Composition operators on vector-valued analytic function spaces: a survey

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Abstract. We survey recent results about composition operators induced by analytic self-maps of the unit disk in the complex plane on various Banach spaces of analytic functions taking values in infinite-dimensional Banach spaces. We mostly concentrate on the research line into qualitative properties such as weak compactness, initiated by Liu, Saksman and Tylli (1998), and continued in several other papers. We discuss composition operators on strong, respectively weak, spaces of vector-valued analytic functions, as well as between weak and strong spaces. As concrete examples, we review more carefully and present some new observations in the cases of vector-valued Hardy and $BMOA$ spaces, though the study of composition operators has been extended to a wide range of spaces of vector-valued analytic functions, including spaces defined on other domains. Several open problems are stated.

1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$, and let $\varphi \colon \mathbb{D} \to \mathbb{D}$ be a fixed analytic map. The classical study of the analytic composition operators C_φ, where

$$f \mapsto C_\varphi(f) = f \circ \varphi,$$

originates from the work of Ryff (1966) and Nordgren (1968). For instance, they observed that any C_φ defines a bounded operator $H^p \to H^p$ as a consequence of the Littlewood subordination principle. Recall that for $1 \le p < \infty$, the analytic function $f \colon \mathbb{D} \to \mathbb{C}$ belongs to the Hardy space H^p if

$$\|f\|_{H^p}^p = \sup_{0 \le r < 1} \int_{\mathbb{T}} |f(r\xi)|^p dm(\xi) < \infty,$$

where $\mathbb{T} = \partial\mathbb{D} = [0, 2\pi]$ and $dm(\mathrm{e}^{it}) = dt/2\pi$. The space H^∞ consists of the bounded analytic functions. Subsequently an extensive literature has

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emerged, where a very wide variety of properties of analytic composition operators has been addressed on a large number of spaces of analytic functions. We refer to [47] and [11] for comprehensive accounts of the theory until ca. 1995.

This survey reviews more recent results about composition operators on various Banach spaces of vector-valued analytic functions including the vector-valued Hardy space $H^p(X)$, where X is a complex Banach space. Let $f \colon \mathbb{D} \to X$ be a vector-valued analytic function and let $1 \le p < \infty$. Then $f \in H^p(X)$, if

$$\|f\|_{H^p(X)}^p = \sup_{0 \le r < 1} \int_{\mathbb{T}} \|f(r\xi)\|_X^p dm(\xi) < \infty.$$

Moreover, $f \in H^\infty(X)$, if $\|f\|_{H^\infty} = \sup_{z \in \mathbb{D}} \|f(z)\|_X < \infty$. In this notation $H^p = H^p(\mathbb{C})$. Above the analyticity of $f \colon \mathbb{D} \to X$ means that the scalar-valued function $x^* \circ f$ is analytic $\mathbb{D} \to \mathbb{C}$ for any functional $x^* \in X^*$ (that is, f is weakly analytic). This is equivalent to the requirement that the X-valued derivative $f'(z)$ exists for all points $z \in \mathbb{D}$ (that is, f is strongly analytic). For the basics of vector-valued analytic functions, see for example [20].

Qualitative properties of the vector-valued composition operators $f \mapsto f \circ \varphi$ on $H^p(X)$ and certain other spaces were first systematically studied by Liu, Saksman and Tylli [35]. Independently Hornor and Jamison [21] considered the operators $f \mapsto f \circ \varphi$ on $H^p(X)$ with different aims, and Sharma and Bhanu [49] looked at some of their basic operator properties on $H^2(X)$, where X is a Hilbert space.

We mostly concentrate on qualitative properties, such as weak compactness, of composition operators on several Banach spaces of vector-valued analytic functions of both strong and weak type defined on $\mathbb{D}$. Weak type spaces were introduced into this context by Bonet, Domanski and Lindström [4], and in this case the techniques differ from those of the strong type spaces. In Section 2 we introduce a general framework for vector-valued composition operators in order to provide a convenient general perspective into the study, and we review results that illustrate both similarities and differences compared to the scalar-valued case $X = \mathbb{C}$. We also highlight new phenomena that do not have any counterparts for scalar composition operators. For instance, composition operators can be studied between a weak and a strong space. In the final section we briefly discuss attempts to generalize the larger class of weighted composition operators to the vector-valued setting. Some vector-valued arguments are sketched, but we mostly assume that the basic scalar theory is known from [47] and [11].

Composition operators of different nature occur in various other settings. For instance, there is a well-developed theory of the composition operators $S \mapsto A \circ S \circ B$, where A and B are fixed bounded operators, on spaces of linear operators, see e.g. the survey [44]. Properties of such composition operators will actually be required in Section 5 below.

2. A general framework

We first introduce a flexible general framework for the study of qualitative properties of vector-valued composition operators, which will facilitate a discussion of some common features.

Suppose that A is a Banach space of analytic functions $\mathbb{D} \to \mathbb{C}$ and let $A(X)$ be an associated vector-valued Banach space of analytic functions $\mathbb{D} \to X$, where X is a complex Banach space. Assume that the following properties hold for the pair $(A, A(X))$ for all Banach spaces X.

(AF1) The constant maps $f(z) \equiv c$ belong to A for all $c \in \mathbb{C}$.

(AF2) $f \mapsto f \otimes x$ defines a bounded linear operator $J_x \colon A \to A(X)$ for any $x \in X$, where $(f \otimes x)(z) = f(z)x$ for $z \in \mathbb{D}$.

(AF3) $g \mapsto x^* \circ g$ defines a bounded linear operator $Q_{x^*} \colon A(X) \to A$ for any $x^* \in X^*$.

(AF4) The point evaluations δ_z, where $\delta_z(f) = f(z)$ for $f \in A(X)$, are bounded $A(X) \to X$ for all $z \in \mathbb{D}$.

It follows from (AF1) and (AF2) that the vector-valued constant maps $z \mapsto f_x(z) \equiv x$, that is $f_x = 1 \otimes x$, belong to $A(X)$ for all $x \in X$. It is easy to check that (AF1)–(AF4) are satisfied for the pair $(H^p, H^p(X))$ for any X. Note that for Banach spaces of analytic functions defined on other domains, such as a half-plane or the plane $\mathbb{C}$, condition (AF1) may not be relevant and the above framework cannot be applied in this form.

Suppose that A and B are Banach spaces of analytic functions $\mathbb{D} \to \mathbb{C}$ so that $(A, A(X))$ and $(B, B(X))$ satisfy (AF1)–(AF4), where $A(X)$ and $B(X)$ are X-valued Banach spaces of analytic functions on $\mathbb{D}$ associated with A, respectively B. Let $\varphi \colon \mathbb{D} \to \mathbb{D}$ be a given analytic self-map, and suppose that the vector-valued composition operator $\widetilde{C_\varphi}$ is bounded $A(X) \to B(X)$, where $f \mapsto \widetilde{C_\varphi}(f) = f \circ \varphi$. In order to distinguish between composition operators acting on different spaces we will in the sequel use $C_\varphi \colon A \to B$ for the composition operator $f \mapsto f \circ \varphi$ in the scalar-valued setting, that is, in the case $X = \mathbb{C}$, and $\widetilde{C_\varphi}$ for its vector-valued version $A(X) \to B(X)$.

The following general formulation is partly motivated by [4, Proposition 1].

Proposition 1. *The following factorizations hold.*

(F1) *Let $x \in X$, $x^* \in X^*$ be norm-1 vectors so that $\langle x^*, x \rangle = 1$. Then*

$$
\begin{array}{ccc}
A(X) & \xrightarrow{\widetilde{C_\varphi}} & B(X) \\
{\scriptstyle J_x}\big\uparrow & & \big\downarrow{\scriptstyle Q_{x^*}} \\
A & \xrightarrow{\;C_\varphi\;} & B
\end{array}
$$

commutes.

(F2) *Let $j(x) = f_x$ for $x \in X$, where $f_x(z) \equiv x$ for all $z \in \mathbb{D}$. Then*

$$
\begin{array}{ccc}
A(X) & \xrightarrow{\ \widetilde{C_\varphi}\ } & B(X) \\
\ \ \uparrow{\scriptstyle j} & & \ \ \downarrow{\scriptstyle \delta_0} \\
X & \xrightarrow{\ \ I_X\ \ } & X
\end{array}
$$

commutes, where I_X is the identity operator on X.

Proof. Note towards (F1) that $x^*(\widetilde{C_\varphi}(f \otimes x)) = x^*((f \circ \varphi) \otimes x) = C_\varphi(f)$ for $f \in A$, while $\delta_0(\widetilde{C_\varphi}(f_x)) = \delta_0(f_x) = x$ for $x \in X$. $\qquad\square$

The above factorizations place some inherent restrictions on possible qualitative properties of the vector-valued operators $\widetilde{C_\varphi}$. Roughly speaking, part (3) below states that $\widetilde{C_\varphi}\colon A(X) \to B(X)$ cannot have any qualitative properties inherited under composition of linear operators that are not shared by $C_\varphi\colon A \to B$ and the identity operator $I_X\colon X \to X$. Thus Banach space properties of X also influence (qualitative) properties of $\widetilde{C_\varphi}$.

Corollary 2. *Let X be a complex Banach space.*

(1) *If $\widetilde{C_\varphi}$ is bounded $A(X) \to B(X)$, then C_φ is bounded $A \to B$.*

(2) *If $\widetilde{C_\varphi}\colon A(X) \to B(X)$ is compact, then C_φ is compact $A \to B$ and X is finite-dimensional. In particular, if X is infinite-dimensional, then $\widetilde{C_\varphi}$ is never compact $A(X) \to B(X)$.*

(3) *Let $\mathcal{I}$ be an operator ideal in the sense of Pietsch [43]. If $\widetilde{C_\varphi}\colon A(X) \to B(X)$ belongs to $\mathcal{I}$, then I_X as well as $C_\varphi\colon A \to B$ belong to $\mathcal{I}$.*

In fact, by (F1) and (F2) the compactness of $\widetilde{C_\varphi}\colon A(X) \to B(X)$ implies that both $C_\varphi\colon A \to B$ and I_X are compact, that is, X is finite-dimensional. Part (3) is verified in a similar fashion. For a converse of (2), see Proposition 7.

3. Weak compactness on $H^1(X)$ and other vector-valued spaces

Let $\varphi\colon \mathbb{D} \to \mathbb{D}$ be any analytic map. It was observed independently in [35] and [21] that $\widetilde{C_\varphi}$ is bounded on $H^p(X)$, while [49] contains the case $H^2(X)$, where X is a Hilbert space. The boundedness can be verified in the following manner by a small modification of an argument for scalar H^p spaces. Note first that $z \mapsto \|f(z)\|_X$ is a subharmonic map on $\mathbb{D}$ for any analytic function $f\colon \mathbb{D} \to X$, since

$$
\|f(z)\|_X = \sup_{\|x^*\| \leq 1} |\langle x^*, f(z) \rangle|, \quad z \in \mathbb{D}.
$$

Consequently, if $\varphi(0) = 0$, then Littlewood's inequality [11, Theorem 2.22] yields that

$$\|\widetilde{C_\varphi}(f)\|_{H^p(X)} \le \|f\|_{H^p(X)}^p$$

for $f \in H^p(X)$.

For the general case let $\sigma_a \colon \mathbb{D} \to \mathbb{D}$ be the Möbius transformation defined by $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$ for $a \in \mathbb{D}$. If $\varphi(0) \ne 0$, let $\psi = \sigma_{\varphi(0)} \circ \varphi$, so that $\psi(0) = 0$ and C_ψ is a contraction $H^p(X) \to H^p(X)$. Since $\sigma_{\varphi(0)}^{-1} = \sigma_{\varphi(0)}$, we get that $C_\varphi = C_\psi \circ C_{\sigma_{\varphi(0)}}$ is bounded on $H^p(X)$ once we have checked that C_ϕ is bounded on $H^p(X)$ for any Möbius transformation ϕ. This can be verified by the change of variables $w = \phi(z)$ inside the integral

$$\int_{\mathbb{T}} \|f(r\phi(\xi))\|_X^p \, dm(\xi)$$

defining $\|C_\phi(f(r\cdot))\|_{H^p(X)}$ for $0 < r < 1$ and letting $r \to 1$.

The minor point of difference between the above scalar- and vector-valued arguments for the Hardy spaces relates to the potential absence of radial limits. In fact, it is well known that any $f \in H^p$ has a.e. radial limits $f(\xi) = \lim_{r \to 1^-} f(r\xi)$ on $\mathbb{T}$, but this is not always true for functions in $H^p(X)$: the bounded analytic function $f \colon \mathbb{D} \to c_0$, where

$$f(z) = (z^n), \quad z \in \mathbb{D},$$

does not have radial limits anywhere on $\mathbb{T}$. In fact, the existence of a.e. radial limits for any $f \in H^p(X)$, and any fixed $1 \le p \le \infty$, characterizes the analytic Radon–Nikodým property (ARNP) of the complex Banach space X. The ARNP and the above result by Bukhvalov and Danilevich (1982) is not needed here, but the reader may keep in mind that e.g. every reflexive Banach space has the ARNP. See, e.g., [28, p. 723] for references and a discussion of the ARNP.

Let X be an infinite-dimensional Banach space. According to Corollary 2 (2) there are no compact compositions $\widetilde{C_\varphi} \colon H^p(X) \to H^p(X)$. This raises the general question of which are the relevant qualitative properties for composition operators on vector-valued spaces such as $H^p(X)$. In [35] the authors considered weak compactness, and related properties, for which there are satisfactory results.

Let X and Y be Banach spaces. Recall that the bounded linear operator $U \colon X \to Y$ is *weakly compact* if there is a weakly convergent subsequence (Ux_{n_k}) for any bounded sequence $(x_n) \subset X$. If X and Y are non-reflexive spaces, then weakly compact $U \colon X \to Y$ are relatively small operators.

A fundamental result of Shapiro [46] (see also [47] and [11]) says that for $1 \le p < \infty$ the composition operator C_φ is compact $H^p \to H^p$ if and only if

$$\lim_{|w| \to 1} \frac{N(\varphi, w)}{\log(1/|w|)} = 0. \tag{1}$$

Above $N(\varphi, w) = \sum_{z \in \varphi^{-1}(w)} \log(1/|z|)$, where $w \in \mathbb{D} \setminus \{\varphi(0)\}$, is the Nevanlinna counting function of φ. Several other equivalent criteria for the compactness of $C_\varphi \colon H^p \to H^p$ are known in the literature, but (1) suffices for our purposes. Littlewood's inequality implies that $N(\varphi, w) \leq C \cdot \log(1/|w|)$ as $|w| \to 1$ for some constant $C = C(\varphi)$ for any analytic map $\varphi \colon \mathbb{D} \to \mathbb{D}$, see [47, 10.4]. Shapiro's condition (1) is interpreted as a little-oh condition describing the rate of decrease of the affinity of φ for the values w as $|w| \to 1$.

There is a precise connection between the weak compactness of $\widetilde{C_\varphi}$ on $H^1(X)$ and the compactness of C_φ on H^1. Note that Corollary 2 (3) implies that X is reflexive, that is, I_X is weakly compact, whenever $\widetilde{C_\varphi}$ is weakly compact on $H^p(X)$. Hence only $p = 1$ or $p = \infty$ are interesting for weak compactness, since $H^p(X)$ is itself reflexive if $1 < p < \infty$ and X is reflexive, because $H^p(X)$ is then a closed subspace of the reflexive space $L^p(\mathbb{T}, X)$. The vector-valued part of the following result comes from [35].

Theorem 3. *Let X be a complex reflexive Banach space, and $\varphi \colon \mathbb{D} \to \mathbb{D}$ be an analytic map. Then the following conditions are equivalent.*
  (1) $\widetilde{C_\varphi} \colon H^1(X) \to H^1(X)$ *is weakly compact.*
  (2) $C_\varphi \colon H^1 \to H^1$ *is weakly compact.*
  (3) $C_\varphi \colon H^1 \to H^1$ *is compact.*
  (4) *Shapiro's condition* (1) *holds.*

Proof. The implication $(1) \Rightarrow (2)$ follows from Corollary 2 (3). Sarason [45] proved that the weak compactness of $C_\varphi \colon H^1 \to H^1$ actually yields the compactness of $C_\varphi \colon H^1 \to H^1$, in other words that $(2) \Rightarrow (3)$. The equivalence of (3) and (4) is contained in Shapiro's theorem. There remains to show that $\widetilde{C_\varphi}$ is weakly compact $H^1(X) \to H^1(X)$ whenever φ satisfies Shapiro's condition.

We outline the proof of the implication $(4) \Rightarrow (1)$. The argument is based on a Littlewood–Paley type formula for $\|\widetilde{C_\varphi}(f)\|_{H^1(X)}$ derived from a formula of Stanton for continuous subharmonic maps. His formula [51, Theorem 2] implies that

$$\|f \circ \varphi\|_{H^1(X)} = \|f(0)\|_X + \frac{1}{2\pi} \int N(\varphi, w) d[\Delta(\|f\|_X)](w) \qquad (2)$$

for $f \in H^1(X)$, where $d[\Delta(\|f\|_X)]$ denotes the distributional Laplacian associated to the subharmonic map $z \mapsto \|f(z)\|_X$ on $\mathbb{D}$.

Define de la Vallée-Poussin operators V_n for any $n \in \mathbb{N}$ by

$$V_n f(z) = \sum_{k=0}^{n} \hat{f}_k z^k + \sum_{k=n+1}^{2n-1} \frac{2n-k}{n} \hat{f}_k z^k$$

for analytic functions $f \colon \mathbb{D} \to X$ having the Fourier expansion $f(z) = \sum_{k=0}^{\infty} \hat{f}_k z^k$. Then (V_n) is a uniformly bounded sequence of operators on

$H^1(X)$ and $V_n\colon H^1(X) \to H^1(X)$ are weakly compact for any n if X is a reflexive Banach space (in fact, V_n factors through a finite direct sum of copies of X). Moreover, given $\varepsilon > 0$ and $0 < r < 1$ there is $n_0 = n_0(\varepsilon, r) \in \mathbb{N}$ so that

$$\|f(z) - V_{n_0}f(z)\|_X \leq \varepsilon \cdot \|f\|_{H^1(X)} \tag{3}$$

holds for all $|z| \leq r$ and $f \in H^1(X)$.

If Shapiro's condition (1) holds and $\varepsilon > 0$ is arbitrary, then there is $r \in (0,1)$ such that $N(\varphi, w) \leq \varepsilon \cdot \log(1/|w|)$ for all $|w| > r$. Fix n_0 as in (3) corresponding to ε and r. By applying (2) to $f - V_{n_0}f$ we get for $f \in H^1(X)$ that

$$\|\widetilde{C_\varphi}(f) - \widetilde{C_\varphi}(V_{n_0}f)\|_{H^1(X)} = \frac{1}{2\pi} \int_{\{r<|z|<1\}} N(\varphi, w)d[\Delta(\|f - V_{n_0}f\|_X)](w)$$

$$+ \frac{1}{2\pi} \int_{\{|z|\leq r\}} N(\varphi, w)d[\Delta(\|f - V_{n_0}f\|_X)](w) \equiv I_1 + I_2.$$

The choice of $r \in (0,1)$ and (2) applied to $\psi(z) = z$ give

$$I_1 \leq \frac{\varepsilon}{2\pi} \int_{\mathbb{D}} \log(1/|w|)d[\Delta(\|f - V_{n_0}f\|_X)](w) = \varepsilon\|f - V_{n_0}f\|_{H^1(X)}$$

$$\leq C \cdot \varepsilon\|f\|_{H^1(X)}$$

for a uniform constant C. Moreover, it can be shown that $I_2 \leq 4\varepsilon\|f\|_{H^1(X)}$ by using the estimates $N(\varphi, w) \leq \log(1/|w|)$ and (3). For this it is convenient to introduce a cut-off function $\psi \in C_0^\infty(\mathbb{D})$ satisfying $0 \leq \psi \leq 1$, $\psi = 1$ on $\{z\colon |z| \leq r\}$ and $\psi = 0$ on $\{z \in \mathbb{D}\colon |z| \geq (1+r)/2\}$. We refer to [35, Proposition 2 and Theorem 3] for the complete technical details.

Thus $\|\widetilde{C_\varphi} - \widetilde{C_\varphi}V_{n_0}\| \leq C' \cdot \varepsilon$, where C' does not depend on ε, so that $\widetilde{C_\varphi}$ is well approximated by the weakly compact operators $\widetilde{C_\varphi}V_{n_0}$ for suitable n_0. This means that $\widetilde{C_\varphi}$ is weakly compact $H^1(X) \to H^1(X)$. $\qquad\square$

Theorem 3 corresponds to the following template for many results about weak compactness, as well as other qualitative properties, of analytic composition operators on vector-valued spaces.

Proposition 4. *Let A be a Banach space of analytic functions on $\mathbb{D}$ and $A(X)$ a vector-valued version of A, such that $(A, A(X))$ satisfies (AF1)–(AF4). Suppose that X is a complex reflexive Banach space and $\varphi\colon \mathbb{D} \to \mathbb{D}$ is an analytic map, so that $\widetilde{C_\varphi}$ is bounded $A(X) \to A(X)$. Assume moreover that the following conditions hold:*

(C1) *if C_φ is weakly compact $A \to A$, then C_φ is compact $A \to A$, and*

(C2) *if C_φ is compact $A \to A$, then the vector-valued composition $\widetilde{C_\varphi}$ is weakly compact $A(X) \to A(X)$.*

Then one has the characterization

(C) $\widetilde{C_\varphi}$ *is weakly compact* $A(X) \to A(X) \Leftrightarrow C_\varphi$ *is compact* $A \to A$.

We stress that the above general scheme is only a guiding principle and in practice the techniques for establishing (C2) depend on A and its vector-valued extension $A(X)$. Moreover, the criteria for the compactness of the operator $C_\varphi\colon A \to A$ usually depend on A. It is straightforward to modify the scheme of Proposition 4 to apply to vector-valued compositions $\widetilde{C_\varphi}\colon A(X) \to B(X)$ between different spaces, where $(A, A(X))$ and $(B, B(X))$ satisfy the properties (AF1)–(AF4).

Condition (C1) is a problem of independent interest for composition operators $A \to A$. Recently Lefevre, Li, Queffelec and Rodriguez-Piazza [31] constructed the first example of a Banach space A of complex-valued analytic functions on $\mathbb{D}$, where (C1) fails for some symbol φ, see Example 10 below.

We next look at cases where Proposition 4 apply. Let $v_\alpha(z) = (1 - |z|^2)^\alpha$ for $z \in \mathbb{D}$ and $\alpha > -1$. The analytic function $f\colon \mathbb{D} \to X$ belongs to the weighted Bergman space $A_p^\alpha(X)$ if

$$\|f\|_{A_p^\alpha(X)}^p = \int_{\mathbb{D}} \|f(z)\|_X^p v_\alpha(z) dA(z) < \infty,$$

where dA is the area Lebesgue measure normalized by $A(\mathbb{D}) = 1$ and $1 \le p < \infty$. The classical Bergman space $A^p(X)$ is obtained for $\alpha = 0$. The following result was established in [4], but the special case $A^1(X)$ was already contained in [35].

Theorem 5. *Let X be a complex reflexive Banach space, $\varphi\colon \mathbb{D} \to \mathbb{D}$ an analytic map and $\alpha > -1$. Then the following conditions are equivalent.*

(1) $\widetilde{C_\varphi}\colon A_1^\alpha(X) \to A_1^\alpha(X)$ *is weakly compact.*
(2) $C_\varphi\colon A_1^\alpha \to A_1^\alpha$ *is compact.*
(3) φ *satisfies the condition*

$$\limsup_{|w| \to 1} \frac{N_{\alpha+2}(\varphi, w)}{(\log(1/|w|))^{\alpha+2}} = 0.$$

Above $N_\beta(\varphi, \cdot)$ is the generalized Nevanlinna counting function defined for $\beta > 0$ by

$$N_\beta(\varphi, w) = \sum_{z \in \varphi^{-1}(w)} (\log(1/|z|))^\beta, \quad w \in D \setminus \{\varphi(0)\},$$

so that $N(\varphi, \cdot) = N_1(\varphi, \cdot)$. Actually, [4, Theorem 8] contains the estimate

$$\text{dist}(\widetilde{C_\varphi}, W(A_1^\alpha(X))) \le C \cdot \limsup_{|w| \to 1} \frac{N_{\alpha+2}(\varphi, w)}{(\log(1/|w|))^{\alpha+2}},$$

where $W(A_1^\alpha(X))$ denotes the linear subspace consisting of the weakly compact operators $A_1^\alpha(X) \to A_1^\alpha(X)$ and C is an absolute constant.

We list some further Banach spaces $A(X)$ for which the characterization (C) for weak compactness of composition operators are known to hold. We emphasize that the arguments establishing (C1) and (C2) usually are specific for $A(X)$, and the relevant compactness conditions for C_φ depend on A. One verifies by inspection that these pairs $(A, A(X))$ satisfy (AF1)–(AF4).

- Let $v \colon \mathbb{D} \to (0, \infty)$ be a bounded continuous weight function, and $f \colon \mathbb{D} \to X$ be an analytic function. Recall that $f \in H_v^\infty(X)$ if

$$\|f\|_{H_v^\infty(X)} = \sup_{|z|<1} v(z)\|f(z)\|_X < \infty.$$

  Let $H_v^\infty = H_v^\infty(\mathbb{C})$. The case $v \equiv 1$ gives the classical spaces $H^\infty(X)$ and H^∞ of bounded analytic functions. It was shown in [35] that (C) holds on $H^\infty(X)$ and this was extended to the case $H_v^\infty(X)$ by different means in [4].
- The vector-valued Bloch space $\mathcal{B}(X)$ [35]. Recall that $f \in \mathcal{B}(X)$ if

$$\sup_{z \in \mathbb{D}}(1 - |z|^2)\|f'(z)\|_X < \infty.$$

  See [4] for an alternative approach via weak spaces (Section 5 below).
- The space $CT(X)$ of vector-valued Cauchy transforms [28]. The argument proceeds via composition operators on the vector-valued harmonic Hardy space $h^1(X)$. The compactness criterion for CT is due to Bourdon, Cima and Matheson.
- Vector-valued $BMOA(X)$-spaces, see Section 4.
- Weak vector-valued versions of the above spaces, see Section 5.

A modified general scheme as in Proposition 4 also applies to other operator ideal properties, namely, just replace weak compactness by the relevant ideal property in (C1) and (C2). We state two results of this kind for $H^1(X)$, respectively $A_\alpha^1(X)$, from [35, Theorem 7] and [4, Corollary 9]. The operator $U \colon X \to Y$ is called *weakly conditionally compact* if (Ux_n) has a weak Cauchy subsequence (Ux_{n_k}) for any bounded sequence $(x_n) \subset X$. Recall that by Rosenthal's ℓ^1-theorem, see [33, 2.e.5], I_X is weakly conditionally compact if and only if X that does not contain any subspaces linearly isomorphic to ℓ^1. By Proposition 1 this is the relevant class of spaces here.

Theorem 6. *Suppose that the Banach space X does not contain any subspaces linearly isomorphic to ℓ^1, and $\varphi \colon \mathbb{D} \to \mathbb{D}$ is an analytic map. Let $A = H^1$ or $A = A_\alpha^1$ for $\alpha > -1$. Then $\widetilde{C_\varphi}$ is weakly conditionally compact $A(X) \to A(X)$ if and only if C_φ is compact $A \to A$.*

We mention for completeness that the cases $\dim(X) < \infty$ are similar to the scalar case.

Proposition 7. *Suppose that* $\dim(X) < \infty$, $(A, A(X))$ *satisfies* (AF1)–(AF4), *and* $\varphi\colon \mathbb{D} \to \mathbb{D}$ *is an analytic map. Then* $\widetilde{C_\varphi}$ *is compact* $A(X) \to A(X)$ *if and only if* C_φ *is compact* $A \to A$.

Proof. Let $n = \dim(X)$ and fix a biorthogonal system $\{(x_r, x_s^*)\colon 1 \le r, s \le n\}$ for X, so that $x = \sum_{k=1}^n x_k^*(x)x_k$ for $x \in X$. Hence any $f \in A(X)$ can be written as $f(z) = \sum_{k=1}^n f_k(z)x_k$, where $f_k = x_k^* \circ f \in A$ for $k = 1, \ldots, n$. Consequently, if C_φ is compact $A \to A$, then $f \mapsto \widetilde{C_\varphi}(f) = \sum_{k=1}^n C_\varphi(f_k)x_k$ is compact $A(X) \to A(X)$. For the converse note that Section 2 applies to this setting. $\qquad\square$

Other results. Hornor and Jamison [21] characterized the isometrically equivalent compositions $\widetilde{C_\varphi}$ and $\widetilde{C_\psi}$ on $H^p(X)$, respectively $S^p(X)$, for $p \ne 2$ and X a Hilbert space. Here $f \in S^p(X)$ if the derivative $f' \in H^p(X)$. Sharma and Bhanu [49] studied e.g. normal and unitary compositions on $H^2(X)$, where X is a Hilbert space. Bonet and Friz [5] characterized the weakly compact compositions on weighted vector-valued locally convex spaces of analytic functions on $\mathbb{D}$. Composition operators on the Hilbert space-valued Fock space of entire functions on $\mathbb{C}$ were considered by Ueki [53]. See also [54] for results on the vector-valued Nevanlinna class.

4. Vector-valued $BMOA$-spaces

In this section we discuss in more detail the case of composition operators on vector-valued $BMOA$ spaces, since there are several natural vector-valued versions of $BMOA$, and condition (C1) is a problem of independent interest.

Recall that the analytic function $f\colon \mathbb{D} \to \mathbb{C}$ belongs to $BMOA$, the space of analytic functions of bounded mean oscillation, if

$$|f|_* = \sup_{a \in \mathbb{D}} \|f \circ \sigma_a - f(a)\|_{H^2} < \infty,$$

where $\sigma_a(z) = (a - z)/(1 - \bar{a}z)$ for $z \in \mathbb{D}$. The Banach space $BMOA$ is equipped with the norm $\|f\|_{BMOA} = |f(0)| + |f|_*$. $BMOA$ is often considered as a Möbius-invariant version of H^2, but its Banach space structure is complicated, see e.g. [41]. Recall also that $(H^1)^* \approx BMOA$.

There are by now several equivalent characterizations of compact compositions $C_\varphi\colon BMOA \to BMOA$, see [27] for a list. The following double criterion due to W. Smith [50] is the most relevant one for our purposes.

Let $\varphi\colon \mathbb{D} \to \mathbb{D}$ *be an analytic map. Then* C_φ *is compact* $BMOA \to BMOA$ *if and only if*

$$\lim_{|\varphi(a)| \to 1} \sup_{0 < |w| < 1} |w|^2 N(\sigma_{\varphi(a)} \circ \varphi \circ \sigma_a, w) = 0, \qquad (S1)$$

$$\lim_{t \to 1} \sup_{\{a\colon |\varphi(a)| \le R\}} m\big(\{\zeta \in \partial\mathbb{D}\colon |(\varphi \circ \sigma_a)(\zeta)| > t\}\big) = 0 \ \ \text{for } 0 < R < 1. \quad (S2)$$

Subsequently it was observed in [26] that (S1) *can be restated as*

$$\lim_{|\varphi(a)|\to 1} \|\sigma_{\varphi(a)} \circ \varphi \circ \sigma_a\|_{H^2} = 0. \tag{L}$$

Let $f \colon \mathbb{D} \to X$ be an analytic function. We say that $f \in BMOA(X)$ if

$$|f|_{*,X} = \sup_{a\in\mathbb{D}} \|f \circ \sigma_a - f(a)\|_{H^2(X)} < \infty,$$

and let $\|f\|_{BMOA(X)} = \|f(0)\|_X + |f|_{*,X}$. There are also other natural possibilities. By departing from a well-known characterization of $BMOA$ in terms of Carleson measures, see [18, Theorem VI.3.4], let $f \in BMOA_{\mathcal{C}}(X)$ if

$$|f|_{\mathcal{C},X} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \|f'(z)\|_X^2 (1 - |\sigma_a(z)|^2) dA(z) < \infty.$$

The norm in $BMOA_{\mathcal{C}}(X)$ is $\|f\|_{BMOA_{\mathcal{C}}(X)} = \|f(0)\|_X + |f|_{\mathcal{C},X}$. Blasco [3] showed that $BMOA(X) = BMOA_{\mathcal{C}}(X)$, with equivalent norms, if and only if X is linearly isomorphic to a Hilbert space. Thus $BMOA(X)$ and $BMOA_{\mathcal{C}}(X)$ are different vector-valued versions of $BMOA$. (In Section 5 we will meet yet another vector-valued version of $BMOA$.)

Laitila [24], [25] initiated the study of composition operators on vector-valued $BMOA$-spaces. He observed that $\widetilde{C_\varphi}$ is bounded $BMOA(X) \to BMOA(X)$ and $BMOA_{\mathcal{C}}(X) \to BMOA_{\mathcal{C}}(X)$ for any self-map $\varphi \colon \mathbb{D} \to \mathbb{D}$. Moreover, if X is a reflexive Banach space and φ satisfies conditions (S1) and (S2), then $\widetilde{C_\varphi}$ is weakly compact both $BMOA(X) \to BMOA(X)$ and $BMOA_{\mathcal{C}}(X) \to BMOA_{\mathcal{C}}(X)$. In order to obtain a complete characterization following Proposition 4 one has to verify condition (C1) for $BMOA$. This was actually a problem stated by Tjani in her Ph.D. thesis [52] and Bourdon, Cima and Matheson [7], which was eventually solved in [27] as follows.

Theorem 8. *The following conditions are equivalent for* $\varphi \colon \mathbb{D} \to \mathbb{D}$.

(1) $C_\varphi \colon BMOA \to BMOA$ *is compact.*
(2) $C_\varphi \colon BMOA \to BMOA$ *is weakly compact.*
(3) (S1) *holds (alternatively,* (L) *holds).*

It is part of the solution that condition (S2) is redundant in Smith's characterization above. The combination of Theorem 8 with [24], [25] completes the following result for these vector-valued $BMOA$-spaces.

Theorem 9. *Let X be a reflexive Banach space, and $\varphi \colon \mathbb{D} \to \mathbb{D}$ an analytic function. Then the following conditions are equivalent.*

(1) $\widetilde{C_\varphi}$ *is weakly compact* $BMOA(X) \to BMOA(X)$.
(2) $\widetilde{C_\varphi}$ *is weakly compact* $BMOA_{\mathcal{C}}(X) \to BMOA_{\mathcal{C}}(X)$.
(3) $C_\varphi \colon BMOA \to BMOA$ *is compact, that is, condition* (S1) *holds.*

The argument for Theorem 8 is quite intricate. It applies measure density ideas for the radial limits of φ combined with a criterion due to Leibov (1986), respectively Müller and Schechtman (1989), which allows to extract copies of the unit vector basis in c_0 from bounded sequences in the subspace $VMOA$ of $BMOA$. We refer to [27] for the full technical details and for the relevant references.

The results of Sections 3 and 4 might suggest that the weak compactness of $C_\varphi \colon A \to A$ always implies its compactness $A \to A$ for any Banach space A of analytic functions on $\mathbb{D}$. However, this is not the case [31, Theorem 4.1]:

Example 10. Let φ be the lens map

$$\varphi(z) = \frac{(1+z)^{1/2} - (1-z)^{1/2}}{(1+z)^{1/2} + (1-z)^{1/2}}, \quad z \in \mathbb{D}.$$

Then there is an Orlicz function ψ so that C_φ is weakly compact $H^\psi \to H^\psi$, but not compact, where H^ψ is the non-reflexive Hardy–Orlicz space of analytic functions of $\mathbb{D}$ defined by ψ.

Question 11. Characterize the weakly compact compositions on the space $H^\psi(X)$ above. The operator C_φ in Example 10 factors through H^4 by construction, and $\widetilde{C_\varphi}$ through the reflexive space $H^4(X)$ for reflexive spaces X. Thus $\widetilde{C_\varphi}$ is weakly compact $H^\psi(X) \to H^\psi(X)$, so that (C) cannot hold for $H^\psi(X)$.

The referee kindly pointed out that the first example of a weakly compact analytic composition operator which is not compact was obtained in the context of uniform algebras defined on infinite-dimensional domains. Let U_E be the open unit ball of the Tsirelson space E and $\varphi \colon U_E \to U_E$ the map $x \mapsto x/2$. It was shown in [1, Example 3] that the composition operator $f \mapsto f \circ \varphi$ is weakly compact, but non-compact, $H^\infty(U_E) \to H^\infty(U_E)$. Here $H^\infty(U_E)$ is the uniform algebra of bounded scalar-valued analytic functions $U_E \to \mathbb{C}$.

5. Weak vector-valued spaces

Bonet, Domanski and Lindström [4] introduced the class of weak spaces of vector-valued analytic functions into the study of vector-valued composition operators. One of their aims was to provide an alternative approach to [35], but the weak spaces are in general different from the spaces considered in Sections 3 and 4. On the other hand, for the class of weak spaces there are some general results concerning vector-valued compositions.

Suppose that E is a Banach space of analytic functions $f \colon \mathbb{D} \to \mathbb{C}$ satisfying the following conditions:

(W1) E contains the constant functions,

(W2) the closed unit ball B_E is compact in the compact open topology τ_{co} of $\mathbb{D}$.

Recall that the vector-valued function $f \colon \mathbb{D} \to X$ is *analytic* if and only if $x^* \circ f$ is analytic $\mathbb{D} \to \mathbb{C}$ for all $x^* \in X^*$. This fact suggests to define $f \in wE(X)$ if

$$\|f\|_{wE(X)} = \sup_{\|x^*\|_{X^*} \leq 1} \|x^* \circ f\|_E < \infty.$$

By the closed graph theorem $\|f\|_{wE(X)}$ is finite if and only if $x^* \circ f \in E$ for all $x^* \in X^*$. We will say that $wE(X)$ is the *weak space* of vector-valued analytic functions $\mathbb{D} \to X$ modelled on E. The spaces appearing in Sections 3 and 4, whose norms involve pointwise norm quantities such as $\|f(z)\|_X$, will in the sequel be called *strong spaces*. Such a distinction between strong and weak spaces is not precise, since e.g. $wH^\infty(X) = H^\infty(X)$. If E is a Banach space of harmonic functions on $\mathbb{D}$ which satisfies (W1) and (W2), then one may similarly define the weak space $wE(X)$ of vector-valued harmonic functions $\mathbb{D} \to X$, see [28]. Weak type spaces first appeared in the theory of vector measures, see e.g. [13, Chapter 13]. The weak Hardy spaces $wH^p(X)$, and in particular their harmonic versions $wh^p(X)$, have been studied by Blasco [2], as well as in [15, 16].

Weak spaces $wE(X)$ have a dual nature, since they also admit a canonical isometrically isomorphic representation as certain spaces of bounded operators. This general fact was observed in [4]. Note first that if E satisfies (W1) and (W2), then

(W3) the evaluation maps $\delta_z \in E^*$ for $z \in \mathbb{D}$, where $\delta_z(f) = f(z)$ for $f \in E$.

The Dixmier–Ng theorem [42] implies that $E = V^*$, where

$$V = \{u^* \in E^* \colon u^* \text{ is } \tau_{co}\text{-continuous on } B_E\}.$$

The identification of $f \in E$ with $u^* \mapsto u^*(f)$ gives the isometric isomorphism $E \to V^*$. In addition, $V = [\delta_z \in E^* \colon z \in \mathbb{D}]$ by the Hahn–Banach theorem, where $[B]$ denotes the closed linear span of the subset $B \subset E^*$. We next formulate the general linearization result from [4], which also implies that $wE(X)$ is a Banach space. An analogue holds for weak harmonic spaces, see [28].

Theorem 12. *Suppose that E satisfies* (W1) *and* (W2), *let $V = [\delta_z \in E^* \colon z \in \mathbb{D}]$ and X be a complex Banach space. Then there is an isometric isomorphism $\chi \colon L(V, X) \to wE(X)$, so that*

$$(\chi(T))(z) = T(\delta_z), \quad (\chi^{-1}(f))(\delta_z) = f(z),$$

hold for $T \in L(V, X)$, $f \in wE(X)$ and $z \in \mathbb{D}$.

Special cases and variants of this linearization result were known earlier. The closest precursor is the general results of Mujica [40] that apply to the case $E = H^\infty$. An explicit operator representation was obtained by Blasco [2] for the weak harmonic spaces $wh^p(X)$, where $1 \le p \le \infty$.

The study of composition operators between weak spaces of analytic functions was initiated by Bonet, Domański and Lindström [4], and this was extended in [28] to weak spaces of vector-valued harmonic functions. Proposition 11 of [4] contains the result on weak compactness stated below in Theorem 14 for the weak spaces, but [4] only explicitly discusses the weighted $wH_v^\infty(X)$-spaces and the weak Bloch space $w\mathcal{B}(X)$ as examples. However, this approach applies to a large class of weak spaces of analytic functions, such as the weak Hardy and weak Bergman spaces, as well as to $wBMOA(X)$, see the discussion below as well as in [28, 24, 25].

Let $\varphi \colon \mathbb{D} \to \mathbb{D}$ be an analytic map. The vector-valued composition $\widetilde{C_\varphi}$ is bounded $wE(X) \to wE(X)$ if and only if C_φ is bounded $E \to E$. In fact, if $x^* \in X^*$, then

$$\|x^* \circ (\widetilde{C_\varphi}f)\|_E = \|C_\varphi(x^* \circ f)\|_E \le \|C_\varphi\| \cdot \|x^* \circ f\|_E,$$

so that $\|\widetilde{C_\varphi}\| \le \|C_\varphi\|$. For the converse it is worthwhile to point out that the framework from Section 2 applies to the weak spaces.

Lemma 13. *If E satisfies* (W1) *and* (W2), *then the pair* $(E, wE(X))$ *satisfies* (AF1)–(AF4) *for any Banach space X.*

Proof. Conditions (AF1)–(AF3) are obvious. Towards (AF4) note that

$$x^*(\widetilde{\delta}_z(f)) = \delta_z(x^* \circ f), \quad f \in wE(X), \ x^* \in X^*,$$

where we momentarily use $\widetilde{\delta}_z$ for the vector-valued evaluations $f \mapsto f(z)$ taking $wE(X)$ to X. $\qquad\square$

We stress that the following basic weak compactness result from [4] for vector-valued compositions holds on all weak spaces $wE(X)$. The proof uses different tools compared to the analytic arguments in Sections 3 and 4. Recall again from Corollary 2 that $\widetilde{C_\varphi}$ is never compact $wE(X) \to wE(X)$ whenever X is infinite-dimensional.

Theorem 14. *Suppose that E is a Banach space of analytic functions on $\mathbb{D}$ that satisfy* (W1) *and* (W2). *Let $\varphi \colon \mathbb{D} \to \mathbb{D}$ be an analytic map and X a reflexive Banach space. If $C_\varphi \colon E \to E$ is compact, then $\widetilde{C_\varphi}$ is weakly compact $wE(X) \to wE(X)$.*

Proof. Assume that $C_\varphi \colon E \to E$ is compact. Its adjoint $(C_\varphi)^* \colon E^* \to E^*$ satisfies

$$(C_\varphi)^*(\delta_z) = \delta_{\varphi(z)}, \quad z \in \mathbb{D},$$

so that $(C_\varphi)^*(V) \subset V$. We obtain the factorization $\widetilde{C_\varphi} = \chi \circ U_\varphi \circ \chi^{-1}$, where U_φ is the operator composition map

$$T \mapsto I_X \circ T \circ (C_\varphi)^*|_V; \quad L(V, X) \to L(V, X),$$

and χ is the isometric isomorphism $L(V, X) \to wE(X)$ from Theorem 12. Since $(C_\varphi)^*|_V$ is a compact operator $V \to V$ by duality, and I_X is weakly compact, it follows from a general result of Saksman and Tylli, see [44, Proposition 2.3], that the operator composition U_φ is weakly compact $L(V, X) \to L(V, X)$. Consequently $\widetilde{C_\varphi}$ is weakly compact $wE(X) \to wE(X)$. $\qquad\square$

Theorem 14 verifies condition (C2) from Proposition 4 for the weak spaces $wE(X)$. The following observation includes many examples.

Proposition 15. *Suppose that E is a Banach space of analytic functions on $\mathbb{D}$ that satisfy (W1) and (W2), let $\varphi\colon \mathbb{D} \to \mathbb{D}$ be an analytic map so that C_φ is bounded $E \to E$ and X a reflexive Banach space. Suppose moreover:*

(C1) *if C_φ is weakly compact $E \to E$, then C_φ is compact $E \to E$.*

Then one has the characterization

(C) *$\widetilde{C_\varphi}$ is weakly compact $wE(X) \to wE(X) \Leftrightarrow C_\varphi$ is compact $E \to E$.*

Moreover, (C1) holds e.g. if E is one of the following spaces: H^1, A_α^1 for $\alpha > -1$, $BMOA$, H_v^∞, where v is a bounded continuous weight on $\mathbb{D}$, or $\mathcal{B}$.

The preceding examples cover results for $wH_v^\infty(X)$ and $w\mathcal{B}(X)$ from [4], $wH^1(X)$ [28], and $wBMOA(X)$ (combine Theorem 8 with [24, 25]).

The results of Sections $3-5$ raise the question of what is the precise connection between these strong and weak spaces of vector-valued analytic functions. Clearly $wH^2(\ell^2) \approx L(\ell^2)$ by Theorem 12, whereas $H^2(\ell^2)$ is a separable Hilbert space, so the difference can be huge. On the other hand, [4] observed that $wH_v^\infty(X) = H_v^\infty(X)$ (equal norms) and $w\mathcal{B}(X) \approx \mathcal{B}(X)$ (equivalent norms). It is evident that e.g. $H^p(X) \subset wH^p(X)$, and

$$\|f\|_{wH^p(X)} \le \|f\|_{H^p(X)}, \quad f \in H^p(X),$$

where $1 \le p < \infty$ and X is any Banach space. Blasco [2] observed that $h^1(C(\mathbb{T})) \subsetneq wh^1(C(\mathbb{T}))$ and $h^p(L^{p'}) \subsetneq wh^p(L^{p'})$ for $1 < p < \infty$. Subsequently Freniche, García-Vázquez and Rodríguez-Piazza [15, 16] exhibited functions $f \in wh^p(X) \setminus h^p(X)$ and $g \in wH^p(X) \setminus H^p(X)$ for $1 \le p < \infty$ and any X. Fairly concrete functions of this kind were provided in [24, 28], and [30] contains the analogous results for the weak vs. strong Bergman norms. In fact, the norms

$$\| \cdot \|_{wH^p(X)} \not\sim \| \cdot \|_{H^p(X)}$$

are non-equivalent on $H^p(X)$. Strict inclusions $BMOA(X) \subsetneq wBMOA(X)$ and $BMOA_\mathcal{C}(X) \subsetneq wBMOA(X)$ for any infinite-dimensional X were obtained in [24, 25]. A common feature of these examples for arbitrary X is

the use of Dvoretzky's ℓ_2^n-theorem to transfer from the Hilbert space setting to X.

The linearization from Theorem 14 can also be used for other purposes. The following result from [4] concerns weak conditional compactness on the spaces $wE(X)$.

Theorem 16. *Suppose that E is a Banach space of analytic functions on $\mathbb{D}$ that satisfy* (W1) *and* (W2). *Let $\varphi\colon \mathbb{D} \to \mathbb{D}$ be an analytic map and X a Banach space that does not contain any subspaces linearly isomorphic to ℓ^1. If $C_\varphi\colon E \to E$ is compact, then $\widetilde{C_\varphi}$ is weakly conditionally compact $wE(X) \to wE(X)$.*

The proof is analogous to that of Theorem 14, but instead apply [34] to deduce the weak conditional compactness of the operator composition U_φ.

Since $wH^2(\ell^2) \approx L(\ell^2)$ is non-reflexive one may also look for a characterization of weakly compact $\widetilde{C_\varphi}\colon wH^2(\ell^2) \to wH^2(\ell^2)$. Note that the following observation is not included in Proposition 15 since H^2 is reflexive.

Proposition 17. *$\widetilde{C_\varphi}$ is weakly compact $wH^2(\ell^2) \to wH^2(\ell^2)$ if and only if φ satisfies Shapiro's condition* (1).

Proof. In view of Theorem 14 there remains to show that the weak compactness of $\widetilde{C_\varphi}\colon wH^2(\ell^2) \to wH^2(\ell^2)$ implies condition (1). As in the proof of Theorem 14 let U_φ be the operator composition map

$$S \mapsto S \circ (C_\varphi)^*|_V, \quad L(V, \ell^2) \to L(V, \ell^2),$$

where $V = [\delta_z : z \in \mathbb{D}] = H^2$. We get that $U_\varphi = \chi^{-1} \circ \widetilde{C_\varphi} \circ \chi$ is weakly compact $L(H^2, \ell^2) \to L(H^2, \ell^2)$. It is known, see e.g. [44, Example 2.6], that for such operator compositions this yields the compactness of $(C_\varphi)^*|_V$ on V. Hence C_φ is compact $H^2 \to H^2$, so that (1) holds. $\square$

The corresponding picture for the general class $wE(X)$ is quite complicated for reflexive E, and remains open, since the spaces $wE(X)$ can also be reflexive. For instance, the weak Hardy spaces $wH^2(\ell^p) \approx L(H^2, \ell^p) = K(H^2, \ell^p)$ are reflexive for $1 < p < 2$ by Pitt's theorem [33, Proposition 2.c.3] and [23, Section 2, Corollary 2]. Here $K(X, Y)$ denotes the space of compact operators $X \to Y$.

6. Compositions from weak to strong spaces

A different line of study concerns the mapping properties of composition operators from weak to strong spaces of analytic functions on $\mathbb{D}$, such as $wH^p(X) \to H^p(X)$. This line was initiated by Laitila, Tylli and Wang in [30] for the Hardy and Bergman spaces, and subsequently the approach has been extended to weighted Bergman and Dirichlet spaces by Wang [55, 56, 57]. The question which motivated [30] came from S. Kaijser for $X = \ell^2$. Recall

from Section 5 that e.g. $wH^2(\ell^2) \approx L(\ell^2)$ while $H^2(\ell^2)$ is a separable Hilbert space, so that boundedness of a composition operator $wH^2(\ell^2) \to H^2(\ell^2)$ entails strong compression.

Somewhat surprisingly, the boundedness of $C_\varphi \colon wH^p(X) \to H^p(X)$ for $2 \le p < \infty$ is related to composition operators in the Hilbert–Schmidt class on H^2. Recall from [48, 11] that C_φ is a Hilbert–Schmidt operator on H^2 precisely when

$$\|C_\varphi\|_{HS}^2 = \int_{\mathbb{T}} \frac{1}{1 - |\varphi(\zeta)|^2} dm(\zeta) < \infty.$$

The following result is taken from [30], which also contains a formally similar result for the vector-valued Bergman spaces. Note that results of this type have no counterparts in the scalar-valued theory.

Theorem 18. *Let X be any infinite-dimensional complex Banach space.*

(1) *If $\|C_\varphi\|_{HS} < \infty$, then C_φ is bounded $wH^p(X) \to H^p(X)$ for any p satisfying $1 \le p < \infty$.*

(2) *The norm $\|C_\varphi \colon wH^p(X) \to H^p(X)\|$ is equivalent to $\|C_\varphi\|_{HS}^{2/p}$ for $2 < p < \infty$.*

(3) *$\|C_\varphi \colon wH^2(X) \to H^2(X)\| = \|C_\varphi\|_{HS}$.*

Parts (2) and (3) are obtained by explicit computations for $X = \ell^2$. The extension to arbitrary Banach spaces X is based on Dvoretzky's ℓ_2^n-theorem and coefficient multiplier results [14] corresponding to bounded operators

$$\sum_k a_k z^k \mapsto \sum_k \lambda_k a_k z^k, \quad H^2 \to H^p,$$

where (λ_k) is a fixed sequence. By contrast to Theorem 18, $\widetilde{C_\varphi}$ is bounded $wBMOA(\ell^2) \to BMOA(\ell^2)$ if and only if the scalar-valued operator C_φ is bounded $\mathcal{B} \to BMOA$, see [30, Example 4.1], where $\mathcal{B}$ is the Bloch space.

Question 19. (a) Does part (2) of Theorem 18 extend to $1 \le p < 2$? The corresponding coefficient multiplier theorems $H^2 \to H^p$ for $1 \le p < 2$ are not readily useful.

(b) Characterize the weakly compact compositions $\widetilde{C_\varphi}$ from $wH^1(X)$ to $H^1(X)$ if X is reflexive. It is possible to show that if

$$\lim_{s \to 1} \sup_{0 < r < 1} \int_{|\varphi(r\zeta)| > s} \frac{1}{1 - |\varphi(\zeta)|^2} dm(\zeta) = 0,$$

then $\widetilde{C_\varphi}$ is weakly compact from $wH^1(X)$ to $H^1(X)$ (details omitted). Note that $\widetilde{C_\varphi}$ is never compact $wH^1(X) \to H^1(X)$ for infinite-dimensional X by Section 2.

One may also consider the composition operators $f \mapsto f \circ \varphi$ from strong to weak spaces, e.g. as acting $H^p(X) \to wH^p(X)$, but this case does not produce new qualitative phenomena. This follows from the factorization

$$H^p(X) \xrightarrow{\quad \widehat{C_\varphi} \quad} wH^p(X)$$

$$\widehat{C_\varphi} \searrow \qquad \nearrow J$$

$$H^p(X)$$

where $\widehat{C_\varphi}$ denotes the composition operator acting $H^p(X) \to wH^p(X)$ and $J: H^p(X) \to wH^p(X)$ is the continuous inclusion. Hence, any $\widehat{C_\varphi}$ is bounded $H^p(X) \to wH^p(X)$, and for $p = 1$ and reflexive spaces X one obtains that $\widehat{C_\varphi}: H^1(X) \to wH^1(X)$ is weakly compact if and only if φ satisfies (1), that is, $C_\varphi: H^1 \to H^1$ is compact. (For the "only if" part note that Section 2 applies here.)

7. Operator-weighted composition operators

In the final section we briefly discuss extensions of weighted composition operators to the vector-valued setting. Let $\psi: \mathbb{D} \to \mathbb{C}$ and $\varphi: \mathbb{D} \to \mathbb{D}$ be given analytic maps. The weighted composition operator

$$W_{\psi,\varphi}: f \mapsto \psi \cdot (f \circ \varphi)$$

defines a linear map $H(\mathbb{D}) \to H(\mathbb{D})$, where $H(\mathbb{D})$ denotes the linear space of analytic functions $\mathbb{D} \to \mathbb{C}$. Clearly $W_{\psi,\varphi} = M_\psi \circ C_\varphi$, where M_ψ is the pointwise multiplier defined by $M_\psi f = \psi \cdot f$ on $H(\mathbb{D})$. Thus $W_{1,\varphi} = C_\varphi$ and $W_{\psi,id} = M_\psi$.

Weighted composition operators $W_{\psi,\varphi}$ have been extensively studied on a range of complex-valued analytic function spaces, and characterizations of e.g. boundedness and compactness are known for many classical spaces. The case of the weighted spaces H_v^∞ was resolved by Contreras and Hernández-Díaz in [9] and Montes-Rodríguez in [39]. For $1 < p < \infty$ and H^p there is a Carleson measure characterization in [10], and the analogous results for the Bergman space A^p are found in [12]. The case of $BMOA$ can be found in [26]. We refer to e.g. [17] and [19] for other types of function-theoretic conditions for the boundedness of $W_{\psi,\varphi}$ on H^p. Moreover, by [10] all weakly compact weighted compositions $W_{\psi,\varphi}: H^1 \to H^1$ are compact.

Independently Manhas [37] and the authors [29] proposed the following natural analogue of weighted composition operators in the vector-valued setting. Let $\varphi: \mathbb{D} \to \mathbb{D}$ be an analytic self-map and $\psi: \mathbb{D} \to L(X, Y)$ an analytic operator-valued map, where X and Y are complex Banach spaces. Here $L(X, Y)$ denotes the space of bounded linear operators $X \to Y$. Define the

operator-weighted composition operator $W_{\psi,\varphi}$ by $f \mapsto \psi(f \circ \varphi)$, that is,

$$(W_{\psi,\varphi}(f))(z) = \psi(z)(f(\varphi(z)), \quad z \in \mathbb{D},$$

for analytic functions $f \colon \mathbb{D} \to X$. Note that $z \mapsto \psi(z)(f(\varphi(z))$ is an analytic map $\mathbb{D} \to Y$, so that $W_{\psi,\varphi}$ is a linear map $H(\mathbb{D}, X) \to H(\mathbb{D}, Y)$. Again $W_{\psi,\varphi} = M_\psi \circ C_\varphi$, where M_ψ denotes the operator-valued pointwise multiplier defined by $(M_\psi f)(z) = \psi(z)(f(z))$ from $H(\mathbb{D}, X)$ to $H(\mathbb{D}, X)$. Thus the operator-weighted composition operators form a much larger class compared to its scalar-valued relative. Operator-weighted compositions appear naturally: for a large class of Banach spaces X all linear onto isometries $H^\infty(X) \to H^\infty(X)$ have the form $W_{\psi,\varphi}$, where $\psi(z) \equiv U$ is a fixed onto isometry of X and φ is an automorphism of $\mathbb{D}$, see [32] and [8]. The above definition of $W_{\psi,\varphi}$ is analogous to that of weighted compositions on spaces $C(K, X)$ of continuous functions, see e.g. [22], where K is a compact Hausdorff space.

The present knowledge of operator-weighted compositions is fairly rudimentary and most of the results deal with weighted $H_v^\infty(X)$ spaces and their locally convex variants. Characterizations of boundedness and (weak) compactness of $W_{\psi,\varphi}$ between $H_v^\infty(X)$ spaces were obtained in [29], and results for certain locally convex spaces of analytic vector-valued functions are found in [37] and [6]. Boundedness of the operator multipliers M_ψ in related settings has been considered e.g. in [36, 37, 38, 57].

We next state the main results from [29], which are vector-valued extensions of scalar results from [9] and [39]. For a bounded continuous weight function $v \colon \mathbb{D} \to (0, \infty)$, put

$$\tilde{v}(z) = (\sup\{|f(z)| \colon \|f\|_{H_v^\infty} \le 1\})^{-1}.$$

If $\psi \colon \mathbb{D} \to L(X, Y)$ is a given analytic operator-valued map, then it defines the auxiliary linear map T_ψ by $x \mapsto \psi(\cdot)x$. It follows that T_ψ is bounded $X \to H_w^\infty(Y)$ whenever $W_{\psi,\varphi}$ is bounded $H_v^\infty(X) \to H_w^\infty(Y)$.

Theorem 20. (1) *Let v and w be weight functions. Then*

$$\|W_{\psi,\varphi} \colon H_v^\infty(X) \to H_w^\infty(Y)\| = \sup_{z \in \mathbb{D}} \frac{w(z)}{\tilde{v}(\varphi(z))} \|\psi(z)\|_{L(X,Y)}.$$

(2) *Assume that v and w are radial weight functions. Then $W_{\psi,\varphi}$ is compact (respectively, weakly compact) $H_v^\infty(X) \to H_w^\infty(Y)$ if and only if both*

$$\limsup_{|\varphi(z)| \to 1} \frac{w(z)}{\tilde{v}(\varphi(z))} \|\psi(z)\|_{L(X,Y)} = 0, \tag{4}$$

and T_ψ is compact (respectively, weakly compact) $X \to H_w^\infty(Y)$.

Parts of Theorem 20 were independently obtained in [37] together with other results. Clearly the case $X = Y$ and $\psi(z) \equiv I_X$ yields the boundedness and weak compactness results from [35, 4] for $\widetilde{C_\varphi}$, since T_ψ is then (essentially) I_X.

Theorem 20 points to some striking differences between scalar- and vector-valued weighted compositions as well as between operator-weighted and standard composition operators. For instance, the auxiliary operators T_ψ play no role for scalar weighted compositions. Moreover, $W_{\psi,\varphi}$ can easily be compact $H_v^\infty(X) \to H_w^\infty(Y)$ for infinite-dimensional spaces X and Y. For instance, let $\|\varphi\|_\infty < 1$ and $\psi(z) \equiv U \in K(X, Y)$, whence (4) holds and T_ψ is compact $X \to H_w^\infty(Y)$. Operator-weighted compositions $W_{\psi,\varphi} \colon H_v^\infty(X) \to H_w^\infty(X)$ do not always factor through I_X as in Proposition 1, but (F2) is replaced by the following factorization for any $z \in \mathbb{D}$:

$$
\begin{array}{ccc}
H_v^\infty(X) & \xrightarrow{\;W_{\psi,\varphi}\;} & H_w^\infty(Y) \\
{\scriptstyle j}\big\uparrow & & \big\downarrow{\scriptstyle \delta_z} \\
X & \xrightarrow{\;\;\psi(z)\;\;} & Y
\end{array}
$$

There are also examples where $W_{\psi,\varphi}$ is weakly compact $H_v^\infty(X) \to H_w^\infty(X)$ but not compact. For this one may use the fact [29, Theorem 4.4] that if w is a radial weight and $\psi \in H_w^0(L(X, Y)))$, then T_ψ is compact (respectively, weakly compact) if and only if

$$
\psi(\mathbb{D}) \subset K(X, Y) \quad (\text{respectively}, \psi(\mathbb{D}) \subset W(X, Y)).
$$

Here $H_w^0(L(X, Y)))$ denotes the closure of the analytic $L(X, Y)$-valued polynomials in $H_w^\infty(L(X, Y)))$, and $W(X, Y)$ the weakly compact operators $X \to Y$. There are further differences between $H_v^\infty(X)$ and the locally convex setting as studied in [6].

The following problem stated in [29] appears quite challenging.

Question 21. Characterize boundedness and (weak) compactness of operator-weighted compositions $W_{\psi,\varphi} \colon H^p(X) \to H^p(Y)$ as well as $A^p(X) \to A^p(Y)$ for $1 \le p < \infty$. The argument from [10] based on Carleson measure techniques does not readily extend to the vector-valued setting.

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