}$$

The following proposition extends to partial rings a result proved by Abian for commutative rings in [1].

Proposition 4.1. *If S is reduced, then the relation $\preceq$ is a partial order.*

Proof. Assume that S is reduced. Evidently, $\preceq$ is reflexive. It is antisymmetric: if $x^2 = xy = y^2$, then $0 = x^2 - xy - yx + y^2 = (x-y)^2$ by distributivity. So $x - y = 0$ and $x = y$. The relation $\preceq$ is also transitive. If $x^2 = xy$ and $y^2 = yz$, then $yD(z-y)$ and $y(z-y) = 0 = x \cdot y(z-y) = xy \cdot (z-y) = x^2 \cdot (z-y)$. So, $xD(z-y)$ and $x(z-y) = 0$ by (3). As also xDy, it follows that xDz and $xz = xy = x^2$, i.e., $x \preceq z$. □

Following the tradition mentioned in the Introduction, let us call the relation $\preceq$ defined as above the *logical order* of a reduced partial ring. (In the 1970s, it was known in arbitrary rings as an Abian order; in commutative rings, this order was studied already by Sussman in [14].) Notice that

$$x \preceq y \iff x \perp (x-y) \iff y = x \oplus z \text{ for some } z \text{ (with } z \perp x). \qquad (5)$$

Therefore, $\preceq$ is actually the so-called natural order of the generalized orthoalgebra induced by S; this was the original definition of the logical order by Gudder in $\mathcal{S}(H)$. See [9, 13] for details.

In the rest of the section, let $\preceq$ be the logical order on S. For lattice operations in S, we use the symbols $\curlyvee$ (join) and $\curlywedge$ (meet); these operations normally are partial.

Lemma 4.2. *In S,*

(a) *$0 \preceq x$ for every x,*
(b) *$\preceq$ extends the natural order of projections,*
(c) *if xDe, then $xe \preceq x$,*
(d) *if $x \preceq e$, then $x \in P$,*
(e) *if one of $e \wedge f$ and $e \curlywedge f$ exists, then the other exists and both are equal,*
(f) *if $y \preceq z$, xDy and xDz, then $xy \preceq xz$,*
(g) *$e \preceq x$ if and only if eDx and $xe = e$.*

If S is unital, then, moreover,

(h) *$x \in P$ if and only if $x \preceq 1$,*
(i) *every invertible element is maximal,*
(j) *S is an upper semilattice only if $P = S$.*

Proof. (c) If xDe, then also $xeDe$ and $xeDx$. Now $(xe)^2 = xe \cdot ex = xe^2 \cdot x = xe \cdot x$.

(d) If $x \preceq e$, then xDe, $xeDe$ and x^2De. Now $x^2 = xe \preceq x$ by (c), and $x^2e = xe \cdot e = xe^2 = xe = x^2$, i.e., $x \preceq xe = x^2$. So, $x \in P$.

(e) If $e \wedge f$ exists in P, then this element is a lower bound of e and f also in S. Suppose that a is another such a lower bound; then $a \in P$ by (d) and, hence $a \leq e \wedge f$ (see (b)). If $e \curlywedge f$ exists in S, then $e \curlywedge f \in P$ by (d); so, this element is the greatest lower bound of e and f also in P.

(f) Suppose that $y \preceq z$, xDy and xDz. Then yDz, $xyDx$ and x^2Dy. Further, $(xy)^2 = x^2y^2 = x^2 \cdot yz = x^2y \cdot z = (xy \cdot x)z = xy \cdot xz$. Thus $xyDxz$ and $xy \preceq xz$.

(i) If xDy, $xy = 1$ and $x \preceq z$, then x^2Dy, xDz, $x^2 = xz$ and $x = x \cdot xy = x^2 \cdot y = xz \cdot y = y \cdot xz = yx \cdot z = z$.

(j) As 1 is invertible and maximal, $x \curlyvee 1$ exists if and only if $x \preceq 1$ if and only if $x \in P$. □

In spite of (f), S cannot be regarded as an ordered partial ring, for the addition need not be isotonic.

Next, we present some partial results concerning joins and meets in S.

Theorem 4.3. *Suppose that, for some $p \in S$ and $e, f \in P$, $x = pe$ and $y = pf$. If eDf, then*

(a) *$x \curlywedge y$ exists, and $x \curlywedge y = p(e \wedge f)$,*
(b) *$x \curlyvee y$ exists, and $x \curlyvee y = p(e \vee f)$.*

Moreover, then

(c) *$x \curlyvee y = x + y - x \curlywedge y$,*
(d) *if $z = pg$, eDg and fDg, then $(x \curlyvee y) \curlywedge z = (x \curlywedge z) \curlyvee (y \curlywedge z)$.*

Proof. The suppositions imply that $x, y \preceq p$ and $e, f \in C(p)$. Then also $(e \wedge f), (e \vee f) \in C(p)$ (see Proposition 3.4(c)) and $peDf$, $pfDe$, p^2De,f.

(a) Put $z := p(e \wedge f)$. Then $z \preceq a$ by Lemma 4.2(b,f), and likewise $z \preceq y$. For any u, if $u \preceq x, y$, then uDx, y and $ux = u^2 = uy$, i.e., $u^2 = u \cdot pe = u \cdot pf$. Now $u^2 = u \cdot pf^2 = u \cdot (pf \cdot f) = (u \cdot pf)f = (u \cdot pe)f = u(pe \cdot f) = u(p \cdot ef) = uz$. Therefore, uDz and $u \preceq z$; so z is the greatest lower bound of x and y.

(b) Put $z := p(e \vee f)$. Then $x \preceq z$ by Lemma 4.2(b,f), and likewise $y \preceq z$. For any u, if $x, y \preceq u$, then uDx, y and $p^2e = x^2 = ux = u \cdot pe$, $p^2f = y^2 = uy = u \cdot pf$. Also, $z^2 = (p(e \vee f))^2 = p^2(e \vee f) = p^2 \cdot (e + f - ef) = u \cdot pe + u \cdot pf - (u \cdot pe)f = u \cdot pe + u \cdot pf - u \cdot p(ef) = u \cdot p(e \vee f) = uz$. Therefore, uDz and $z \preceq u$; so z is the least upper bound of x and y.

(c) follows from (a) and (b) by Proposition 3.4.

(d) By Proposition 3.4(d), the suppositions imply that $(e \vee f)Dg$. By (b), (a), Proposition 3.4(e) and again (b), (a), further $(x \curlyvee y) \curlywedge z = p(e \vee f) \curlywedge pg = p((e \vee f) \wedge g) = p((e \wedge g) \vee (f \wedge g)) = (x \curlywedge z) \curlyvee (y \curlywedge z)$. □

To proceed, we need some relationships involving both relations $\preceq$ and $\perp$.

Lemma 4.4. *Suppose that S is reduced. Then the following holds:*

(a) *if $x \preceq y$ and $y \perp z$, then $x \perp z$,*
(b) *if $x \perp y$, then $0 = x \curlywedge y$ and $x + y = x \curlyvee y$,*
(c) *if $x \perp y$ and $y \preceq x + z$, then $y \preceq z$.*

Proof. (a) If $y = x \oplus u$ for some u, and $y \perp z$, then $x \perp z$ by Theorem 3.8(b).

(b) Suppose that xDy and $xy = 0$. If $z \preceq x, y$, then zDy and, by Lemma 4.2(f), $zy \preceq xy = 0$, i.e., $z^2 = 0$, and then $z = 0$ by (1). So, 0 is the single lower bound of x and y. Further, $x, y \preceq x \oplus y$ (see (5)); so, $x, yD(x+y)$ and $(x+y)D(x+y)$. If also $x, y \preceq z$ for some z, then $(x+y)^2 = x^2 + y^2 + 2xy = xz + yz + 0 = (x+y)z$. Thus, $x + y \preceq z$; so, $x + y$ is the least upper bound of x and y.

(c) If xDy, $yD(x+z)$, $xy = 0$ and $y^2 = y(x+z)$, then yDz and $y^2 = yx + yz = yz$. $\square$

This lemma together with (5) and Theorem 3.8(a,c) leads us to the following conclusion, which is another significant result of the section.

Theorem 4.5. *A reduced partial ring is a quasi-orthomodular poset relatively to the logical order and the ring orthogonality.*

So, we may apply to S the structure theorems of quasi-orthomodular posets stated in [3]. The next result rests on Theorems 2.1 and 3.8(f) above; it corresponds to Theorem 4.3 of [13] dealing with the partial ring $\mathcal{S}(H)$ for some Hilbert space H. Corollary 4.9 in [9] states that any interval $[O, A]$ with $A \in S(H)$ is even σ-orthomodular, but this was proved referring to the spectral theorem.

Proposition 4.6. *Let S be a reduced partial ring, let $\perp$ be its orthogonality relation, and let $\preceq$ be the logical order on S. Then $(S, \preceq)$ is a weak generalized orthomodular poset with $\perp$ as its induced orthogonality. In particular,*

(a) *every initial segment $[0, p]$ of S is an orthomodular poset with the orthocomplementation $^{\perp}_{p}$ given by $x^{\perp}_{p} = p - x$,*
(b) *if $x \preceq p \preceq q$, then $x^{\perp}_{p} = p \curlywedge x^{\perp}_{q}$,*
(c) *for all $x, y \in [0, p]$, $x \perp y$ if and only if $y \preceq p - x$.*

Proposition 3.5 may be regarded as a consequence of this result; however, it did not require S to be reduced.

Acknowledgements

This research is supported by Latvian Science Council Grant No. 271/2012. The author thanks the anonymous referee for careful reading the text and for several suggestions, which have improved the presentation.

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