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A note on spark varieties

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Abstract. We study basic properties (e.g., algebraicity, reducibility, and dimension) of certain sets of matrices defined by means of the spark.

0. Introduction and preliminaries

Throughout the text $\mathbb{F}$ stands for a field, and $\mathbb{N}$ for the set of positive integers. The present paper is about certain algebraic subsets of $\mathcal{M}_{m\times n}(\mathbb{F})$, the vector space of all $m \times n$ matrices over $\mathbb{F}$. We start by recalling a few definitions and properties.

Consider a nonzero finite dimensional vector space V over $\mathbb{F}$ and a linear isomorphism $\varphi : V \longrightarrow \mathbb{F}^d$, where $d = \dim_{\mathbb{F}} V$. A set $E \subseteq V$ is said to be algebraic, if

$$\exists\, s \in \mathbb{N}\, \exists\, f_1, \ldots, f_s \in \mathbb{F}[x_1, \ldots, x_d]\colon$$
$$E = \{v \in V\colon\ f_1(\varphi(v)) = \ldots = f_s(\varphi(v)) = 0\}.$$

A set $Q \subseteq V$ is said to be quasi-algebraic, if $Q = E_1 \setminus E_2$ for some algebraic sets $E_1, E_2 \subseteq V$ (i.e., if it is locally closed in the Zariski topology on V). The linear capacity of an algebraic set $E \subseteq V$ is defined by

$$\Lambda(E) = \sup\{\dim_{\mathbb{F}} L\colon\ L \text{ is a linear subspace of } V,\ L \subseteq E\}.$$

The linear capacity was introduced in [5]. The following properties are quite obvious (cf. [5, Proposition 1.1]).

Proposition 0.1. *Let V and W be nonzero finite dimensional vector spaces over $\mathbb{F}$, let $E_1 \subseteq V$ and $E_2, G \subseteq W$ be algebraic sets, and let $\psi\colon V \longrightarrow W$ be a linear map such that $\psi(E_1) \subseteq G$. Then*

(i) $\Lambda(E_1) = -\infty$ *if and only if the zero vector does not belong to* E_1*,*

(ii) $\Lambda(E_1 \times E_2) = \Lambda(E_1) + \Lambda(E_2)$*,*

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(iii) $\Lambda(E_1) \leq \Lambda(G)$ *whenever* $E_1 \neq \emptyset$ *and the restriction* $\psi|_{E_1}$ *is injective,*
(iv) $\Lambda(E_1) = \Lambda(G)$ *whenever* $E_1 \neq \emptyset$, $\psi|_{E_1} \colon E_1 \longrightarrow G$ *is bijective, and* $(\psi|_{E_1})^{-1}$ *is the restriction of a linear map* $\xi \colon W \longrightarrow V$.

Finally, let us assume that the field $\mathbb{F}$ is algebraically closed. An algebraic set $E \subseteq V$ is said to be normal, if it is irreducible and the coordinate ring $\mathbb{F}[E]$ is integrally closed in the function field $\mathbb{F}(E)$. We refer to [3] for more information on algebraic geometry.

Let $m, n \in \mathbb{N}$ and r be a non-negative integer such that $r \leq \min\{m, n\}$. The generic determinantal variety $\mathcal{H}^r_{m\times n}$ defined by

$$\mathcal{H}^r_{m\times n} = \{A \in \mathcal{M}_{m\times n}(\mathbb{F}) \colon \operatorname{rank}(A) \leq r\}$$

is an important classical example of an algebraic subset of $\mathcal{M}_{m\times n}(\mathbb{F})$. The Flanders–Meshulam theorem [4] says that $\Lambda(\mathcal{H}^r_{m\times n}) = r \max\{m, n\}$. Moreover, it is well known [1] that if the field $\mathbb{F}$ is algebraically closed, then $\mathcal{H}^r_{m\times n}$ is normal and $\dim \mathcal{H}^r_{m\times n} = r(m + n - r)$.

In [2], Donoho and Elad introduced the notion of spark of a matrix.

Definition 0.2. Let $C_1, \ldots, C_n \in \mathbb{F}^m$ be the columns of a matrix $A \in \mathcal{M}_{m\times n}(\mathbb{F})$. The *spark* of A is defined to be the infimum of the set of all positive integers ℓ such that

$$\exists\, j_1, \ldots, j_\ell \in \{1, \ldots, n\} : \begin{cases} j_1 < \ldots < j_\ell, \\ C_{j_1}, \ldots, C_{j_\ell} \text{ are linearly dependent.} \end{cases}$$

The definition of the spark is similar to the definition of the rank of a matrix. However, algebraic and computational properties of the spark are very different from those of the rank.

In the present note we will look at the notion of spark of a matrix from a geometric point of view. Namely, for a positive integer k we define the spark variety $\mathcal{S}^k_{m\times n}$ by

$$\mathcal{S}^k_{m\times n} = \{A \in \mathcal{M}_{m\times n}(\mathbb{F}) \colon \operatorname{spark}(A) \leq k\},$$

and our goal is to give a geometric characterization of the sets $\mathcal{S}^k_{m\times n}$ (analogous to the characterization of the generic determinantal varieties).

1. Basic properties

First, let us collect some remarks about the family of all spark varieties in $\mathcal{M}_{m\times n}(\mathbb{F})$.

Proposition 1.1. (i) *The zero matrix belongs to* $\mathcal{S}^k_{m\times n}$ *and* $\lambda\mathcal{S}^k_{m\times n} \subseteq \mathcal{S}^k_{m\times n}$ *for all* $\lambda \in \mathbb{F}$.

(ii) $\mathcal{S}^k_{m\times n} \subseteq \mathcal{S}^{k+1}_{m\times n}$.

(iii) *If* $m < n$, *then* $\mathcal{S}^{m+1}_{m\times n} = \mathcal{M}_{m\times n}(\mathbb{F})$.

(iv) *If* $r \in \mathbb{N} \cup \{0\}$ *is such that* $r \leq m$ *and* $r < n$, *then* $\mathcal{H}^r_{m\times n} \subseteq \mathcal{S}^{r+1}_{m\times n}$.
(v) *If* $\min\{k, m\} \geq n$, *then* $\mathcal{S}^k_{m\times n} = \mathcal{H}^{n-1}_{m\times n}$.

Proof. Observe that the spark of the zero matrix is equal to 1. Moreover,

$$\forall\, A \in \mathcal{M}_{m\times n}(\mathbb{F})\, \forall\, \lambda \in \mathbb{F} \setminus \{0\}\colon\ \operatorname{spark}(\lambda A) = \operatorname{spark}(A).$$

Property (i) follows. Inclusion (ii) is obvious.

It is easy to see that

$$\forall\, A \in \mathcal{M}_{m\times n}(\mathbb{F}) : \begin{cases} \operatorname{spark}(A) \neq +\infty \Rightarrow \operatorname{spark}(A) \leq \operatorname{rank}(A) + 1, \\ \operatorname{spark}(A) = +\infty \Leftrightarrow \operatorname{rank}(A) = n. \end{cases}$$

Consequently, if $m < n$, then $\operatorname{spark}(A) \leq m + 1$ for all $A \in \mathcal{M}_{m\times n}(\mathbb{F})$. Property (iii) follows.

If $r \in \mathbb{N} \cup \{0\}$, $r \leq m$, $r < n$, and $A \in \mathcal{H}^r_{m\times n}$, then $\operatorname{spark}(A) \leq r+1$. This yields property (iv).

Suppose finally that $\min\{k, m\} \geq n$. Then, by (iv) and (ii), we have $\mathcal{H}^{n-1}_{m\times n} \subseteq \mathcal{S}^n_{m\times n} \subseteq \mathcal{S}^k_{m\times n}$. On the other hand,

$$\forall\, A \in \mathcal{M}_{m\times n}(\mathbb{F})\colon\ \operatorname{spark}(A) \neq +\infty \Leftrightarrow \operatorname{rank}(A) \leq n - 1,$$

and hence $\mathcal{S}^k_{m\times n} \subseteq \mathcal{H}^{n-1}_{m\times n}$. Property (v) follows. □

Let $m, n, \ell \in \mathbb{N}$ be such that $\ell \leq \min\{m, n\}$. For $A = [a_{ij}] \in \mathcal{M}_{m\times n}(\mathbb{F})$, a strictly increasing sequence $(i_1, \ldots, i_\ell)$ of elements of $\{1, \ldots, m\}$ and a strictly increasing sequence $(j_1, \ldots, j_\ell)$ of elements of $\{1, \ldots, n\}$ we define $\mu^{i_1,\ldots,i_\ell}_{j_1,\ldots,j_\ell}(A)$ to be the determinant of the matrix $[a_{i_u j_v}] \in \mathcal{M}_{\ell\times\ell}(\mathbb{F})$.

Theorem 1.2. *Every spark variety* $\mathcal{S}^k_{m\times n}$ *is an algebraic subset of the space* $\mathcal{M}_{m\times n}(\mathbb{F})$.

Proof. If $\ell := \min\{k, n\} > m$, then by Proposition 1.1 we have $\mathcal{S}^k_{m\times n} = \mathcal{M}_{m\times n}(\mathbb{F})$. Suppose therefore that $\ell \leq m$. Let $(j_1, \ldots, j_\ell)$ be a strictly increasing sequence of elements of $\{1, \ldots, n\}$. We define

$$\mathcal{D}_{j_1,\ldots,j_\ell} = \Big\{A \in \mathcal{M}_{m\times n}(\mathbb{F})\colon\ \mu^{i_1,\ldots,i_\ell}_{j_1,\ldots,j_\ell}(A) = 0 \text{ for all } i_1, \ldots, i_\ell \in \{1, \ldots, m\} \text{ such that } i_1 < \ldots < i_\ell\Big\}.$$

Notice that $\mathcal{D}_{j_1,\ldots,j_\ell}$ is an algebraic subset of $\mathcal{M}_{m\times n}(\mathbb{F})$. Moreover, $\mathcal{D}_{j_1,\ldots,j_\ell}$ is equal to the totality of matrices in $\mathcal{M}_{m\times n}(\mathbb{F})$ whose columns with indices $j_1, \ldots, j_\ell$ are linearly dependent. If $s \in \{1, \ldots, \ell\}$, then each s-element set of linearly dependent columns of a matrix $A \in \mathcal{M}_{m\times n}(\mathbb{F})$ is contained in an ℓ-element set of linearly dependent columns of A. Thus,

$$\mathcal{S}^k_{m\times n} = \mathcal{S}^\ell_{m\times n} = \bigcup \{\mathcal{D}_{j_1,\ldots,j_\ell}\colon\ j_1, \ldots, j_\ell \in \{1, \ldots, n\},\ j_1 < \ldots < j_\ell\}.$$

The algebraicity follows. □

The sets $\mathcal{D}_{j_1,\ldots,j_\ell}$ are examples of so-called linear determinantal varieties.

Corollary 1.3. (i) *For any* $k \in \mathbb{N} \cup \{+\infty\}$, *the set* $\big\{A \in \mathcal{M}_{m\times n}(\mathbb{F})\colon \operatorname{spark}(A) = k\big\}$ *is quasi-algebraic.*

(ii) *If* $m < n$, *then* $\{A \in \mathcal{M}_{m\times n}(\mathbb{F})\colon \operatorname{spark}(A) = m+1\}$ *is open in the Zariski topology on* $\mathcal{M}_{m\times n}(\mathbb{F})$.

Recall that if $m < n$, then $\operatorname{spark}(A) \le m+1$ for all $A \in \mathcal{M}_{m\times n}(\mathbb{F})$. Moreover, if $m \ge n$, then $\max_{A\in\mathcal{M}_{m\times n}(\mathbb{F})} \operatorname{spark}(A) = +\infty$ and the set

$$\{A \in \mathcal{M}_{m\times n}(\mathbb{F})\colon \operatorname{spark}(A) = +\infty\} = \mathcal{M}_{m\times n}(\mathbb{F}) \setminus \mathcal{H}^{n-1}_{m\times n}$$

is open in the Zariski topology on $\mathcal{M}_{m\times n}(\mathbb{F})$.

2. Main results

We are in a position to describe the geometric structure of a spark variety.

Theorem 2.1. *Suppose that the field* $\mathbb{F}$ *is algebraically closed. Let* $m, n, k \in \mathbb{N}$ *be such that* $k \le m$ *and* $k < n$. *Then the family of all irreducible components of* $\mathcal{S}^k_{m\times n}$ *coincides with*

$$\big\{\mathcal{D}_{j_1,\ldots,j_k}\colon\ j_1,\ldots,j_k \in \{1,\ldots,n\},\, j_1 < \ldots < j_k\big\}.$$

Moreover,

- *the above sets* $\mathcal{D}_{j_1,\ldots,j_k}$ *are normal and have dimension* $m(n-1)+k-1$,
- $\bigcap\big\{\mathcal{D}_{j_1,\ldots,j_k}\colon\ j_1,\ldots,j_k \in \{1,\ldots,n\},\, j_1 < \ldots < j_k\big\} = \mathcal{H}^{k-1}_{m\times n}$.

Proof. Recall from the proof of Theorem 1.2 that

$$\mathcal{S}^k_{m\times n} = \bigcup\big\{\mathcal{D}_{j_1,\ldots,j_k}\colon\ j_1,\ldots,j_k \in \{1,\ldots,n\},\, j_1 < \ldots < j_k\big\}$$

and the sets $\mathcal{D}_{j_1,\ldots,j_k}$ are algebraic.

Let $(j'_1,\ldots,j'_k)$ and $(j''_1,\ldots,j''_k)$ be two distinct strictly increasing sequences of elements of $\{1,\ldots,n\}$. Since $k \le m$, there exists a matrix in $\mathcal{M}_{m\times n}(\mathbb{F})$ such that its columns with indices $j''_1,\ldots,j''_k$ are linearly independent while its columns with indices belonging to $\{j'_1,\ldots,j'_k\} \setminus \{j''_1,\ldots,j''_k\}$ are not. Thus, $\mathcal{D}_{j'_1,\ldots,j'_k}$ is not contained in $\mathcal{D}_{j''_1,\ldots,j''_k}$.

Pick a strictly increasing sequence $(j_1,\ldots,j_k)$ of elements of $\{1,\ldots,n\}$. Define $A' \in \mathcal{M}_{m\times k}(\mathbb{F})$ to be the matrix that consists of the columns of a matrix $A \in \mathcal{M}_{m\times n}(\mathbb{F})$ with indices $j_1,\ldots,j_k$, and $A'' \in \mathcal{M}_{m\times(n-k)}(\mathbb{F})$ to be the matrix that consists of all other columns of A. The map

$$\mathcal{D}_{j_1,\ldots,j_k} \ni A \longmapsto (A', A'') \in \mathcal{H}^{k-1}_{m\times k} \times \mathcal{M}_{m\times(n-k)}(\mathbb{F})$$

is an isomorphism of algebraic sets. Therefore, since $\mathcal{H}^{k-1}_{m\times k}$ is normal, so is $\mathcal{D}_{j_1,\ldots,j_k}$. (In particular, $\mathcal{D}_{j_1,\ldots,j_k}$ is irreducible.) Moreover, since $\dim \mathcal{H}^{k-1}_{m\times k} =$

$(k-1)(m+k-k+1)$, we have

$$\begin{aligned}\dim \mathcal{D}_{j_1,\dots,j_k} &= \dim \mathcal{H}^{k-1}_{m\times k} + \dim \mathcal{M}_{m\times(n-k)}(\mathbb{F})\\ &= (k-1)(m+1) + m(n-k)\\ &= m(n-1) + k - 1.\end{aligned}$$

Finally, a matrix A belongs to all components $\mathcal{D}_{j_1,\dots,j_k}$ if and only if every k-element set of its columns is linearly dependent, which means exactly that $\operatorname{rank}(A) \le k-1$. ☐

Corollary 2.2. *Let $\mathbb{F}$ be algebraically closed. Then every spark variety $\mathcal{S}^k_{m\times n}$ is pure dimensional and all its irreducible components are normal. Moreover, $\mathcal{S}^k_{m\times n}$ is irreducible if and only if $k > m$ or $k \ge n$.*

Proof. If $\min\{k,n\} > m$, then by Proposition 1.1 we have that $\mathcal{S}^k_{m\times n} = \mathcal{M}_{m\times n}(\mathbb{F})$. Similarly, if $\min\{k,m\} \ge n$, then $\mathcal{S}^k_{m\times n} = \mathcal{H}^{n-1}_{m\times n}$. Thus, $\mathcal{S}^k_{m\times n}$ is normal whenever $k > m$ or $k \ge n$. On the other hand, by Theorem 2.1, if $k \le m$ and $k < n$, then $\mathcal{S}^k_{m\times n}$ is reducible and pure dimensional, and all its components are normal. ☐

We will conclude the note by the formula for the linear capacity of spark varieties.

Lemma 2.3. *Suppose that the field $\mathbb{F}$ is infinite. Let V be a nonzero finite dimensional vector space over $\mathbb{F}$, let $s \in \mathbb{N}$, and let $E_1,\dots,E_s \subseteq V$ be algebraic sets. Then $\Lambda(E_1 \cup \ldots \cup E_s) = \max\{\Lambda(E_1),\dots,\Lambda(E_s)\}$.*

Proof. We define $\lambda = \Lambda(E_1 \cup \ldots \cup E_s)$. Then, obviously, $\max\{\Lambda(E_1),\dots,\Lambda(E_s)\} \le \lambda$. Since $\mathbb{F}$ is infinite, every linear subspace of V is irreducible. Therefore, if L is a linear subspace of V such that $L \subseteq E_1 \cup \ldots \cup E_s$ and $\dim_{\mathbb{F}} L = \lambda$, then $L \subseteq E_{i_0}$ for some $i_0 \in \{1,\dots,s\}$, and hence $\lambda \le \Lambda(E_{i_0}) \le \max\{\Lambda(E_1),\dots,\Lambda(E_s)\}$. ☐

Theorem 2.4. *Suppose that $\mathbb{F}$ is infinite. Let $m,n,k \in \mathbb{N}$ be such that $\min\{k,n\} \le m$. Then*

$$\Lambda(\mathcal{S}^k_{m\times n}) = m(n-1).$$

(Recall that $\mathcal{S}^k_{m\times n} = \mathcal{M}_{m\times n}(\mathbb{F})$ whenever $\min\{k,n\} > m$.)

Proof. If $\min\{k,m\} \ge n$, then $\mathcal{S}^k_{m\times n} = \mathcal{H}^{n-1}_{m\times n}$, and hence the assertion follows from the Flanders–Meshulam theorem. Let us therefore assume that $k \le m$ and $k < n$. The isomorphism considered in the proof of Theorem 2.1 satisfies the assumptions of Proposition 0.1, (iv). Thus, for an arbitrary strictly increasing sequence $(j_1,\dots,j_k)$ of elements of $\{1,\dots,n\}$, we have $\Lambda(\mathcal{D}_{j_1,\dots,j_k}) = \Lambda\big(\mathcal{H}^{k-1}_{m\times k} \times \mathcal{M}_{m\times(n-k)}(\mathbb{F})\big)$. By Proposition 0.1, (ii), and the

Flanders–Meshulam theorem,

$$\begin{aligned}\Lambda\big(\mathcal{H}^{k-1}_{m\times k}\times\mathcal{M}_{m\times(n-k)}(\mathbb{F})\big)&=\Lambda(\mathcal{H}^{k-1}_{m\times k})+\Lambda\big(\mathcal{M}_{m\times(n-k)}(\mathbb{F})\big)\\&=(k-1)\max\{k,m\}+m(n-k)\\&=m(n-1).\end{aligned}$$

Since $\mathcal{S}^k_{m\times n}$ coincides with the union of all sets $\mathcal{D}_{j_1,\dots,j_k}$, the assertion follows now from Lemma 2.3. □

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