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Semi-symmetric metric connections on pseudosymmetric Lorentzian α-Sasakian manifolds

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Abstract. We consider semi-symmetric metric connections on pseudosymmetric Lorentzian α-Sasakian manifolds. We study some properties of Weyl pseudosymmetric and Ricci pseudosymmetric Lorentzian α-Sasakian manifolds. We also give an example of a pseudosymmetric Lorentzian α-Sasakian manifold with a semi-symmetric metric connection.

1. Introduction

The concept of a pseudosymmetric manifold was introduced by M.C. Chaki and B. Chaki (see [3]) and R. Deszcz (see [8]) in two different ways. Various properties of pseudosymmetric manifolds in various metric structures have been studied in both senses (see [3] – [7]). The two types of pseudosymmetric manifolds are different in their nature. We shall study properties of pseudosymmetric manifolds and Ricci pseudosymmetric manifolds with a semi-symmetric metric connection in the Deszcz sense.

A Riemannian manifold (M, g) of dimension n is called *pseudosymmetric* if the Riemannian curvature tensor R satisfies, for all vector fields X, Y, U, V, W on M, the conditions (see [1])

1. $(R(X,Y).R)(U,V,W) = L_R\left[((X\wedge Y).R)(U,V,W)\right]$,
2. $(R(X,Y).R)(U,V,W) = R(X,Y)(R(U,V)W) - R(R(X,Y)U,V)W - R(U,R(X,Y)V)W - R(U,V)(R(X,Y)W)$,
3. $((X\wedge Y).R)(U,V,W) = (X\wedge Y)(R(U,V)W) - R((X\wedge Y)U,V)W - R(U,(X\wedge Y)V)W - R(U,V)((X\wedge Y)W)$,

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where $L_R \in C^\infty(M)$,

$$R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]\, Z,$$

and $X \wedge Y$ is an endomorphism which is defined by

$$(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y. \tag{1}$$

The manifold M is said to be *pseudosymmetric of constant type* if L is constant. A Riemannian manifold (M,g) is called *semi-symmetric* if $R.R = 0$, where $R.R$ is the derivative of R by R.

Remark 1.1. From [2] and [8], we know that, for a $(0,k)$-tensor field T, the $(0,k+2)$ tensor fields $R.T$ and $Q(g,T)$ are defined by

$$\begin{aligned}(R.T)(X_1,\dots,X_k;X,Y) &= (R(X,Y).T)(X_1,\dots,X_k)\\ &= -T(R(X,Y)X_1,\dots,X_k) - \cdots - T(X_1,\dots,R(X,Y)X_k)\end{aligned}$$

and

$$\begin{aligned}Q(g,T)(X_1,\dots,X_k;X,Y) &= -((X\wedge Y).T)(X_1,\dots,X_k)\\ &= T((X\wedge Y)X_1,\dots,X_k) + \cdots + T(X_1,\dots,(X\wedge Y)X_k).\end{aligned}$$

Let S and r denote the Ricci tensor and the scalar curvature tensor of M respectively. The operator Q and the $(0,2)$-tensor S^2 are defined by

$$S(X,Y) = g(QX,Y)$$

and

$$S^2(X,Y) = S(QX,Y). \tag{2}$$

The Weyl conformal curvature operator C is defined by

$$C(X,Y) = R(X,Y) - \frac{1}{n-2}\left[X\wedge QY + QX\wedge Y - \frac{r}{n-1}X\wedge Y\right].$$

If $C = 0$, $n \geq 3$, then M is called *conformally flat*. If the tensors $R.C$ and $Q(g,C)$ are linearly dependent, then M is called *Weyl pseudosymmetric*. This is equivalent to the statement that

$$(R.C)(U,V,W;X,Y) = L_C\left[((X\wedge Y).C)(U,V)W\right]$$

holds on the set

$$U_C = \{x \in M \colon\ C \neq 0 \text{ at } x\},$$

where L_C is defined on U_C. If $R.C = 0$, then M is called *Weyl semi-symmetric*. If $\nabla C = 0$, then M is called *conformally symmetric* (see [10], [9]).

2. Preliminaries

A differentiable manifold M of dimension n is said to be a *Lorentzian α-Sasakian manifold* if it admits a $(1,1)$-tensor field ϕ, a vector field ξ, a one-form η, and Lorentzian metric g which satisfy the conditions

$$\begin{aligned} \phi^2 &= I + \eta \otimes \xi, \\ \eta(\xi) &= -1,\ \phi\xi = 0,\ \eta \circ \phi = 0, \\ g(\phi X, \phi Y) &= g(X,Y) + \eta(X)\eta(Y), \\ g(X,\xi) &= \eta(X), \\ (\nabla_X \phi)(Y) &= \alpha\{g(X,Y)\xi + \eta(Y)X\}, \end{aligned} \tag{3}$$

for all X, $Y \in \chi(M)$, where $\chi(M)$ is the Lie algebra of smooth vector fields on M, α is smooth functions on M, and ∇ denotes the covariant differentiation operator of Lorentzian metric g (see [11], [10]).

On a Lorentzian α-Sasakian manifold, it can be shown that (see [11], [10])

$$\begin{aligned} \nabla_X \xi &= \alpha\phi X, \\ (\nabla_X \eta)Y &= \alpha g(\phi X, Y). \end{aligned}$$

Moreover, on a Lorentzian α-Sasakian manifold the following relations hold (see [10]):

$$\eta(R(X,Y)Z) = \alpha^2 \left[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\right], \tag{4}$$

$$R(\xi, X)Y = \alpha^2 \left[g(X,Y)\xi - \eta(Y)X\right], \tag{5}$$

$$R(X,Y)\xi = \alpha^2 \left[\eta(Y)X - \eta(X)Y\right], \tag{6}$$

$$S(\xi, X) = S(X,\xi) = (n-1)\alpha^2\eta(X), \tag{7}$$

$$S(\xi,\xi) = -(n-1)\alpha^2, \tag{8}$$

$$Q\xi = (n-1)\alpha^2\xi.$$

The equalities (4) – (8) will be required in the next section.

3. Semi-symmetric metric connection on a Lorentzian α-Sasakian manifold

Let M be a Lorentzian α-Sasakian manifold with Levi–Civita connection ∇ and let X, Y, $Z \in \chi(M)$. We define a linear connection D on M by

$$D_X Y = \nabla_X Y + \eta(Y)X - g(X,Y)\xi, \tag{9}$$

where η is 1-form and ϕ is a tensor field of type $(1,1)$. The connection D is said to be *semi-symmetric* if $\bar{T}$, the torsion tensor of the connection D, satisfies

$$\bar{T}(X,Y) = \eta(Y)X - \eta(X)Y, \tag{10}$$

and *metric* if

$$(D_X g)(Y, Z) = 0. \tag{11}$$

The connection D is said to be *semi-symmetric metric* if it satisfies (9), (10) and (11).

We shall show the existence of a semi-symmetric metric connection D on a Lorentzian α-Sasakian manifold M.

Theorem 3.1. *Let X, Y, Z be vector fields on a Lorentzian α-Sasakian manifold M. Define a connection D by*

$$\begin{aligned} 2g(D_X Y, Z) = {} & Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) + g([X, Y], Z) \\ & - g([Y, Z], X) + g([Z, X], Y) + g(\eta(Y)X - \eta(X)Y, Z) \\ & + g(\eta(X)Z - \eta(Z)X, Y) + g(\eta(Y)Z - \eta(Z)Y, X). \end{aligned} \tag{12}$$

Then D is a semi-symmetric metric connection on M.

Proof. It can be verified that D is a linear connection on M. From (12), we have

$$g(D_X Y, Z) - g(D_Y X, Z) = g([X, Y], Z) + \eta(Y)g(X, Z) - \eta(X)g(Y, Z)$$

or

$$D_X Y - D_Y X - [X, Y] = \eta(Y)X - \eta(X)Y$$

or

$$\bar{T}(X, Y) = \eta(Y)X - \eta(X)Y.$$

Again, from (12), we get

$$2g(D_X Y, Z) + 2g(D_X Z, Y) = 2Xg(Y, Z)$$

or

$$(D_X g)(Y, Z) = 0.$$

This shows that D is a semi-symmetric metric connection on M. □

4. Curvature tensor and Ricci tensor of semi-symmetric metric connection D on a Lorentzian α-Sasakian manifold

Let $\bar{R}(X, Y)Z$ and $R(X, Y)Z$ be the curvature tensors on a Lorentzian α-Sasakian manifold M of a semi-symmetric metric connection D and of the Riemannian connection ∇, respectively. A relation between $\bar{R}(X, Y)Z$ and $R(X, Y)Z$ is given by

$$\begin{aligned} \bar{R}(X, Y)Z = {} & R(X, Y)Z + \alpha\,[g(\phi X, Z)Y - g(\phi Y, Z)X \\ & + g(X, Z)\phi Y - g(Y, Z)\phi X] + \eta(Z)\,[\eta(Y)X - \eta(X)Y] \\ & + g(Y, Z)\phi^2 X - g(X, Z)\phi^2 Y. \end{aligned} \tag{13}$$

From (13) we obtain

$$\bar{S}(X, Y) = S(X, Y) + (n-2)\,[g(\phi X, \phi Y) - \alpha g(\phi X, Y)], \tag{14}$$

where $\bar{S}$ and S are the Ricci tensors of the connections D and ∇, respectively. Again

$$\begin{aligned}\bar{S}^2(X,Y) = S^2(X,Y) &+ (n-2)\left[\{S(\phi X,\phi Y) + S(\phi^2 X,Y)\}\right.\\ &\left.-2\alpha S(\phi X,Y)\right] + (n-2)^2\left[(\alpha^2+1)g(\phi X,\phi Y)\right.\\ &\left.-2\alpha g(\phi X,Y)\right].\end{aligned} \tag{15}$$

Contracting (15), we get

$$\bar{r} = r + (n-1)(n-2), \tag{16}$$

where $\bar{r}$ and r are the scalar curvatures of the connections D and ∇, respectively.

Let $\bar{C}$ be the conformal curvature tensor of the connection D. Then

$$\begin{aligned}\bar{C}(X,Y)Z = \bar{R}(X,Y)Z &- \frac{1}{n-2}\left[\bar{S}(Y,Z)X\right.\\ &\left.-g(X,Z)\bar{Q}Y + g(Y,Z)\bar{Q}X - \bar{S}(X,Z)Y\right]\\ &+\frac{\bar{r}}{(n-1)(n-2)}\left[g(Y,Z)X - g(X,Z)Y\right],\end{aligned} \tag{17}$$

where $\bar{Q}$ is the Ricci operator of the connection D on M and

$$\bar{S}(X,Y) = g(\bar{Q}X,Y), \tag{18}$$

$$\bar{S}^2(X,Y) = \bar{S}(\bar{Q}X,Y). \tag{19}$$

Now we shall prove the following theorem.

Theorem 4.1. *Let M be a Lorentzian α-Sasakian manifold with a semi-symmetric metric connection D. Then the following relations hold:*

$$\bar{R}(\xi,X)Y = \alpha^2\left[g(X,Y)\xi - \eta(Y)X\right] + \alpha\left[\eta(Y)\phi X - g(\phi X,Y)\xi\right], \tag{20}$$

$$\begin{aligned}\eta(\bar{R}(X,Y)Z) = \alpha^2&\left[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\right]\\ &+\alpha\left[g(\phi X,Z)\eta(Y) - g(\phi Y,Z)\eta(X)\right],\end{aligned} \tag{21}$$

$$\bar{R}(X,Y)\xi = \alpha^2\left[\eta(Y)X - \eta(X)Y\right] - \alpha\left[\eta(Y)\phi X - \eta(X)\phi Y\right], \tag{22}$$

$$\bar{S}(X,\xi) = \bar{S}(\xi,X) = (n-1)\alpha^2\eta(X), \tag{23}$$

$$\bar{S}^2(X,\xi) = \bar{S}^2(\xi,X) = \alpha^4(n-1)^2\eta(X), \tag{24}$$

$$\bar{S}(\xi,\xi) = -(n-1)\alpha^2, \tag{25}$$

$$\bar{Q}X = QX + (n-2)\left[\phi^2 X - \alpha\phi X\right], \tag{26}$$

$$\bar{Q}\xi = (n-1)\alpha^2\xi. \tag{27}$$

Proof. Since M is a Lorentzian α-Sasakian manifold and D is a semi-symmetric metric connection, replacing $X=\xi$ in (13) and using (3) and (5), we get (20). Using (3) and (4), from (13), we get (21). To prove (22), we put $Z=\xi$ in (13) and then we use (6). Replacing $Y=\xi$ in (14) and using

(7), we get (23). Putting $Y = \xi$ in (15) and using (2) and (7), we get (24). Again, putting $X = Y = \xi$ in (14) and using (8), we get (25). Using (18) and (23), we get (26). Now, putting $X = \xi$ in (26), we get (27). □

5. Lorentzian α-Sasakian manifold with a semi-symmetric metric connection D satisfying the condition $\bar{C}.\bar{S} = 0$

In this section we shall find out characterization of Lorentzian α-Sasakian manifold with a semi-symmetric metric connection D satisfying the condition $\bar{C}.\bar{S} = 0$, where

$$(\bar{C}(X,Y).\bar{S})(Z,W) = -\bar{S}(\bar{C}(X,Y)Z,W) - \bar{S}(Z,\bar{C}(X,Y)W) \tag{28}$$

with $X, Y, Z, W \in \chi(M)$.

Theorem 5.1. *Let M be an n-dimensional Lorentzian α-Sasakian manifold with a semi-symmetric metric connection D. If $\bar{C}.\bar{S} = 0$, then*

$$\begin{aligned}\frac{1}{n-2}\bar{S}^2(X,Y) = &\left[\alpha^2 + 1 + \frac{r}{(n-1)(n-2)}\right]\left[\bar{S}(\phi X,Y)\right.\\ &\left.-\alpha^2(n-1)g(\phi X,Y)\right] - \alpha\left[\bar{S}(X,Y)\right.\\ &\left.-\alpha^2(n-1)g(X,Y)\right] + \frac{\alpha^4(n-1)^2}{n-2}g(X,Y).\end{aligned} \tag{29}$$

Proof. From (28), we get

$$\bar{S}(\bar{C}(X,Y)Z,W) + \bar{S}(Z,\bar{C}(X,Y)W) = 0, \tag{30}$$

where $X, Y, Z, W \in \chi(M)$. Now, putting $X = \xi$ in (29), we get

$$\bar{S}(\bar{C}(\xi,X)Y,Z) + \bar{S}(Y,\bar{C}(\xi,X)Z) = 0. \tag{31}$$

Using (17), (19), (20) and (23), we have that

$$\begin{aligned}\bar{S}(\bar{C}(\xi,X)Y,Z) = &\left[\alpha^2 - \frac{(n-1)\alpha^2}{n-2} + \frac{\bar{r}}{(n-1)(n-2)}\right]\\ &\times\left[(n-1)\alpha^2\eta(Z)g(X,Y) - \eta(Y)\bar{S}(X,Z)\right]\\ &-\alpha\left[(n-1)\alpha^2\eta(Z)g(\phi X,Y) - \eta(Y)\bar{S}(\phi X,Z)\right]\\ &-\frac{1}{n-2}\left[(n-1)\alpha^2\eta(Z)\bar{S}(X,Y) - \bar{S}^2(X,Z)\eta(Y)\right]\end{aligned} \tag{32}$$

and

$$\begin{aligned}\bar{S}(Y,\bar{C}(\xi,X)Z) = &\left[\alpha^2 - \frac{(n-1)\alpha^2}{n-2} + \frac{\bar{r}}{(n-1)(n-2)}\right]\\ &\times\left[(n-1)\alpha^2\eta(Y)g(X,Z) - \eta(Z)\bar{S}(X,Y)\right]\\ &-\alpha\left[(n-1)\alpha^2\eta(Y)g(\phi X,Z) - \eta(Z)\bar{S}(\phi X,Y)\right]\\ &-\frac{1}{n-2}\left[(n-1)\alpha^2\eta(Y)\bar{S}(X,Z) - \bar{S}^2(X,Y)\eta(Z)\right].\end{aligned} \tag{33}$$

Using (31) and (32) in (30), we get

$$\begin{aligned}
&\left[\alpha^2 - \frac{(n-1)\alpha^2}{n-2} + \frac{\bar{r}}{(n-1)(n-2)}\right]\left[(n-1)\alpha^2\left\{\eta(Z)g(X,Y)\right.\right.\\
&\quad \left.\left.+\eta(Y)g(X,Z)\right\} - \{\eta(Y)\bar{S}(X,Z) + \eta(Z)\bar{S}(X,Y)\}\right]\\
&\quad - \alpha\left[(n-1)\alpha^2\{\eta(Z)g(\phi X,Y) + \eta(Y)g(\phi X,Z)\}\right.\\
&\quad \left.-\{\eta(Y)\bar{S}(\phi X,Z) + \eta(Z)\bar{S}(\phi X,Y)\}\right]\\
&\quad - \frac{1}{n-2}\left[(n-1)\alpha^2\{\eta(Z)\bar{S}(X,Y) + \eta(Y)\bar{S}(X,Z)\}\right.\\
&\quad \left.-\{\bar{S}^2(X,Z)\eta(Y) + \bar{S}^2(X,Y)\eta(Z)\}\right] = 0.
\end{aligned} \tag{34}$$

Finally, replacing $Z = \xi$ in (33) and using (23) and (24), we get (29). $\square$

An n-dimensional Lorentzian α-Sasakian manifold M with a semi-symmetric metric connection D is said to be η-*Einstein* if its Ricci tensor $\bar{S}$ is of the form

$$\bar{S}(X,Y) = Ag(X,Y) + B\eta(X)\eta(Y), \tag{35}$$

where A, B are smooth functions on M. We consider the vector fields e_i, $i = 1, 2, \ldots, n$, which forms an orthonormal basis for the tangent space T_xM of M.

Now, putting $X = Y = e_i$, $i = 1, 2, \ldots, n$, in (35) and summing over $i = 1, \ldots, n$, we get

$$An - B = \bar{r}. \tag{36}$$

Again, replacing $X = Y = \xi$ in (35), we have that

$$A - B = (n-1)\alpha^2. \tag{37}$$

Solving (36) and (37), we obtain

$$A = \frac{\bar{r}}{n-1} - \alpha^2 \quad \text{and} \quad B = \frac{\bar{r}}{n-1} - n\alpha^2.$$

Thus the Ricci tensor of an η-Einstein manifold with a semi-symmetric metric connection D is given by

$$\bar{S}(X,Y) = \left[\frac{\bar{r}}{n-1} - \alpha^2\right] g(X,Y) + \left[\frac{\bar{r}}{n-1} - n\alpha^2\right]\eta(X)\eta(Y). \tag{38}$$

6. η-Einstein Lorentzian α-Sasakian manifold with a semi-symmetric metric connection D satisfying the condition $\bar{C}.\bar{S} = 0$

Theorem 6.1. *Let M be an η-Einstein Lorentzian α-Sasakian manifold of dimension n with the restriction $X = Z = \xi$. Then $\bar{C}.\bar{S} = 0$ if and only if*

$$g(\phi Y, \phi W) = -\alpha g(\phi Y, W), \quad Y, W \in \chi(M).$$

Proof. Let M be an η-Einstein Lorentzian α-Sasakian manifold of the semi-symmetric metric connection D satisfying $\bar{C}.\bar{S} = 0$. Using (38) in (30), we get

$$\eta(\bar{C}(X,Y)Z)\eta(W) + \eta(\bar{C}(X,Y)W)\eta(Z) = 0.$$

Further, using (16), (21) and (23) in the above equation, we obtain that

$$\begin{aligned}&\{g(Y,Z)\eta(X)\eta(W) - g(X,Z)\eta(Y)\eta(W)\\&\quad + g(Y,W)\eta(X)\eta(Z) - g(X,W)\eta(Y)\eta(Z)\}\\&\quad - \alpha\{g(\phi Y,Z)\eta(X)\eta(W) - g(\phi X,Z)\eta(Y)\eta(W)\\&\quad + g(\phi Y,W)\eta(X)\eta(Z) - g(\phi X,W)\eta(Y)\eta(Z)\} = 0.\end{aligned}$$

Putting here $X = Z = \xi$, we get

$$g(\phi Y,\phi W) = -\alpha g(\phi Y,W).$$

Conversely,

$$\begin{aligned}\bar{C}.\bar{S} = &\{g(Y,Z)\eta(X)\eta(W) - g(X,Z)\eta(Y)\eta(W)\\&+ g(Y,W)\eta(X)\eta(Z) - g(X,W)\eta(Y)\eta(Z)\}\\&- \alpha\{g(\phi Y,Z)\eta(X)\eta(W) - g(\phi X,Z)\eta(Y)\eta(W)\\&+ g(\phi Y,W)\eta(X)\eta(Z) - g(\phi X,W)\eta(Y)\eta(Z)\}.\end{aligned}$$

Using $X = Z = \xi$ in this equation, we get

$$\bar{C}.\bar{S} = g(Y,W) + \eta(Y)\eta(W) + \alpha g(\phi Y,W).$$

Thus $\bar{C}.\bar{S} = 0$. □

7. Ricci pseudosymmetric Lorentzian α-Sasakian manifold with a semi-symmetric metric connection D

Theorem 7.1. *A Ricci pseudosymmetric Lorentzian α-Sasakian manifold M with a semi-symmetric metric connection D and with restrictions $Y = W = \xi$, $L_{\bar{S}} = \alpha^2$ is an Einstein manifold.*

Proof. Recall that a Lorentzian α-Sasakian manifold M with a semi-symmetric metric connection D is called Ricci pseudosymmetric if

$$(\bar{R}(X,Y).\bar{S})(Z,W) = L_{\bar{S}}\left[((X\wedge Y).\bar{S})(Z,W)\right]$$

or

$$\begin{aligned}&\bar{S}(\bar{R}(X,Y)Z,W) + \bar{S}(Z,\bar{R}(X,Y)W)\\&\quad = L_{\bar{S}}\left[\bar{S}((X\wedge Y)Z,W) + \bar{S}(Z,(X\wedge Y)W)\right].\end{aligned} \tag{39}$$

Putting $Y = W = \xi$, in (39) and using (1), (20) and (23), we have

$$\begin{aligned}&\left[L_{\bar{S}} - \alpha^2\right]\left[\bar{S}(Z,X) - (n-1)\alpha^2 g(Z,X)\right]\\&\quad = -\alpha\left[\bar{S}(Z,\phi X) - (n-1)\alpha^2 g(Z,\phi X)\right].\end{aligned} \tag{40}$$

Then, for $L_{\bar{S}} = \alpha^2$,

$$\bar{S}(Z, \phi X) = (n-1)\alpha^2 g(Z, \phi X).$$

Thus M is an Einstein manifold. □

Corollary 7.1. *If M is a Ricci pseudosymmetric Lorentzian α-Sasakian manifold with a semi-symmetric metric connection D and with restriction $Y = W = \xi$, then*

$$\alpha \left[\bar{S}(Z, X) - (n-1)\alpha^2 g(Z, X)\right] = \bar{S}(Z, \phi X) - (n-1)\alpha^2 g(Z, \phi X). \quad (41)$$

Proof. If M is a Ricci pseudosymmetric Lorentzian α-Sasakian manifold with a semi-symmetric metric connection D, then $L_{\bar{S}} = 0$. Putting $L_{\bar{S}} = 0$ in (40), we get (41). □

8. Pseudosymmetric Lorentzian α-Sasakian manifold and Weyl pseudosymmetric Lorentzian α-Sasakian manifold with semi-symmetric metric connections

In the present section, we shall give the definitions of a pseudosymmetric and a Weyl-pseudosymmetric Lorentzian α-Sasakian manifolds with semi-symmetric metric connections and discuss their properties.

Definition 8.1. A Lorentzian α-Sasakian manifold M with a semi-symmetric metric connection D is said to be *pseudosymmetric* if the curvature tensor $\bar{R}$ of D satisfies the condition

$$(\bar{R}(X, Y).\bar{R})(U, V, W) = L_{\bar{R}} \left[((X \wedge Y).\bar{R})(U, V, W)\right], \quad (42)$$

where

$$\begin{aligned}(\bar{R}(X, Y).\bar{R})(U, V, W) = {} & \bar{R}(X, Y)(\bar{R}(U, V)W) - \bar{R}(\bar{R}(X, Y)U, V)W \\ & - \bar{R}(U, \bar{R}(X, Y)V)W - \bar{R}(U, V)(R(X, Y)W)\end{aligned} \quad (43)$$

and

$$\begin{aligned}((X \wedge Y).\bar{R})(U, V, W) = {} & (X \wedge Y)(\bar{R}(U, V)W) - \bar{R}((X \wedge Y)U, V)W \\ & - \bar{R}(U, (X \wedge Y)V)W - \bar{R}(U, V)((X \wedge Y)W).\end{aligned} \quad (44)$$

Definition 8.2. A Lorentzian α-Sasakian manifold M with a semi-symmetric metric connection D is said to be *Weyl pseudosymmetric* if the curvature tensor $\bar{R}$ of D satisfies the condition

$$(\bar{R}(X, Y).\bar{C})(U, V, W) = L_{\bar{C}} \left[((X \wedge Y).\bar{C})(U, V, W)\right], \quad (45)$$

where

$$\begin{aligned}(\bar{R}(X, Y).\bar{C})(U, V, W) = {} & \bar{R}(X, Y)(\bar{C}(U, V)W) - \bar{C}(\bar{R}(X, Y)U, V)W \\ & - \bar{C}(U, \bar{R}(X, Y)V)W - \bar{C}(U, V)(R(X, Y)W)\end{aligned} \quad (46)$$

and

$$\begin{aligned}((X \wedge Y).\bar{C})(U,V,W) = (X \wedge Y)(\bar{C}(U,V)W) - \bar{C}((X \wedge Y)U,V)W \\ - \bar{C}(U,(X \wedge Y)V)W - \bar{C}(U,V)((X \wedge Y)W).\end{aligned} \tag{47}$$

Theorem 8.1. *Let M be an n-dimensional Lorentzian α-Sasakian manifold. If M is Weyl pseudosymmetric, then M is either conformally flat or $L_{\bar{C}} = \alpha^2$.*

Proof. Let M be Weyl pseudosymmetric and $X, Y, U, V, W \in \chi(M)$. Then, using (45) and (46) in (44), we have

$$\begin{aligned}&\bar{R}(X,Y)(\bar{C}(U,V)W) - \bar{C}(\bar{R}(X,Y)U,V)W \\ &\quad - \bar{C}(U,\bar{R}(X,Y)V)W - \bar{C}(U,V)(R(X,Y)W) \\ &= L_{\bar{C}}\left[(X \wedge Y)(\bar{C}(U,V)W) - \bar{C}((X \wedge Y)U,V)W\right. \\ &\quad \left.-\bar{C}(U,(X \wedge Y)V)W - \bar{C}(U,V)((X \wedge Y)W)\right].\end{aligned}$$

Replacing here X with ξ, we obtain

$$\begin{aligned}&\bar{R}(\xi,Y)(\bar{C}(U,V)W) - \bar{C}(\bar{R}(\xi,Y)U,V)W \\ &\quad - \bar{C}(U,\bar{R}(\xi,Y)V)W - \bar{C}(U,V)(R(\xi,Y)W) \\ &= L_{\bar{C}}\left[(\xi \wedge Y)(\bar{C}(U,V)W) - \bar{C}((\xi \wedge Y)U,V)W\right. \\ &\quad \left.-\bar{C}(U,(\xi \wedge Y)V)W - \bar{C}(U,V)((\xi \wedge Y)W)\right].\end{aligned} \tag{48}$$

Using (1) and (20) in (47), and taking inner product of (47) with ξ, we get

$$\begin{aligned}&\alpha^2\left[-\bar{C}(U,V,W,Y) - \eta(\bar{C}(U,V)W)\eta(Y)\right. \\ &\quad +g(Y,U)\eta(\bar{C}(\xi,V)W) - \eta(U)\eta(\bar{C}(Y,V)W) \\ &\quad \left.+g(Y,V)\eta(\bar{C}(U,\xi)W) - \eta(V)\eta(\bar{C}(U,Y)W) - \eta(W)\eta(\bar{C}(U,V)Y)\right] \\ &\quad + \alpha\left[\bar{C}(U,V,W,\phi Y) + \eta(U)\eta(\bar{C}(\phi Y,V)W) - g(\phi Y,U)\eta(\bar{C}(\xi,V)W)\right. \\ &\quad \left.+\eta(V)\eta(\bar{C}(U,\phi Y)W) - g(\phi Y,V)\eta(\bar{C}(U,\xi)W) + \eta(W)\eta(\bar{C}(U,V)\phi Y)\right] \\ &= L_{\bar{C}}\left[-\bar{C}(U,V,W,Y) - \eta(Y)\eta(\bar{C}(U,V)W) + g(Y,U)\eta(\bar{C}(\xi,V)W)\right. \\ &\quad -\eta(U)\eta(\bar{C}(Y,V)W + g(Y,V)\eta(\bar{C}(U,\xi)W) \\ &\quad \left.-\eta(V)\eta(\bar{C}(U,Y)W) - \eta(W)\eta(\bar{C}(U,V)Y)\right].\end{aligned}$$

Then, putting $Y = U = \xi$, we get

$$\left[L_{\bar{C}} - \alpha^2\right]\eta(\bar{C}(\xi,V)W) = 0.$$

This shows that either $\eta(\bar{C}(\xi,V)W) = 0$ or $L_{\bar{C}} - \alpha^2 = 0$.

Now, if $L_{\bar{C}} - \alpha^2 \neq 0$, then $\eta(\bar{C}(\xi,V)W) = 0$, i.e., M is conformally flat and

$$\bar{S}(V,W) = Ag(V,W) + B\eta(V)\eta(W) - \alpha g(\phi V,W),$$

with

$$A = \left[\alpha^2 - \frac{(n-1)\alpha^2}{n-2} + \frac{\bar{r}}{(n-1)(n-2)}\right](n-2),$$

and

$$B = \left[\alpha^2 - \frac{2(n-1)\alpha^2}{n-2} + \frac{\bar{r}}{(n-1)(n-2)}\right](n-2).$$

But if $\eta(\bar{C}(\xi, V)W) \neq 0$, then we have $L_{\bar{C}} = \alpha^2$. □

Theorem 8.2. *Let M be an n-dimensional Lorentzian α-Sasakian manifold. If M is pseudosymmetric, then either M is a space of constant curvature and $F(X,Y) = \alpha g(\phi X, \phi Y)$ for $\alpha \neq 0$, or $L_{\bar{R}} = \alpha^2$ for $X, Y \in \chi(M)$.*

Proof. Let M be pseudosymmetric and let $X, Y, U, V, W \in \chi(M)$. Then, using (42) and (43) in (41), we have that

$$\begin{aligned}
&\bar{R}(X,Y)(\bar{R}(U,V)W) - \bar{R}(\bar{R}(X,Y)U,V)W \\
&\qquad - \bar{R}(U,\bar{R}(X,Y)V)W - \bar{R}(U,V)(R(X,Y)W) \\
&\quad = L_{\bar{R}}\left[(X \wedge Y)(\bar{R}(U,V)W) - \bar{R}((X \wedge Y)U,V)W\right. \\
&\qquad \left. - \bar{R}(U,(X \wedge Y)V)W - \bar{R}(U,V)((X \wedge Y)W)\right].
\end{aligned}$$

Replacing here X with ξ, we obtain

$$\begin{aligned}
&\bar{R}(\xi,Y)(\bar{R}(U,V)W) - \bar{R}(\bar{R}(\xi,Y)U,V)W \\
&\qquad - \bar{R}(U,\bar{R}(\xi,Y)V)W - \bar{R}(U,V)(R(\xi,Y)W) \\
&\quad = L_{\bar{R}}\left[(\xi \wedge Y)(\bar{R}(U,V)W) - \bar{R}((\xi \wedge Y)U,V)W\right. \\
&\qquad \left. - \bar{R}(U,(\xi \wedge Y)V)W - \bar{R}(U,V)((\xi \wedge Y)W)\right].
\end{aligned} \tag{49}$$

Using (1), (20) in (48) and taking inner product of (48) with ξ, we get

$$\begin{aligned}
&\alpha^2\left[-\bar{R}(U,V,W,Y) - \eta(\bar{R}(U,V)W)\eta(Y)\right. \\
&\qquad + g(Y,U)\eta(\bar{R}(\xi,V)W) - \eta(U)\eta(\bar{R}(Y,V)W) \\
&\qquad \left. + g(Y,V)\eta(\bar{R}(U,\xi)W) - \eta(V)\eta(\bar{R}(U,Y)W) - \eta(W)\eta(\bar{R}(U,V)Y)\right] \\
&\qquad + \alpha\left[\bar{R}(U,V,W,\phi Y) + \eta(U)\eta(\bar{R}(\phi Y,V)W) - g(\phi Y,U)\eta(\bar{R}(\xi,V)W)\right. \\
&\qquad \left. + \eta(V)\eta(\bar{R}(U,\phi Y)W) - g(\phi Y,V)\eta(\bar{R}(U,\xi)W) + \eta(W)\eta(\bar{R}(U,V)\phi Y)\right] \\
&\quad = L_{\bar{R}}\left[-\bar{R}(U,V,W,Y) - \eta(Y)\eta(\bar{R}(U,V)W) + g(Y,U)\eta(\bar{R}(\xi,V)W)\right. \\
&\qquad - \eta(U)\eta(\bar{R}(Y,V)W + g(Y,V)\eta(\bar{R}(U,\xi)W) \\
&\qquad \left. - \eta(V)\eta(\bar{R}(U,Y)W) - \eta(W)\eta(\bar{R}(U,V)Y)\right].
\end{aligned}$$

Then, putting $Y = U = \xi$, we get

$$\left[L_{\bar{R}} - \alpha^2\right]\eta(\bar{R}(\xi,V)W) = 0.$$

This shows that either $\eta(\bar{R}(\xi,V)W) = 0$ or $L_{\bar{R}} - \alpha^2 = 0$.

Now, if $L_{\bar{R}} - \alpha^2 \neq 0$, then $\eta(\bar{R}(\xi, V)W) = 0$ which implies that M is a space of constant curvature and

$$\alpha g(\phi V, \phi W) = g(\phi V, W)$$

or

$$F(V, W) = \alpha g(\phi V, \phi W).$$

If $\eta(\bar{R}(\xi, V)W) \neq 0$, then we have $L_{\bar{R}} = \alpha^2$. □

9. Example of a pseudosymmetric Lorentzian α-Sasakian manifold with a semi-symmetric metric connection D

Let us consider a three-dimensional manifold

$$M = \{(x_1, x_2, x_3) \in R^3 : x_1, x_2, x_3 \in R\},$$

where (x_1, x_2, x_3) are the standard coordinates of R^3. We consider the vector fields

$$e_1 = e^{x_3}\frac{\partial}{\partial x_2}, \;\; e_2 = e^{x_3}(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}) \;\text{ and }\; e_3 = \alpha\frac{\partial}{\partial x_3},$$

where α is a constant.

Clearly, $\{e_1, e_2, e_3\}$ is a set of linearly independent vector fields for each point of M and hence a basis of T_xM. The Lorentzian metric g is defined by

$$g(e_1, e_2) = g(e_2, e_3) = g(e_1, e_3) = 0,$$
$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = -1.$$

Then the form of metric becomes

$$g = -\frac{1}{(e^{x_3})^2}(dx_2)^2 - \frac{1}{\alpha^2}(dx_3)^2$$

which is a Lorentzian metric.

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$, and let ϕ be the $(1,1)$-tensor field defined by

$$\phi e_1 = -e_1, \;\; \phi e_2 = -e_2, \;\; \phi e_3 = 0.$$

From the linearity of ϕ and g, we have that

$$\eta(e_3) = -1, \;\; \phi^2(X) = X + \eta(X)e_3 \;\text{ and }\; g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y)$$

for any $X \in \chi(M)$. Then, for $e_3 = \xi$, the structure (ϕ, ξ, η, g) defines a Lorentzian paracontact structure on M.

Let ∇ be the Levi–Civita connection of the Lorentzian metric g. Then

$$[e_1, e_2] = 0, \;\; [e_1, e_3] = -\alpha e_1, \;\; [e_2, e_3] = -\alpha e_2.$$

Recall Koszul's formula:

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &\quad - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned}$$

From the above formula, we can calculate the following:

$$\begin{aligned}&\nabla_{e_1}e_1 = -\alpha e_3,\ \ \nabla_{e_1}e_2 = 0,\ \ \nabla_{e_1}e_3 = -\alpha e_1,\\ &\nabla_{e_2}e_1 = 0,\ \ \nabla_{e_2}e_2 = -\alpha e_3,\ \ \nabla_{e_2}e_3 = -\alpha e_2,\\ &\nabla_{e_3}e_1 = 0,\ \ \nabla_{e_3}e_2 = 0,\ \ \nabla_{e_3}e_3 = 0.\end{aligned}$$

Hence the structure (ϕ, ξ, η, g) is a Lorentzian α-Sasakian manifold (see [11]).

Using (9), we find D, the semi-symmetric metric connection on M:

$$\begin{aligned}&D_{e_1}e_1 = (1-\alpha)e_3,\ +, D_{e_1}e_2 = 0,\ \ D_{e_1}e_3 = -(1+\alpha)e_1,\\ &D_{e_2}e_1 = 0,\ \ D_{e_2}e_2 = (1-\alpha)e_3,\ \ D_{e_2}e_3 = -(1+\alpha)e_1,\\ &D_{e_3}e_1 = 0,\ \ D_{e_3}e_2 = 0,\ \ D_{e_3}e_3 = 0.\end{aligned}$$

Using (10), the torson tensor $\bar{T}$ of the semi-symmetric metric connection D may be expressed as follows:

$$\begin{aligned}&\bar{T}(e_i, e_i) = 0, \quad i = 1, 2, 3,\\ &\bar{T}(e_1, e_2) = 0,\ \ \bar{T}(e_1, e_3) = -e_1, ++, \bar{T}(e_2, e_3) = -e_2.\end{aligned}$$

Also,

$$(D_{e_1}g)(e_2, e_3) = (D_{e_2}g)(e_3, e_1) = (D_{e_3}g)(e_1, e_2) = 0.$$

Thus M is a Lorentzian α-Sasakian manifold with a semi-symmetric metric connection D.

Now, we calculate the curvature tensor $\bar{R}$ and the Ricci tensor $\bar{S}$ as follows:

$$\begin{aligned}&\bar{R}(e_1, e_2)e_3 = 0,\ \ \bar{R}(e_1, e_3)e_3 = -(\alpha^2 + \alpha)e_1,\\ &\bar{R}(e_3, e_2)e_2 = (\alpha^2 - \alpha)e_3,\ \ \bar{R}(e_3, e_1)e_1 = (\alpha^2 - \alpha)e_3,\\ &\bar{R}(e_2, e_1)e_1 = (\alpha^2 - 2\alpha - 1)e_2,\ \ \bar{R}(e_2, e_3)e_3 = -(\alpha^2 + \alpha)e_2,\\ &\bar{R}(e_1, e_2)e_2 = (\alpha^2 - 2\alpha - 1)e_1,\ \ \bar{S}(e_3, e_3) = -2\alpha^2,\\ &\quad \bar{S}(e_1, e_1) = \bar{S}(e_2, e_2) = -(n-2)(\alpha + 1).\end{aligned}$$

Again, using (1), we get

$$\begin{aligned}&(e_1, e_2)e_3 = 0,\ \ (e_i \wedge e_i)e_j = 0, \quad i, j = 1, 2, 3,\\ &(e_1 \wedge e_2)e_2 = (e_1 \wedge e_3)e_3 = -e_1,\ \ (e_2 \wedge e_1)e_1 = (e_2 \wedge e_3)e_3 = -e_2,\\ &(e_3 \wedge e_2)e_2 = (e_3 \wedge e_1)e_1 = -e_3.\end{aligned}$$

Now,

$$\begin{aligned}\bar{R}(e_1, e_2)(\bar{R}(e_3, e_1)e_2) = 0,\ \ \bar{R}(\bar{R}(e_1, e_2)e_3, e_1)e_2 = 0,\\ \bar{R}(e_3, \bar{R}(e_1, e_2)e_1)e_2 = (1 + 2\alpha - \alpha^2)(\alpha^2 - \alpha)e_3,\\ (\bar{R}(e_3, e_1)(\bar{R}(e_1, e_2)e_2) = (\alpha^2 - 2\alpha - 1)(\alpha^2 - \alpha)e_3.\end{aligned}$$

Then $(\bar{R}(e_1, e_2).\bar{R})(e_3, e_1, e_2) = 0$.

Again,

$$\begin{aligned}
(e_1 \wedge e_2)(\bar{R}(e_3, e_1)e_2) &= 0, \\
\bar{R}((e_1 \wedge e_2)e_3, e_1)e_2 &= 0, \\
\bar{R}(e_3, (e_1 \wedge e_2)e_1)e_2 &= (\alpha^2 - \alpha)e_3, \\
\bar{R}(e_3, e_1)((e_1 \wedge e_2)e_2) &= (\alpha - \alpha^2)e_3.
\end{aligned}$$

Consequently, $((e_1, e_2).\bar{R})(e_3, e_1, e_2) = 0$. Thus

$$(\bar{R}(e_1, e_2).\bar{R})(e_3, e_1, e_2) = L_{\bar{R}}\left[((e_1, e_2).\bar{R})(e_3, e_1, e_2)\right]$$

for any function $L_{\bar{R}} \in C^\infty(M)$. Similarly, for any combination of e_1, e_2 and e_3, we can show (45). Hence M is a pseudosymmetric Lorentzian α-Sasakian manifold with semi-symmetric metric connection.

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