ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 18, Number 1, June 2014
Available online at http://acutm.math.ut.ee

Geometry of Banach spaces with an octahedral norm

Rainis Haller and Johann Langemets

Abstract. We discuss the geometry of Banach spaces whose norm is octahedral or, more generally, locally or weakly octahedral. Our main results characterize these spaces in terms of covering of the unit ball.

1. Introduction

Let X be a Banach space over the field of real numbers $\mathbb{R}$. A norm on X is called *octahedral* (see, e.g., [9, 8, 4, 11]) if, for every finite-dimensional subspace E of X and $\varepsilon > 0$, there exists an element y of X with $\|y\| = 1$ and

$$\|x - y\| \geq (1 - \varepsilon)(\|x\| + \|y\|) \quad \text{for all } x \in E.$$

Note that only infinite-dimensional Banach spaces may possess an octahedral norm.

In addition to the octahedral norms, two wider classes called locally octahedral norms and weakly octahedral norms (see definitions below in Section 3) were introduced and studied in a very recent preprint [11] (cf. [4]), whereas these three octahedrality conditions came up in relation to the diameter two properties [1].

In this paper, we shall study the geometry of those Banach spaces whose norm is octahedral or, more generally, locally or weakly octahedral. Our main results characterize these spaces in terms of covering of the unit ball.

The starting point of our investigation is the following observation made by G. Godefroy in [9]:

The norm on X is octahedral if and only if B_X, the closed unit ball of X, has the property that in every finite covering of B_X with closed balls at least one member of the covering itself contains B_X.

Received January 7. 2014.
2010 *Mathematics Subject Classification.* 46B20, 46B22.
Key words and phrases. Octahedral norm, thickness.
http://dx.doi.org/10.12097/ACUTM.2014.18.13

We shall recall the result in Proposition 2.3 below and provide its proof for completeness. In the special case when X is separable, the preceding equivalence is contained in [10, Lemma 9.1].

We consider only infinite-dimensional real Banach spaces. Let us fix some notation. By $B(x,r)$ we denote the closed ball in X with center at $x \in X$ and radius $r > 0$. The closed unit ball of a Banach space X is denoted by B_X and its unit sphere by S_X. The dual space of X is denoted by X^*.

2. Octahedral spaces

We recall a simple criterion for the octahedral norm.

Lemma 2.1 (see [11, Proposition 2.4]). *Let X be a Banach space. The following assertions are equivalent:*

(i) *the norm on X is octahedral;*
(ii) *for every $n \in \mathbb{N}$, $x_1, \dots, x_n \in S_X$, and $\varepsilon > 0$ there is a $y \in S_X$ such that*
$$\|x_i - y\| \geq 2 - \varepsilon \quad \text{for all } i \in \{1, \dots, n\}.$$

The following characterization of octahedral norms is given by P. L. Papini in [13]. We shall give its proof for completeness. Our proof is not essentially different from the one in [13], but the benefit of Lemma 2.1 is clear.

Proposition 2.2 (see [13, Theorem 2.1]). *Let X be a Banach space. The following assertions are equivalent:*

(i) *the norm on X is octahedral;*
(ii) *$\mu_2(X) = 2$, where*
$$\mu_2(X) = \inf_{\substack{x_1,\dots,x_n \in S_X \\ n \in \mathbb{N}}} \sup_{x \in S_X} \frac{1}{n} \sum_{i=1}^{n} \|x_i - x\|.$$

Proof. (i) $\Rightarrow$ (ii). Assume that the norm on X is octahedral. By Lemma 2.1 (ii), we have $\sup_{x \in S_X} \sum_{i=1}^{n} \|x_i - x\| = 2n$ for every $x_1, \dots, x_n$ in S_X. Thus $\mu_2(X) = 2$.

(ii) $\Rightarrow$ (i). Assume that $\mu_2(X) = 2$. Let $x_1, \dots, x_n \in S_X$ and let $\varepsilon > 0$. Since $\mu_2(X) = 2$, we have
$$\sup_{x \in S_X} \frac{1}{n} \sum_{i=1}^{n} \|x_i - x\| = 2.$$
It follows that there is an element x in S_X with
$$\frac{1}{n} \sum_{i=1}^{n} \|x_i - x\| \geq 2 - \frac{\varepsilon}{n}.$$
This yields
$$\|x_i - x\| \geq 2 - \varepsilon \quad \text{for all } i \in \{1, \dots, n\}.$$

By Lemma 2.1 (ii), the norm on X is octahedral. $\square$

For a Banach space X, R. Whitley [15] introduced the *thickness* parameter $T(X)$, defined by

$$T(X) = \inf\{\varepsilon > 0\colon \text{there exists a finite } \varepsilon\text{-net for } S_X \text{ in } S_X\}.$$

For finite-dimensional Banach spaces X, one has $T(X) = 0$. If X is an infinite-dimensional Banach space, then Whitley showed that $1 \leq T(X) \leq 2$.

Remark. In an infinite-dimensional Banach space X a finite covering of S_X with closed balls covers the entire unit ball B_X (for the proof see, e.g., [2], [14, Proposition 3ter], [6, Proposition 2]). Therefore, for an infinite-dimensional X,

$$T(X) = \inf\{\varepsilon > 0\colon \exists\{x_1, \dots, x_n\} \subset S_X \text{ such that } B_X \subset \bigcup_{i=1}^{n} B(x_i, \varepsilon)\}.$$

We shall soon see that the octahedrality of the norm on X is also characterized by the condition $T(X) = 2$. In fact, this is a direct consequence of the following observation.

Proposition 2.3 (cf. [9, p. 12]). *Let X be a Banach space. The following assertions are equivalent:*

(i) *the norm on X is octahedral;*

(ii) *if $x_1, \dots, x_n \in X$ and $r_1, \dots, r_n > 0$ are such that*

$$S_X \subset \bigcup_{i=1}^{n} B(x_i, r_i)$$

then $S_X \subset B(x_i, r_i)$ for some i in $\{1, \dots, n\}$;

(iii) *if $x_1, \dots, x_n \in S_X$ and $r_1, \dots, r_n > 0$ are such that*

$$S_X \subset \bigcup_{i=1}^{n} B(x_i, r_i)$$

then $S_X \subset B(x_i, r_i)$ for some i in $\{1, \dots, n\}$.

Proof. (i) $\Rightarrow$ (ii). Assume that the norm on X is octahedral, and consider a finite number of closed balls $B(x_1, r_1), \dots, B(x_n, r_n)$ in X with

$$S_X \subset \bigcup_{i=1}^{n} B(x_i, r_i).$$

Since the norm on X is octahedral, for every $\varepsilon > 0$ there are i in $\{1, \dots, n\}$ and norm one y in $B(x_i, r_i)$ with

$$\|x_i - y\| \geq (1 - \varepsilon)(\|x_i\| + 1),$$

which yields $r_i \geq (1 - \varepsilon)(\|x_i\| + 1)$. Consequently, $r_i \geq \|x_i\| + 1$ for at least one i in $\{1, \dots, n\}$.

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i). Assume that (iii) holds. Let $n \in \mathbb{N}$, $x_1, \dots, x_n \in S_X$, and $\varepsilon > 0$. By Lemma 2.1 it suffices to find a $y \in S_X$ such that

$$\|x_i - y\| \geq 2 - \varepsilon \quad \text{for all } i \in \{1, \dots, n\}.$$

Suppose that, on the contrary, for every $y \in S_X$ there is an x_i such that $\|x_i - y\| < 2 - \varepsilon$. Then

$$S_X \subset \bigcup_{i=1}^{n} B(x_i, 2 - \varepsilon).$$

By our assumption, $S_X \subset B(x_i, 2 - \varepsilon)$ for some i in $\{1, \dots, n\}$, which is a contradiction.

□

Condition (iii) of Proposition 2.3 is clearly equivalent to $T(X) = 2$.

Corollary 2.4. *Let X be a Banach space. The norm on X is octahedral if and only if $T(X) = 2$.*

Remark. In general $T(X) \geq T(X^{**})$. Since $C[0,1]$ is octahedral and $C[0,1]^{**}$ fails to be octahedral (see [9, p. 12]), we have $T(X) > T(X^{**})$ for $X = C[0,1]$ (cf. last paragraph in [7]).

Banach spaces that admit octahedral norms are exactly the spaces containing isomorphic copies of ℓ_1.

Theorem 2.5 (see [8, Theorem 2.5, p. 106])**.** *Let X be a Banach space. The following assertions are equivalent:*

(a) *X contains a subspace isomorphic to ℓ_1;*

(b) *there is an equivalent octahedral norm on X.*

Combining Theorem 2.5 and Corollary 2.4 leads to the following result.

Theorem 2.6 (cf. [12, Theorem 1.2])**.** *A Banach space X can be equivalently renormed to have thickness $T(X) = 2$ if and only if X contains an isomorphic copy of ℓ_1.*

Theorem 2.6 strengthens Theorem 1.2 of [12], which asserts that a separable Banach space X can be equivalently renormed to have thickness $T(X) = 2$ if and only if X contains an isomorphic copy of ℓ_1. Moreover, it justifies Corollary 6 in [6]. It also improves Proposition 3.1 in [3], which asserts that a Banach space X with $\mu_2(X) = T(X) = 2$ contains an isomorphic copy of ℓ_1.

The behavior of Whitley's thickness with respect to ℓ_p-products has been recently studied in [5] and [7]. If X and Y are Banach spaces and $1 < p < \infty$, then $T(X \oplus_p Y) \leq \max\{T(X), T(Y)\}$ (see [7, Theorem 2]), and for finite-dimensional X one has $T(X \oplus_p Y) \leq 2^{1/p}$ (see [5, Lemma 2.3]). We have the following estimation for $X \oplus_p Y$.

Proposition 2.7. *If X and Y are Banach spaces and $1 < p < \infty$, then*

$$T(X \oplus_p Y) \leq \sqrt[p]{\frac{(\sqrt[p]{2}+1)^p + 1}{2}}.$$

Thus

$$T(X \oplus_2 Y) \leq \sqrt{2+\sqrt{2}}.$$

Remark. This estimation is sharp since $T(\ell_1 \oplus_2 \ell_1) = \sqrt{2+\sqrt{2}}$ (see [7, Lemma 2]). On the other hand, $T(\ell_p \oplus_p Y) = \sqrt[p]{2}$ (see [7, Proposition 1]) which is strictly less than our estimation shows.

Proof. Denote $Z = X \oplus_p Y$ and $r = \sqrt[p]{\frac{(\sqrt[p]{2}+1)^p + 1}{2}}$. Fix arbitrarily $x_1 \in S_X$ and $y_1 \in S_Y$. We will show that

$$S_Z \subset B((x_1,0),r) \cup B((0,y_1),r).$$

Let $(x,y) \in S_Z$ and set

$$m = \min\{\|(x_1,0)-(x,y)\|_p, \|(0,y_1)-(x,y)\|_p\}.$$

Then

$$\begin{aligned} m^p &= \min\{\|x_1 - x\|^p + \|y\|^p, \|x\|^p + \|y_1 - y\|^p\} \\ &\leq \frac{\|x_1 - x\|^p + \|y\|^p + \|x\|^p + \|y_1 - y\|^p}{2} \\ &= \frac{\|(x_1 - x, y_1 - y)\|^p + 1}{2}. \end{aligned}$$

Since

$$\|(x_1 - x, y_1 - y)\| \leq \|(x_1,y_1)\| + \|(x,y)\| = \sqrt[p]{2} + 1,$$

we get

$$m^p \leq \frac{(\sqrt[p]{2}+1)^p + 1}{2}.$$

□

Corollary 2.8. *If X and Y are Banach spaces and $1 < p < \infty$, then $X \oplus_p Y$ is never octahedral.*

Proof. By Proposition 2.7 we have

$$T(X \oplus_p Y)^p \leq \frac{(\sqrt[p]{2}+1)^p + 1}{2} < 2^p.$$

The last inequality is easily obtained from the Minkowski's inequality by considering $(\sqrt[p]{2},0),(1,1) \in \mathbb{R}^2$. □

33

3. Locally and weakly octahedral spaces

According to the terminology in [11], the norm on a Banach space X is

- *locally octahedral* if, for every $x \in X$ and $\varepsilon > 0$, there is a $y \in S_X$ such that
$$\|sx - y\| \geq (1-\varepsilon)\big(|s|\|x\| + \|y\|\big) \quad \text{for all } s \in \mathbb{R};$$
- *weakly octahedral* if, for every finite-dimensional subspace E of X, $x^* \in B_{X^*}$, and $\varepsilon > 0$, there is a $y \in S_X$ such that
$$\|x - y\| \geq (1-\varepsilon)\big(|x^*(x)| + \|y\|\big) \quad \text{for all } x \in E.$$

Our aim in this section is to find analogous results to Proposition 2.3 for locally and weakly octahedral spaces.

We recall a simple criterion for the locally octahedral norm.

Lemma 3.1 (see [11, Proposition 2.1])**.** *Let X be a Banach space. The following assertions are equivalent:*

(i) *the norm on X is locally octahedral;*
(ii) *for every $x \in S_X$ and $\varepsilon > 0$, there is a $y \in S_X$ such that*
$$\|x \pm y\| \geq 2 - \varepsilon.$$

For locally octahedral spaces we have the following geometric characterization.

Proposition 3.2. *Let X be a Banach space. The following assertions are equivalent:*

(i) *the norm on X is locally octahedral;*
(ii) *if $S_X \subset B(x,r) \cup B(-x,r)$ for some $x \in X$ and $r > 0$ then $S_X \subset B(x,r)$;*
(iii) *if $S_X \subset B(x,r) \cup B(-x,r)$ for some $x \in S_X$ and $r > 0$ then $S_X \subset B(x,r)$.*

Proof. (i) $\Rightarrow$ (ii). Assume that the norm on X is locally octahedral, and $S_X \subset B(x,r) \cup B(-x,r)$ for some $x \in X$ and $r > 0$. We have to show that $S_X \subset B(x,r)$. Suppose that, contrary to the claim, $r < \|x\| + 1$. Let $\varepsilon > 0$ be such that $r < (1-\varepsilon)(\|x\|+1)$. Since the norm on X is locally octahedral, there is a $y \in S_X$ such that
$$(1-\varepsilon)(\|x\| + 1) \leq \|x \pm y\|.$$
Thus $\|x \pm y\| > r$, which is a contradiction.

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i). Assume that (iii) holds. Let $x \in S_X$ and $\varepsilon > 0$. By Lemma 3.1 it suffices to find a $y \in S_X$ such that
$$\|x \pm y\| \geq 2 - \varepsilon.$$

Suppose that for every $y \in S_X$ we have $\|x+y\| < 2-\varepsilon$ or $\|x-y\| < 2-\varepsilon$. Thus $S_X \subset B(x, 2-\varepsilon) \cup B(-x, 2-\varepsilon)$. By the assumption, $S_X \subset B(x, 2-\varepsilon)$, which is a contradiction. $\square$

Condition (iii) of Proposition 3.2 is clearly equivalent to $g'(X) = 2$, where the constant $g'(X)$ is defined by

$$g'(X) = \inf\{\varepsilon > 0 \colon S_X \subset B(x, \varepsilon) \cup B(-x, \varepsilon) \text{ for some } x \text{ in } S_X\}.$$

The interested reader can find more about this constant $g'(X)$ in [14], where Papini has compared it with Whitley's thickness constant.

Corollary 3.3. *Let X be a Banach space. The norm on X is locally octahedral if and only if $g'(X) = 2$.*

For weakly octahedral spaces we have the following geometric characterization.

Proposition 3.4. *Let X be a Banach space. The following assertions are equivalent:*

(i) *the norm on X is weakly octahedral;*
(ii) *if $x_1, \dots, x_n \in X$ and $r_1, \dots, r_n > 0$ are such that*

$$S_X \subset \bigcup_{i=1}^{n} B(x_i, r_i)$$

then for every $x^* \in S_{X^*}$ one has $S_X \subset \{x \in X \colon |x^*(x - x_i)| \le r_i\}$ for some i in $\{1, \dots, n\}$.

Proof. (i) $\Rightarrow$ (ii). Assume that the norm on X is weakly octahedral, and $S_X \subset \bigcup_{i=1}^{n} B(x_i, r_i)$ for some $x_1, \dots, x_n \in X$ and $r_1, \dots, r_n > 0$. Let $x^* \in S_{X^*}$. We have to show that for some $i \in \{1, \dots, n\}$, one has

$$S_X \subset \{x \in X \colon |x^*(x - x_i)| \le r_i\}.$$

Suppose that, on the contrary, for every $i \in \{1, \dots, n\}$ there is an $x \in S_X$ such that $|x^*(x - x_i)| > r_i$. Pick $\varepsilon > 0$ satisfying

$$r_i < (1-\varepsilon)(1 + |x^*(x_i)|) \quad \text{for all } i \in \{1, \dots, n\}.$$

Since the norm on X is weakly octahedral, there is a $y \in S_X$ such that

$$(1-\varepsilon)(1 + |x^*(x_i)|) \le \|x_i - y\| \quad \text{for all } i \in \{1, \dots, n\}.$$

This yields $r_i < \|x_i - y\|$ for every $i \in \{1, \dots, n\}$, which is a contradiction.

(ii) $\Rightarrow$ (i). Assume that (ii) holds. Let E be a finite-dimensional subspace of X. Let $x^* \in S_{X^*}$ and let $0 < \varepsilon \le 1$. Suppose that, for every $y \in S_X$, there is an $x \in E$ such that

$$\|x - y\| < (1-\varepsilon)(|x^*(x)| + 1). \tag{3.1}$$

Then $\|x\| < \frac{2-\varepsilon}{\varepsilon}$. Denote $\delta = \varepsilon/2$. Consider now a finite δ-net $\{x_1, \dots, x_n\}$ for $\frac{2-\varepsilon}{\varepsilon} B_E$. If $y \in S_X$, then find a corresponding $x \in E$ such that (3.1) holds, and choose x_i such that $\|x - x_i\| < \delta$. By (3.1), we have

$$\begin{aligned}
\|x_i - y\| &\leq \|x - y\| + \delta \\
&< (1-\varepsilon)(|x^*(x)| + 1) + \delta \\
&\leq (1-\varepsilon)(|x^*(x_i)| + \delta + 1) + \delta \\
&= (1-\varepsilon)|x^*(x_i)| + 1 - \varepsilon\delta \\
&\leq (1 - \varepsilon^2/2)(|x^*(x_i)| + 1).
\end{aligned}$$

Thus $S_X \subset \bigcup_{i=1}^n B(x_i, r_i)$, where $r_i = (1-\varepsilon^2/2)(|x^*(x_i)|+1)$. On the other hand,

$$S_X \not\subset \{x \in X \colon |x^*(x - x_i)| \leq r_i\} \quad \text{for all } i \in \{1, \dots, n\}.$$

This contradicts (ii). $\square$

We recall a simple criterion for the weakly octahedral norm.

Lemma 3.5 (see [11, Proposition 2.2]). *Let X be a Banach space. The following assertions are equivalent:*

(i) *the norm on X is weakly octahedral;*

(ii) *for every $n \in \mathbb{N}$, $x_1, \dots, x_n \in S_X$, $x^* \in B_{X^*}$, and $\varepsilon > 0$, there is a $y \in S_X$ such that*

$$\|x_i + ty\| \geq (1-\varepsilon)(|x^*(x_i)| + t) \quad \textit{for all } i \in \{1, \dots, n\} \textit{ and } t > 0.$$

Remark. It is not clear whether it suffices to take $t = 1$ in the preceding condition, i.e., whether condition (ii) is equivalent to the following:

(iii) for every $n \in \mathbb{N}$, $x_1, \dots, x_n \in S_X$, $x^* \in B_{X^*}$, and $\varepsilon > 0$, there is a $y \in S_X$ such that

$$\|x_i + y\| \geq (1-\varepsilon)(|x^*(x_i)| + 1) \quad \text{for all } i \in \{1, \dots, n\}.$$

However, similarly to the proof of Proposition 3.4 we observe that condition (iii) is equivalent to the following:

(iv) if $x_1, \dots, x_n \in S_X$ and $r_1, \dots, r_n > 0$ are such that

$$S_X \subset \bigcup_{i=1}^n B(x_i, r_i),$$

then for every $x^* \in S_{X^*}$ one has $S_X \subset \{x \in X \colon |x^*(x - x_i)| \leq r_i\}$ for some i in $\{1, \dots, n\}$.

Acknowledgements

This research was supported by Estonian Targeted Financing Project SF0180039s08, Estonian Science Foundation Grant 8976, and Estonian Institutional Research Project IUT20-57. The authors are grateful to the referee for a careful reading of the paper and several useful comments.

References

[1] T. Abrahamsen, V. Lima, and O. Nygaard, *Remarks on diameter 2 properties*, J. Conv. Anal. **20** (2013), 439–452.

[2] B. Bagchi, G. Misra, and N. S. N. Sastry, Florida State University Mail Room Problem Group, O. P. Lossers and J. H. Steelman, *Problems and Solutions: Solutions of Advanced Problems: 6577*, Amer. Math. Monthly **97** (1990), 436–437.

[3] M. Baronti, E. Casini, and P. L. Papini, *On average distances and the geometry of Banach spaces*, Nonlinear Anal. **42** (2000), 533–541.

[4] J. Becerra Guerrero, G. López Pérez, and A. Rueda Zoca, *Octahedral norms and convex combination of slices in Banach spaces*, arXiv:math/1309.3866.

[5] F. Botelho and J. Jamison, *Some results on the thickness and thinness of Banach spaces*, Acta Sci. Math. (Szeged), **78** (2012), 173–186.

[6] J. Castillo and P. L. Papini, *Smallness and the covering of a Banach space*, Milan J. Math. **80** (2012), 251–263.

[7] J. Castillo, P. L. Papini, and M. Simões, *Thick coverings for the unit ball of a Banach space*, arXiv:math/1307.4385.

[8] R. Deville, G. Godefroy, and V. Zizler, *Smoothness and Renormings in Banach Spaces*, Pitman Monographs and Surveys in Pure and Applied Mathematics **64**, Longman Scientific & Technical, Harlow, 1993.

[9] G. Godefroy, *Metric characterization of first Baire class linear forms and octahedral norms*, Studia Math. **95** (1989), 1–15.

[10] G. Godefroy and N. J. Kalton, *The ball topology and its applications*, in: Banach Space Theory (Iowa City, IA, 1987), Contemp. Math. **85**, Amer. Math. Soc., Providence, RI, 1989, pp. 195–237.

[11] R. Haller, J. Langemets, and M. Põldvere, *On duality of diameter* 2 *properties*, arXiv:math/1311.2177.

[12] V. Kadets, V. Shepelska, and D. Werner, *Thickness of the unit sphere, ℓ_1-types, and the almost Daugavet property*, Houston J. Math. **37** (2011), no. 3, 867–878.

[13] P. L. Papini, *Average distances and octahedral norms*, Bull. Korean Math. Soc. **36** (1999), 259–272.

[14] P. L. Papini, *Covering the sphere and the ball in Banach spaces*, Commun. Appl. Anal. **13** (2009), 579–586.

[15] R. Whitley, *The size of the unit sphere*, Canad. J. Math. **20** (1968), 450–455.

Institute of Mathematics, University of Tartu, J. Liivi 2, 50409 Tartu, Estonia

E-mail address: rainis.haller@ut.ee

E-mail address: johann.langemets@ut.ee