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On (a, B, c)-ideals in Banach spaces

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Abstract. In this paper we focus on subspaces of Banach spaces that are (a, B, c)-ideals. We study (a, B, c)-ideals in ℓ_∞^2 and present easily verifiable conditions for a subspace of ℓ_∞^2 to be an (a, B, c)-ideal. Our main results concern the transitivity of (a, B, c)-ideals. We show that if X is an (a, B, c)-ideal in Y and Y is a (d, E, f)-ideal in Z, then X is a certain type of ideal in Z. Relying on this result, we show that if X is an (a, B, c)-ideal in its bidual, then X is a certain type of ideal in $X^{(2n)}$ for every $n \in \mathbb{N}$.

1. Introduction

Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$. Throughout this paper, $B \subset \mathbb{K}$ will be a compact set and $a, c \geqslant 0$.

A closed subspace Y of a Banach space X is said to be an *ideal* in X if there is a norm one projection P on X^* such that $\ker P = Y^\perp$, where $Y^\perp$ denotes the annihilator of Y. In this case the projection P is called an *ideal projection*. If, in addition,

$$\|ax^* + bPx^*\| + c\|Px^*\| \leqslant \|x^*\| \quad \forall b \in B$$

holds for all x^* in X^*, then Y is said to be an (a, B, c)-*ideal* in X.

This approach was first suggested by Eve Oja in [10] (see also [9]) and later formalized in [11]. It is meant to encompass all previously studied special cases of ideals: M-ideals (which are $(1, \{-1\}, 1)$-ideals; first introduced in [1]), u-ideals ($(1, \{-2\}, 0)$-ideals; [2]), h-ideals ($(1, \{-(1 + \lambda) : \lambda \in S_{\mathbb{C}}\}, 0)$-ideals; [5], see also [4]), $M(r, s)$-ideals ($(s, \{-s\}, r)$-ideals; [7], [12], introduced as *ideals satisfying the $M(r, s)$-inequality* in [3]).

For every $n \in \mathbb{N}$, we denote by $X^{(n)} = \left(X^{(n-1)}\right)^*$, where $X^{(0)} = X$. We denote the closed unit ball of a Banach space X by B_X.

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2. (a, B, c)-structure of ℓ_∞^2

In this section, we study the (a, B, c)-structure of ℓ_∞^2. We present necessary and sufficient conditions for a one-dimensional subspace of ℓ_∞^2 to be an (a, B, c)-ideal in ℓ_∞^2. Throughout this section, we shall denote $Y_k = \{(\xi, k\xi)\colon \xi \in \mathbb{R})\}$ and $Y_\infty = \{(0, \xi)\colon \xi \in \mathbb{R})\}$.

In order to obtain these results we first focus on ideal projections in ℓ_∞^2.

Proposition 2.1. *Let $k \in \mathbb{R}$. If Y_k is an ideal in ℓ_∞^2 with respect to some ideal projection P, then $P \in \{P_{x,k}, P_{y,k}\} \cup \{P_{d^+} : d \geqslant 0\} \cup \{P_{d^-} : d \leqslant 0\}$, where*

$$P_{x,k} : \ell_2^1 \ni (\alpha_1, \alpha_2) \mapsto (\alpha_1 + k\alpha_2, 0) \in \ell_2^1,$$

$$P_{y,k} : \ell_2^1 \ni (\alpha_1, \alpha_2) \mapsto \left(0, \frac{\alpha_1 + k\alpha_2}{k}\right) \in \ell_2^1,$$

$$P_{d^+} : \ell_2^1 \ni (\alpha_1, \alpha_2) \mapsto \left(\frac{\alpha_1 + \alpha_2}{d+1}, \frac{d(\alpha_1 + \alpha_2)}{d+1}\right) \in \ell_2^1, \quad d \geqslant 0,$$

$$P_{d^-} : \ell_2^1 \ni (\alpha_1, \alpha_2) \mapsto \left(\frac{\alpha_1 - \alpha_2}{1-d}, \frac{d(\alpha_1 - \alpha_2)}{1-d}\right) \in \ell_2^1, \quad d \leqslant 0.$$

Proof. Let Y_k be an ideal in ℓ_∞^2 and let P be the corresponding ideal projection, then $\ker P = Y_k^\perp = \{(-k\alpha, \alpha)\colon \alpha \in \mathbb{R}\}$. By the rank-nullity theorem, we can choose $(u, v) \in \ell_2^1$ such that $\operatorname{ran} P = \{\lambda(u, v)\colon \lambda \in \mathbb{R}\}$. For every $(\alpha_1, \alpha_2) \in \ell_1^2$, we can write

$$(\alpha_1, \alpha_2) = \frac{\alpha_2 u - \alpha_1 v}{u + kv}(-k, 1) + \frac{\alpha_1 + k\alpha_2}{u + kv}(u, v), \quad (\alpha_1, \alpha_2) \in \ell_2^1$$

and hence

$$P(\alpha_1, \alpha_2) = \frac{\alpha_1 + k\alpha_2}{u + kv}(u, v), \quad (\alpha_1, \alpha_2) \in \ell_2^1.$$

1. In case $v = 0$, we have

$$P(\alpha_1, \alpha_2) = (\alpha_1 + k\alpha_2, 0), \quad (\alpha_1, \alpha_2) \in \ell_2^1.$$

Since an ideal projection has norm one, we demand that

$$\sup_{|\alpha_1| + |\alpha_2| \leqslant 1} |\alpha_1 + k\alpha_2| = 1.$$

It is easy to see that this is true if and only if $|k| \leqslant 1$, which means that $P = P_{x,k}$ is an ideal projection if and only if $|k| \leqslant 1$.

2. Proceeding similarly in case $u = 0$, we obtain that $P_{y,k}$ is an ideal projection if and only if $|k| \geqslant 1$.

3. Assume $u \neq 0$, $v \neq 0$ and denote $d = \dfrac{v}{u}$. We can write

$$P(\alpha_1, \alpha_2) = \left(\frac{\alpha_1 + k\alpha_2}{1 + kd}, \frac{d(\alpha_1 + k\alpha_2)}{1 + kd}\right), \quad (\alpha_1, \alpha_2) \in \ell_2^1.$$

It is easy to see that in case $|k| \geqslant 1$ ($|k| \leqslant 1$) demanding $\|P\| = 1$ yields $|k|(1+|d|) = |1+kd|$ ($1+|d| = |1+kd|$), which, in turn, means that $|k| = 1$.

Now if $k = 1$ we have

$$1 = \|P\| = \sup_{|\alpha_1|+|\alpha_2|\leqslant 1} \left\| \left(\frac{d(\alpha_1+\alpha_2)}{d+1}, \frac{\alpha_1+\alpha_2}{d+1} \right) \right\| = \frac{|d|+1}{|d+1|}$$

which holds only for $d \geqslant 0$. Hence if $k = 1$, P is an ideal projection if and only if $d \geqslant 0$ and in this case $P = P_{d^+}$. The case $k = -1$ is analogous. □

Remark 2.2. As we saw in the proof of Proposition 2.1, the following assertions hold.

(1) $P_{x,k}$ is an ideal projection if and only if $|k| \geqslant 1$,
(2) $P_{y,k}$ is an ideal projection if and only if $|k| \leqslant 1$,
(3) P_{d^+} is an ideal projection for some $d \geqslant 0$ if and only if $k = 1$,
(4) P_{d^-} is an ideal projection for some $d \leqslant 0$ if and only if $k = -1$.

The following proposition can be proven in a manner similar to Proposition 2.1.

Proposition 2.3. *If Y_∞ is an ideal in ℓ_∞^2 with respect to an ideal projection P, then $P = P_\infty$, where*

$$P_\infty : \ell_2^1 \ni (\alpha_1, \alpha_2) \mapsto (0, \alpha_2) \in \ell_2^1.$$

Knowing the form of ideal projections in ℓ_∞^2, we can now derive necessary and sufficient conditions for a subspace of ℓ_∞^2 to be an (a,B,c)-ideal. For the sake of convenience, we shall handle each ideal projection separately.

Proposition 2.4. *Y_k is an (a,B,c)-ideal in ℓ_∞^2 with respect to ideal projection $P_{x,k}$ if and only if $|k| \leqslant 1$ and*

$$\begin{cases} a + |b||k| + c|k| \leqslant 1 & \forall b \in B, \\ |a+b| + c \leqslant 1 & \forall b \in B. \end{cases} \tag{1}$$

Proof. By Remark 2.2, $P_{x,k}$ is an ideal projection if and only if $|k| \leqslant 1$.

Let $|k| \leqslant 1$. Note that for every $(\alpha_1, \alpha_2) \in \ell_1^2$ and every $b \in B$ we have

$$\begin{aligned} &\|a(\alpha_1,\alpha_2) + bP_{x,k}(\alpha_1,\alpha_2)\| + c\|P_{x,k}(\alpha_1,\alpha_2)\| \\ &= \|(a\alpha_1, a\alpha_2) + (b\alpha_1 + bk\alpha_2, 0)\| + c\,\|(\alpha_1 + k\alpha_2, 0)\| \\ &= |a\alpha_1 + b\alpha_1 + bk\alpha_2| + a|\alpha_2| + c|\alpha_1 + k\alpha_2|. \end{aligned}$$

Necessity. Let Y_k be an (a,B,c)-ideal in ℓ_∞^2 with respect to the ideal projection $P_{x,k}$.

For every $(\alpha_1, \alpha_2) \in \ell_1^2$ and every $b \in B$ we have

$$|a\alpha_1 + b\alpha_1 + bk\alpha_2| + a|\alpha_2| + c|\alpha_1 + k\alpha_2| \leqslant |\alpha_1| + |\alpha_2|.$$

Choosing $(\alpha_1, \alpha_2) = (1, 0)$, and $(\alpha_1, \alpha_2) = (0, 1)$, we obtain that condition (1) holds.

Sufficiency. Note that in case conditions (1) hold, we have

$$\begin{aligned}
&\|a(\alpha_1,\alpha_2)+bP_{x,k}(\alpha_1,\alpha_2)\|+c\|P_{x,k}(\alpha_1,\alpha_2)\| \\
&\quad = |a\alpha_1+b\alpha_1+bk\alpha_2|+a|\alpha_2|+c|\alpha_1+k\alpha_2| \\
&\quad \leqslant (|a+b|+c)|\alpha_1|+(a+b|k|+c|k|)|\alpha_2| \\
&\quad \leqslant |\alpha_1|+|\alpha_2| \\
&\quad = \|(\alpha_1,\alpha_2)\|,
\end{aligned}$$

hence Y_k is an (a,B,c)-ideal in ℓ_∞^2. □

The following assertions can be proven similarly.

Proposition 2.5. *Y_k is an (a,B,c)-ideal in ℓ_∞^2 with respect to an ideal projection $P_{y,k}$ if and only if $|k|\geqslant 1$ and*

$$\begin{cases} |a+b|+c\leqslant 1 & \forall b\in B,\\ \dfrac{a|k|+|b|+c}{|k|}\leqslant 1 & \forall b\in B.\end{cases}$$

Proposition 2.6. *Y_k is an (a,B,c)-ideal in ℓ_∞^2 with respect to an ideal projection P_{d^+} if and only if*

$$\begin{cases} k=1,\\ d\geqslant 0,\\ \dfrac{|ad+a+bd|+|b|+cd+c}{d+1}\leqslant 1 & \forall b\in B,\\ \dfrac{|ad+a+b|+|b|d+cd+c}{d+1}\leqslant 1 & \forall b\in B.\end{cases}$$

Proposition 2.7. *Y_k is an (a,B,c)-ideal in ℓ_∞^2 with respect to an ideal projection P_{d^-} if and only if*

$$\begin{cases} k=-1,\\ d\leqslant 0,\\ \dfrac{|a-ad-bd|+|b|-cd+c}{1-d}\leqslant 1 & \forall b\in B,\\ \dfrac{|a-ad+b|-|b|d-cd+c}{1-d}\leqslant 1 & \forall b\in B.\end{cases}$$

Proposition 2.8. *Y_∞ is an (a,B,c)-ideal in ℓ_∞^2 if and only if*

$$\begin{cases} a\leqslant 1,\\ |a+b|+c\leqslant 1 \quad \forall b\in B.\end{cases}$$

From Propositions 2.4–2.8, one obtains the following corollaries.

Corollary 2.9. *Y_∞ and Y_0 are the only M-ideals in ℓ_∞^2.*

Corollary 2.10. *Y_0, Y_∞, Y_1, Y_{-1} are the only u-ideals in ℓ_∞^2.*

3. Transitivity of (a, B, c)-ideals

In this section, we rely on [6] and extend its results to a more general (a, B, c)-setting. We obtain the following results. If X is an (a, B, c)-ideal in Y and Y is a (d, E, f)-ideal in Z, then X is a certain type of ideal in Z (see Theorem 3.3). If X is an (a, B, c)-ideal in its bidual, then X is a certain type of ideal in $X^{(2n)}$ for every $n \in \mathbb{N}$ (see Theorem 3.13).

If Y is a subspace of a Banach space Z, a linear operator $\varphi\colon Y^* \to Z^*$ is called a *Hahn-Banach extension operator* if φy^* is a norm-preserving extension of y^* for all y^* in Y^*. The following propositions are well known and straightforward to prove.

Proposition 3.1. *If $\varphi\colon X^* \to Y^*$ and $\psi\colon Y^* \to Z^*$ are Hahn-Banach extension operators, then $\psi\varphi\colon X^* \to Z^*$ is also a Hahn-Banach extension operator.*

Proposition 3.2. *Y is an ideal in X with respect to an ideal projection P if and only if there is a Hahn-Banach extension operator $\varphi\colon Y^* \to X^*$ such that $P = \varphi i_{YX}^*$.*

Assume that X and Y are closed subspaces of a Banach space Z such that $X \subset Y \subset Z$. The first of our two main results is the following theorem.

Theorem 3.3. *Let X be an (a, B, c)-ideal in Y.*

(1) *If Y is an ideal in Z, then X is an $\left(\frac{a}{2a+1}, \frac{B}{2a+1}, \frac{c}{2a+1}\right)$-ideal in Z.*
(2) *Assume that $d > 0$, $f \geqslant 0$ and $a|d + e| + d \geqslant af$ for all $e \in E$, where E is a compact set of scalars. If Y is a (d, E, f)-ideal, then X is an $\left(\frac{ad}{\gamma}, \frac{dB}{\gamma}, \frac{cd}{\gamma}\right)$-ideal in Z, where $\gamma := a + d - af + a \min|d + E|$.*

Proof. Let P and Q be corresponding ideal projections on X^* and Y^* respectively. By Propositions 3.1 and 3.2, we have that $P = \varphi i_{XY}^*$ and $Q = \psi i_{YZ}^*$ for some Hahn-Banach extension operators $\varphi\colon X^* \to Y^*$ and $\psi\colon Y^* \to Z^*$, therefore $R = \psi\varphi i_{XZ}^*$ is an ideal projection with $\ker R = X^\perp$. Note that one can write $R = \psi P i_{YZ}^*$.

(1) For every $z^* \in Z^*$, we have

$$\begin{aligned}
\|az^* + bRz^*\| + c\|Rz^*\| &= \|az^* + b\psi P i_{YZ}^* z^* + a\psi i_{YZ}^* z^* - a\psi i_{YZ}^* z^*\| \\
&\quad + c\|\psi P i_{YZ}^* z^*\| \\
&\leqslant \|az^* - a\psi i_{YZ}^* z^*\| + \|\psi(bP i_{YZ}^* z^* + a i_{YZ}^* z^*)\| \\
&\quad + c\|\psi P i_{YZ}^* z^*\| \\
&= \|az^* - a\psi i_{YZ}^* z^*\| + \|a i_{YZ}^* z^* + bP i_{YZ}^* z^*\| \\
&\quad + c\|P i_{YZ}^* z^*\|
\end{aligned}$$

$$\leqslant a\|z^* - Qz^*\| + \|i_{YZ}^* z^*\|$$
$$\leqslant (2a+1)\|z^*\|,$$

since X is an (a,B,c)-ideal in Y. This result yields that X is an $\left(\frac{a}{2a+1}, \frac{B}{2a+1}, \frac{c}{2a+1}\right)$-ideal in Z.

(2) In case Y is a (d,E,f)-ideal in Z, we have

$$\|dz^* + eQz^*\| + f\|Qz^*\| \leqslant \|z^*\| \quad \forall e \in E, \forall z^* \in Z^*.$$

Since $d > 0$, we can write

$$\left\|z^* + \frac{e}{d}Qz^*\right\| \leqslant \frac{\|z^*\|}{d} - \frac{f}{d}\|Qz^*\| \quad \forall e \in E, \forall z^* \in Z^*.$$

We proceed similarly to part (1). For every $z^* \in Z^*$, we have

$$\begin{aligned}\|az^* + bRz^*\| + c\|Rz^*\| &\leqslant a\|z^* - Qz^*\| + \|i_{YZ}^* z^*\| \\ &= a\left\|z^* + \frac{e}{d}Qz^* - \left(1+\frac{e}{d}\right)Qz^*\right\| + \|i_{YZ}^* z^*\| \\ &\leqslant a\left\|z^* + \frac{e}{d}Qz^*\right\| + a\left\|\left(1+\frac{e}{d}\right)Qz^*\right\| + \|i_{YZ}^* z^*\| \\ &\leqslant \frac{a}{d}\|z^*\| - \frac{af}{d}\|Qz^*\| + a\left|1+\frac{e}{d}\right|\|Qz^*\| + \|i_{YZ}^* z^*\| \\ &\leqslant \left(1 + \frac{a}{d} - \frac{af}{d} + a\left|1+\frac{e}{d}\right|\right)\|z^*\|,\end{aligned}$$

hence X is an $\left(\frac{ad}{\gamma}, \frac{dB}{\gamma}, \frac{cd}{\gamma}\right)$-ideal in Z. □

From Theorem 3.3, one immediately obtains the following results.

Corollary 3.4 (cf. [6, Theorem 1])**.** *If X is an $M(r,s)$-ideal in Y, Y is an $M(u,v)$-ideal in Z and $v \geqslant su$, then X is an $M\left(\frac{rv}{s(1-u)+v}, \frac{sv}{s(1-u)+v}\right)$-ideal in Z.*

Corollary 3.5. *If X is an h-ideal in Y and Y is an h-ideal in Z, then X is an $\left(\frac{1}{3}, \left\{-\frac{1+\lambda}{3} : \lambda \in S_{\mathbb{C}}\right\}, 0\right)$-ideal in Z.*

Corollary 3.6. *If X is a u-ideal in Y and Y is u-ideal in Z, then X is an $\left(\frac{1}{3}, \left\{-\frac{2}{3}\right\}, 0\right)$-ideal in Z.*

Corollary 3.7. *If X is an M-ideal in Y and Y is a u-ideal in Z, then X is an $\left(\frac{1}{3}, \left\{-\frac{1}{3}\right\}, \frac{1}{3}\right)$-ideal in Z.*

Corollary 3.8. *If X is an (a,B,c)-ideal in Y, $a \leqslant 1$, and Y is an M-ideal in Z, then X is an (a,B,c)-ideal in Z.*

Corollary 3.9 (cf., e.g., [8, Proposition 1.17])**.** *If X is an M-ideal in Y and Y is an M-ideal in Z, then X is an M-ideal in Z.*

The following propositions are preliminary work for the proof of our second main result, Theorem 3.13. We generalize analogous results from [6], which concerned $M(r,s)$-ideals.

Proposition 3.10. *Let Y be a closed subspace of a Banach space X. If there is a norm one projection $Q\colon X \to X$ such that $\operatorname{ran} Q = Y$ and*

$$\|ax + bQx + cQz\| \leqslant \max\{\|x\|, \|z\|\} \quad \forall b \in B, \quad \forall x, z \in X,$$

then Y is an (a,B,c)-ideal in X.

Proof. Consider the ideal projection $P := Q^*$. Choose (x_n), $(z_n) \subset B_X$ so that

$$\operatorname{Re}((ax^* + bPx^*)(x_n)) \to \|ax^* + bPx^*\|,$$
$$\operatorname{Re}(cPx^*(z_n)) \to \|cPx^*\|$$

then by assumption $(ax_n + bQx_n + cQz_n) \subset B_X$. For every $x^* \in X^*$, we have

$$\begin{aligned}\|x^*\| &\geqslant |x^*(ax_n + bQx_n + cQz_n)| \\ &\geqslant \operatorname{Re}((ax^* + bPx^*)(x_n)) + \operatorname{Re}(cPx^*(z_n)) \\ &\to \|ax^* + bPx^*\| + c\|Px^*\|\end{aligned}$$

and hence

$$\|ax^* + bPx^*\| + c\|Px^*\| \leqslant \|x^*\| \quad \forall b \in B, \quad \forall x^* \in X^*,$$

which means that Y is an (a,B,c)-ideal in X. □

Proposition 3.11. *If Y is an (a,B,c)-ideal in X, then $Y^{\perp\perp}$ is an (a,B,c)-ideal in X^{**}.*

Proof. Let P be a corresponding ideal projection on X^*. Consider a norm one projection $P^*\colon X^{**} \to X^{**}$. For every y^{**}, $z^{**} \in X^{**}$, and $x^* \in X^*$, we have

$$\begin{aligned}\|(ay^{**} + bP^*y^{**} + cP^*z^{**})(x^*)\| &\leqslant \|(ay^{**} + bP^*y^{**})x^*\| + \|cP^*z^{**}(x^*)\| \\ &\leqslant \|y^{**}\|\|ax^* + bPx^*\| + \|z^{**}\|\|cP(x^*)\| \\ &\leqslant \max\{\|y^{**}\|, \|z^{**}\|\}\,(\|ax^* + bPx^*\| \\ &\quad + c\|P(x^*)\|) \\ &\leqslant \max\{\|y^{**}\|, \|z^{**}\|\}\|x^*\|,\end{aligned}$$

hence

$$\|ay^{**} + bP^*y^{**} + cP^*z^{**}\| \leqslant \max\{\|y^{**}\|, \|z^{**}\|\}.$$

Note that $\operatorname{ran} P^* = (\ker P)^\perp = (Y^\perp)^\perp = Y^{\perp\perp}$. By Proposition 3.10, $Y^{\perp\perp}$ is an (a,B,c)-ideal in X^{**}. □

29

Proposition 3.12. *If a closed subspace Y of a Banach space X is an (a,B,c)-ideal in X and T is a linear isometry from X onto a Banach space W, then $T(Y)$ is an (a,B,c)-ideal in W.*

Proof. Let P be a corresponding ideal projection on X^*, then $P = \varphi i^*_{YX}$ for some Hahn-Banach extension operator $\varphi\colon Y^* \to X^*$. Let $R = (T^{-1})^*\varphi S^* i^*_{T(Y)W}$, where $S\colon Y \ni y \mapsto Ty \in T(Y)$.

Note that $(T^{-1})^*\varphi S^*\colon T(Y)^* \to W^*$ is a Hahn-Banach extension operator, therefore R is an ideal projection on W^*. Since $i^*_{T(Y)W} = (S^{-1})^* i^*_{YX} T^*$, we can write $R = (T^{-1})^* P T^*$.

For every $w^* \in W^*$, we have

$$\begin{aligned}\|aw^* + bRw^*\| + c\|Rw^*\| &= \|aw^* + b(T^{-1})^*PT^*w^*\| \\ &\quad + c\|(T^{-1})^*PT^*w^*\| \\ &\leqslant \|(T^{-1})^*\|\|aT^*w^* + bPT^*w^*\| \\ &\quad + c\|(T^{-1})^*\|\|\varphi PT^*w^*\| \\ &= \|aT^*w^* + bPT^*w^*\| + c\|PT^*w^*\| \\ &\leqslant \|T^*w^*\| \leqslant \|T^*\|\|w^*\| = \|w^*\|,\end{aligned}$$

hence, $T(Y)$ is an (a,B,c)-ideal in W. □

The following is our second main result.

Theorem 3.13. *If X is an (a,B,c)-ideal in X^{**}, $a>0$, and $|a+b|+1 \geqslant c$ for all $b \in B$, then X is an $\left(\frac{a}{\gamma_n}, \frac{B}{\gamma_n}, \frac{c}{\gamma_n}\right)$-ideal is $X^{(2n)}$ for every $n \in \mathbb{N}$, where $\gamma_n = n + (n-1)\min|a+B| - (n-1)c$.*

Proof. We prove the assertion by induction on n. Assume that X (that is, $(j_{X^{(2n-2)}} \dots j_X)(X))$ is an $\left(\frac{a}{\gamma_n}, \frac{B}{\gamma_n}, \frac{c}{\gamma_n}\right)$-ideal in $X^{(2n)}$. Note that this holds for $n=1$.

Let $A = j_{X^{(2n-2)}} \dots j_X\colon X \to X^{(2n)}$. Consider a linear onto isometry $T\colon X^{**} \to \operatorname{ran} A^{**}$ defined by $Tx^{**} = A^{**}x^{**}$ for all $x^{**} \in X^{**}$.

Also note that

$$\operatorname{ran} A^{**} = (\ker A^*)^\perp = (\operatorname{ran} A)^{\perp\perp} = ((j_{X^{(2n-2)}} \dots j_X(X))^{\perp\perp}$$

and

$$\begin{aligned}T(j_X(X)) &= (j_{X^{(2n-2)}} \dots j_X)^{**}(j_X(X)) \\ &= j^{**}_{X^{(2n-2)}} \dots j^{**}_X j_X(X) \\ &= j_{X^{(2n)}} \dots j_{X^{**}} j_X(X).\end{aligned}$$

Since $j_X(X)$ is an (a, B, c)-ideal in X^{**} and T is a linear isometry from X^{**} onto $((j_{X^{(2n-2)}} \dots j_X(X))^{\perp\perp}$, we obtain by Proposition 3.12 that $T(j_X(X)) = j_{X^{(2n)}} \dots j_{X^{**}} j_X(X)$ is an (a, B, c)-ideal in $((j_{X^{(2n-2)}} \dots j_X(X))^{\perp\perp}$.

By the inductive assumption, $(j_{X^{(2n-2)}} \dots j_X)(X)$ is an $\left(\frac{a}{\gamma_n}, \frac{B}{\gamma_n}, \frac{c}{\gamma_n}\right)$-ideal in $X^{(2n)}$ and by Proposition 3.11 we get that $((j_{X^{(2n-2)}} \dots j_X)(X))^{\perp\perp}$ is an $\left(\frac{a}{\gamma_n}, \frac{B}{\gamma_n}, \frac{c}{\gamma_n}\right)$-ideal in $X^{(2n+2)}$.

We can now apply Theorem 3.3, which yields that $j_{X^{(2n)}} \dots j_{X^{**}} j_X(X)$ is an $\left(\frac{ad}{\gamma}, \frac{dB}{\gamma}, \frac{cd}{\gamma}\right)$-ideal in $X^{(2n+2)}$, where

$$\begin{aligned} a &= a, \quad B = B, \quad c = c, \\ d &= \frac{a}{n + (n-1)\min|a + B| - (n-1)c}, \\ E &= \frac{B}{n + (n-1)\min|a + B| - (n-1)c}, \\ f &= \frac{c}{n - (n-1)c + (n-1)\min|a + B|}. \end{aligned}$$

Hence

$$\begin{aligned} \gamma &= a + d - af + a\min|d + E| \\ &= a + \frac{a}{n + (n-1)\min|a + B| - (n-1)c} \\ &\quad - a \cdot \frac{c}{n + (n-1)\min|a + B| - (n-1)c} \\ &\quad + a\min\left|\frac{a}{n + (n-1)\min|a + B| - (n-1)c}\right. \\ &\quad \left. + \frac{B}{n + (n-1)\min|a + B| - (n-1)c}\right| \\ &= a \cdot \frac{(n+1) + n\min|a + B| - nc}{n + (n-1)\min|a + B| - (n-1)c} \end{aligned}$$

and

$$\frac{ad}{\gamma} = \frac{a}{\gamma_{n+1}}, \quad \frac{dB}{\gamma} = \frac{B}{\gamma_{n+1}}, \quad \frac{cd}{\gamma} = \frac{c}{\gamma_{n+1}},$$

which means that $j_{X^{(2n)}} \dots j_{X^{**}} j_X(X)$ is an $\left(\frac{a}{\gamma_{n+1}}, \frac{B}{\gamma_{n+1}}, \frac{c}{\gamma_{n+1}}\right)$-ideal in $X^{(2n+2)}$, as desired. $\square$

The following results are immediate from Theorem 3.13.

Corollary 3.14 (cf. [6, Theorem 6]). *If X is an $M(r,s)$-ideal in X^{**}, then X is an $M\left(\dfrac{r}{r+n(1-r)}, \dfrac{s}{r+n(1-r)}\right)$-ideal in $X^{(2n)}$.*

Corollary 3.15. *If X is an h-ideal in X^{**}, then X is an $\left(\dfrac{1}{2n-1}, \left\{-\dfrac{1+\lambda}{2n-1} \colon \lambda \in S_{\mathbb{C}}\right\}, 0\right)$-ideal in $X^{(2n)}$.*

Corollary 3.16. *If X is a u-ideal in X^{**}, then X is an $\left(\dfrac{1}{2n-1}, \left\{-\dfrac{2}{2n-1}\right\}, 0\right)$-ideal in $X^{(2n)}$.*

Corollary 3.17 (cf. [13, Theorem 2]). *If X is an M-ideal in X^{**}, then X is an M-ideal in $X^{(2n)}$ for every $n \in \mathbb{N}$.*

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