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On the location of zeros of a polynomial with restricted coefficients

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Abstract. Let $P(z) = \sum_{j=0}^{n} a_j z^j$, where $a_0 > 0$ and $a_j \geq a_{j-1}, j = 1, 2, \cdots, n$. Then, by a classical result of Eneström–Kakeya, all the zeros of $P(z)$ lie in $|z| \leq 1$. In this paper, we prove some extensions and generalizations of this result.

1. Introduction and statements of results

Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n. Then concerning the distribution of zeros of $P(z)$, Eneström and Kakeya [10, 11] proved the following interesting result.

Theorem A. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *such that*

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0. \tag{1}$$

Then $P(z)$ *has all its zeros in* $|z| \leq 1$.

In the literature [1–11] there exist several extensions and generalizations of this theorem. Joyal et al. [9] extended Theorem A to the polynomials whose coefficients are monotonic but not necessarily non-negative. In fact they proved the following result.

Theorem B. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *such that*

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0.$$

Then $P(z)$ *has all its zeros in the disk*

$$|z| \leq \frac{1}{|a_n|} \left(|a_n| - a_0 + |a_0|\right).$$

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Govil and Rahman [8] extended the result to the class of polynomials with complex coefficients by proving the following result.

Theorem C. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients such that for some real β,*

$$|\arg a_j - \beta| \le \alpha \le \frac{\pi}{2}, \quad 0 \le j \le n,$$

and

$$|a_n| \ge |a_{n-1}| \ge \cdots \ge |a_1| \ge |a_0|.$$

Then $P(z)$ has all its zeros in the disk

$$|z| \le (\sin\alpha + \cos\alpha) + \frac{2\sin\alpha}{|a_n|} \sum_{j=0}^{n-1} |a_j|.$$

Aziz and Zargar [2] relaxed the hypothesis of Theorem A and proved the following extension of Theorem A.

Theorem D. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $k \ge 1$,*

$$k a_n \ge a_{n-1} \ge \cdots \ge a_1 \ge a_0 > 0. \tag{2}$$

Then $P(z)$ has all its zeros in $|z + k - 1| \le k$.

In this paper we prove some generalizations and extensions of the above theorems. In this direction we first present the following result which is a generalization of Theorem B.

Theorem 1. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients. For $j = 0, 1, \cdots, n$, let $\operatorname{Re} a_j = \alpha_j$ and $\operatorname{Im} a_j = \beta_j$. If for some real t and for some $\lambda \in \{0, 1, \cdots, n-1\}$,*

$$t + \alpha_n \le \alpha_{n-1} \le \cdots \le \alpha_\lambda \ge \alpha_{\lambda-1} \ge \cdots \ge \alpha_1 \ge \alpha_0$$

and

$$\beta_n \ge \beta_{n-1} \ge \cdots \ge \beta_1 \ge \beta_0 > 0,$$

then all the zeros of $P(z)$ lie in the union of disks $|z| \le 1$ and

$$\left| z + \frac{t}{a_n} \right| \le \frac{1}{|a_n|} \left\{ 2\alpha_\lambda - (\alpha_n + t) - \alpha_0 + |\alpha_0| + \beta_n \right\}. \tag{3}$$

Taking $t = -(1-k)\alpha_n,\ 0 < k \le 1$, in Theorem 1, we obtain the following result.

Corollary 1. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients. For $j = 0, 1, \cdots, n$, let $\operatorname{Re} a_j = \alpha_j$ and $\operatorname{Im} a_j = \beta_j$. If for some $0 < k \le 1$ and for some $\lambda \in \{0, 1, \cdots, n-1\}$,*

$$k\alpha_n \le \alpha_{n-1} \le \cdots \le \alpha_\lambda \ge \alpha_{\lambda-1} \ge \cdots \ge \alpha_1 \ge \alpha_0$$

and

$$\beta_n \geq \beta_{n-1} \geq \cdots \geq \beta_1 \geq \beta_0 > 0,$$

then all the zeros of $P(z)$ *lie in the union of disks* $|z| \leq 1$ *and*

$$\left| z - \frac{\alpha_n}{a_n}(1-k) \right| \leq \frac{1}{|a_n|} \left\{ 2\alpha_\lambda - k\alpha_n - \alpha_0 + |\alpha_0| + \beta_n \right\}. \tag{4}$$

If $\alpha_0 > 0$, then we get the following result.

Corollary 2. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *with complex coefficients. For* $j = 0, 1, \cdots, n$, *let* $\operatorname{Re} a_j = \alpha_j$ *and* $\operatorname{Im} a_j = \beta_j$. *If for some real* t *and for some* $\lambda \in \{0, 1, \cdots, n-1\}$,

$$t + \alpha_n \leq \alpha_{n-1} \leq \cdots \leq \alpha_\lambda \geq \alpha_{\lambda-1} \geq \cdots \geq \alpha_1 \geq \alpha_0 > 0$$

and

$$\beta_n \geq \beta_{n-1} \geq \cdots \geq \beta_1 \geq \beta_0 > 0,$$

then all the zeros of $P(z)$ *lie in the union of disks* $|z| \leq 1$ *and*

$$\left| z + \frac{t}{a_n} \right| \leq \frac{1}{|a_n|} \left\{ 2\alpha_\lambda - (\alpha_n + t) + \beta_n \right\}. \tag{5}$$

Instead of proving Theorem 1, we prove the following more general result.

Theorem 2. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *with complex coefficients. For* $j = 0, 1, \cdots, n$, *let* $\operatorname{Re} a_j = \alpha_j$ *and* $\operatorname{Im} a_j = \beta_j$. *If for some reals* t, s, *and for some* $\lambda \in \{0, 1, \cdots, n-1\}$,

$$t + \alpha_n \leq \alpha_{n-1} \leq \cdots \leq \alpha_\lambda \geq \alpha_{\lambda-1} \geq \cdots \geq \alpha_1 \geq \alpha_0 - s$$

and

$$\beta_n \geq \beta_{n-1} \geq \cdots \geq \beta_1 \geq \beta_0 > 0,$$

then all the zeros of $P(z)$ *lie in the union of disks* $|z| \leq 1$ *and*

$$\left| z + \frac{t}{a_n} \right| \leq \frac{1}{|a_n|} \left\{ 2\alpha_\lambda - (\alpha_n + t) - \alpha_0 + 2s) + |\alpha_0| + \beta_n \right\}. \tag{6}$$

For $s = 0$, Theorem 2 reduces to Theorem 1. For $t = 0$, Theorem 2 reduces to the following result.

Corollary 3. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *with complex coefficients. For* $j = 0, 1, \cdots, n$, *let* $\operatorname{Re} a_j = \alpha_j$ *and* $\operatorname{Im} a_j = \beta_j$. *If for some real* s *and for some* $\lambda \in \{0, 1, \cdots, n-1\}$,

$$\alpha_n \leq \alpha_{n-1} \leq \cdots \leq \alpha_\lambda \geq \alpha_{\lambda-1} \geq \cdots \geq \alpha_1 \geq \alpha_0 - s$$

and

$$\beta_n \geq \beta_{n-1} \geq \cdots \geq \beta_1 \geq \beta_0 > 0,$$

then all the zeros of $P(z)$ *lie in the union of disks* $|z| \leq 1$ *and*

$$|z| \leq \frac{1}{|a_n|} \left\{ 2\alpha_\lambda - \alpha_n - \alpha_0 + 2s + |\alpha_0| + \beta_n \right\}. \tag{7}$$

Finally we present the following result for the polynomials with real coefficients.

Theorem 3. *Let* $P(z)=\sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n*. If for some positive numbers* λ *and* μ

$$\lambda+a_n\geq a_{n-1}\geq\cdots\geq a_0-\mu\geq 0,$$

then all the zeros of $P(z)$ *lie in the closed disk*

$$\left|z+\frac{\lambda}{a_n}\right|\leq\frac{1}{a_n}(a_n+\lambda+2\mu). \tag{8}$$

For $\lambda=(k-1)a_n,\ k\geq 1$, and $\mu=(1-\rho)a_0,\ 0<\rho\leq 1$, Theorem 2 gives a generalization of Theorem C. Also, for $\lambda=0=\mu$, it reduces to the Eneström–Kakeya theorem.

2. Proofs of the theorems

Proof of Theorem 2. Consider the polynomial

$$\begin{aligned}F(z)&=(1-z)P(z)\\&=(1-z)\left(a_nz^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0\right)\\&=-a_nz^{n+1}+(a_n-a_{n-1})z^n+(a_{n-1}-a_{n-2})z^{n-1}+\cdots\\&\quad+(a_1-a_0)z+a_0\\&=-z^n\left(a_nz+t\right)+\left\{\left(\alpha_n+t-\alpha_{n-1}\right)z^n+\cdots\right.\\&\quad\left.+\left(\alpha_1-\alpha_0+s\right)z-sz+\alpha_0\right\}\\&\quad+i\left\{\left(\beta_n-\beta_{n-1}\right)z^n+\cdots+\left(\beta_1-\beta_0\right)z+\beta_0\right\}.\end{aligned}$$

This gives

$$\begin{aligned}|F(z)|&\geq|z|^n\left|a_nz+t\right|-\big\{\left|\alpha_n+t-\alpha_{n-1}\right||z|^n+\left|\alpha_{n-1}-\alpha_{n-2}\right||z|^{n-1}+\cdots\\&\quad+\left|\alpha_{\lambda+1}-\alpha_\lambda\right||z|^{\lambda+1}+\left|\alpha_\lambda-\alpha_{\lambda-1}\right||z|^\lambda+\cdots+\left|\alpha_1-(\alpha_0-s)\right||z|\\&\quad+s|z|+|\alpha_0|+\left|\beta_n-\beta_{n-1}\right||z|^n+\left|\beta_{n-1}-\beta_{n-2}\right||z|^{n-1}+\cdots\\&\quad+\left|\beta_1-\beta_0\right||z|+|\beta_0|\big\}\\&=|z|^n\Bigg[\left|a_nz+t\right|-\Bigg\{\left|\alpha_n+t-\alpha_{n-1}\right|+\frac{\left|\alpha_{n-1}-\alpha_{n-2}\right|}{|z|}+\cdots\\&\quad+\frac{\left|\alpha_{\lambda+1}-\alpha_\lambda\right|}{|z|^{n-\lambda-1}}+\frac{\left|\alpha_\lambda-\alpha_{\lambda-1}\right|}{|z|^{n-\lambda}}+\cdots\\&\quad+\frac{\left|\alpha_1-(\alpha_0-s)\right|}{|z|^{n-1}}+\frac{s}{|z|^{n-1}}+\frac{|\alpha_0|}{|z|^n}+\left|\beta_n-\beta_{n-1}\right|\\&\quad+\frac{\left|\beta_{n-1}-\beta_{n-2}\right|}{|z|}+\cdots+\frac{\left|\beta_1-\beta_0\right|}{|z|^{n-1}}+\frac{|\beta_0|}{|z|^n}\Bigg\}\Bigg].\end{aligned}$$

Now, let $|z| \geq 1$, so that $\frac{1}{|z|^{n-j}} \leq 1,\ 0 \leq j \leq n$. Then we have

$$\begin{aligned}
|F(z)| \geq |z|^n & [|a_n z + t| - \{|\alpha_n + t - \alpha_{n-1}| + |\alpha_{n-1} - \alpha_{n-2}| + \cdots \\
& + |\alpha_{\lambda+1} - \alpha_\lambda| + |\alpha_\lambda - \alpha_{\lambda-1}| + \cdots + |\alpha_1 - (\alpha_0 - s)| + (s + |\alpha_0|) \\
& + |\beta_n - \beta_{n-1}| + |\beta_{n-1} - \beta_{n-2}| + \cdots + |\beta_1 - \beta_0| + |\beta_0|\}] \\
= |z|^n & [|a_n z + t| - \{-\alpha_n - t + \alpha_{n-1} - \alpha_{n-1} + \alpha_{n-2} - \cdots \\
& - \alpha_{\lambda+1} + \alpha_\lambda + \alpha_\lambda - \alpha_{\lambda-1} + \cdots + \alpha_1 - (\alpha_0 - s) + (s + |\alpha_0|) \\
& + \beta_n - \beta_{n-1} + \beta_{n-1} - \beta_{n-2} + \cdots + \beta_1 - \beta_0 + \beta_0\}] \\
= |z|^n & [|a_n z + t| - \{-\alpha_n - t + 2\alpha_\lambda - (\alpha_0 - s) + (s + |\alpha_0|) + \beta_n\}] \\
> 0 &
\end{aligned}$$

if

$$|a_n z + t| > \{-\alpha_n - t + 2\alpha_\lambda - (\alpha_0 - s) + (s + |\alpha_0|) + \beta_n\},$$

i.e., if

$$\left| z + \frac{t}{a_n} \right| > \frac{1}{|a_n|} \{2\alpha_\lambda - t - \alpha_n - \alpha_0 + s + s + |\alpha_0| + \beta_n\}.$$

Thus all the zeros of $F(z)$ whose modulus is greater than or equal to 1 lie in

$$\left| z + \frac{t}{a_n} \right| \leq \frac{1}{|a_n|} \{2\alpha_\lambda - t - \alpha_n - \alpha_0 + 2s + |\alpha_0| + \beta_n\}.$$

But all the zeros of $P(z)$ are also the zeros of $F(z)$. Hence it follows that all the zeros of $F(z)$ and hence of $P(z)$ lie in the union of disks $|z| \leq 1$ and

$$\left| z + \frac{t}{a_n} \right| \leq \frac{1}{|a_n|} \{2\alpha_\lambda - t - \alpha_n - \alpha_0 + 2s + |\alpha_0| + \beta_n\}.$$

This completes the proof of Theorem 2. □

Proof of Theorem 3. Consider the polynomial

$$\begin{aligned}
F(z) &= (1 - z)P(z) \\
&= -a_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots \\
&\quad + (a_1 - a_0)z + a_0 \\
&= -a_n z^{n+1} - \lambda z^n + (a_n + \lambda - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} \\
&\quad + \cdots + (a_1 - a_0 + \mu)z - \mu z + a_0 \\
&= -z^n (a_n z + \lambda) + (a_n + \lambda - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} \\
&\quad + \cdots + (a_1 - a_0 + \mu)z - \mu z + a_0.
\end{aligned}$$

This gives

$$\begin{aligned}|F(z)| &\geq |z|^n\,|a_n z+\lambda| - \{|a_n+\lambda-a_{n-1}|\,|z|^n \\ &\quad + |a_{n-1}-a_{n-2}|\,|z|^{n-1} + \cdots + |a_1-a_0+\mu||z| + \mu|z| + |a_0|\} \\ &= |z|^n\left\{|a_n z+\lambda| - \left(|a_n+\lambda-a_{n-1}| + \frac{|a_{n-1}-a_{n-2}|}{|z|} + \cdots \right.\right. \\ &\quad \left.\left. + \frac{|a_1-a_0+\mu|}{|z|^{n-1}} + \frac{\mu}{|z|^{n-1}} + \frac{|a_0|}{|z|^n}\right)\right\}.\end{aligned}$$

Now, let $|z| \geq 1$, so that $\frac{1}{|z|^{n-j}} \leq 1,\ 0 \leq j \leq n$. Then we have

$$\begin{aligned}|F(z)| &\geq |z|^n\,\{|a_n z+\lambda| - (|a_n+\lambda-a_{n-1}| + |a_{n-1}-a_{n-2}| + \cdots \\ &\quad + |a_1-a_0+\mu| + \mu + |a_0|)\} \\ &= |z|^n\,\{|a_n z+\lambda| - (a_n+\lambda-a_{n-1}+a_{n-1}-a_{n-2}+\cdots \\ &\quad + a_1-a_0+\mu+\mu+a_0)\} \\ &= |z|^n\,\{|a_n z+\lambda| - (a_n+\lambda+2\mu)\} > 0\end{aligned}$$

if

$$|a_n z+\lambda| > (a_n+\lambda+2\mu)\,,$$

i.e., if

$$\left|z+\frac{\lambda}{a_n}\right| > \frac{1}{a_n}(a_n+\lambda+2\mu)\,.$$

Thus all the zeros of $F(z)$ whose modulus is greater than or equal to 1 lie in

$$\left|z+\frac{\lambda}{a_n}\right| \leq \frac{1}{a_n}(a_n+\lambda+2\mu)\,.$$

But those zeros of $F(z)$ whose modulus is less than 1 already satisfy the above inequality. Indeed, for $|z| \leq 1$, we have

$$\left|z+\frac{\lambda}{a_n}\right| \leq |z| + \frac{\lambda}{|a_n|} \leq 1 + \frac{\lambda}{a_n} + \frac{2\mu}{a_n} = \frac{1}{a_n}(a_n+\lambda+2\mu)\,.$$

Also all the zeros of $P(z)$ are the zeros of $F(z)$. Hence it follows that all the zeros of $F(z)$ and hence of $P(z)$ lie in

$$\left|z+\frac{\lambda}{a_n}\right| \leq \frac{1}{a_n}(a_n+\lambda+2\mu)$$

This completes the proof of Theorem 3. □

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