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Global behavior of a fourth order difference equation

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Abstract. We determine the forbidden set, introduce an explicit formula for the solutions, and discuss the global behavior of solutions of a fourth order difference equation.

1. Introduction

Difference equations have played an important role in analysis of mathematical models of biology, physics and engineering. Recently, there has been a great interest in studying properties of nonlinear and rational difference equations (see [1] and [4] − [13], and references therein). In [5], the authors showed that the second order rational difference equation

$$x_{n+1} = \frac{A{x_n}^2 + Bx_nx_{n-1} + C{x_{n-1}}^2}{\alpha x_n + \beta x_{n-1}}, \quad n = 0, 1, \ldots,$$

has several qualitatively different types of positive solutions, where A, B, C, α, β are nonnegative real numbers. Amleh et al. [2, 3] studied in details the difference equation

$$x_{n+1} = \frac{\alpha + \beta x_nx_{n-1} + \gamma x_{n-1}}{A + Bx_nx_{n-1} + Cx_{n-1}}, \quad n = 0, 1, \ldots,$$

with nonnegative parameters and initial conditions such that $A+B+C > 0$. Sedaghat [12] determined the global behavior of all solutions of the rational difference equations

$$x_{n+1} = \frac{ax_{n-1}}{x_nx_{n-1} + b}, \quad x_{n+1} = \frac{ax_nx_{n-1}}{x_n + bx_{n-2}}, \quad n = 0, 1, \ldots,$$

where $a, b > 0$.

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In this paper, we determine the forbidden set, introduce an explicit formula for the solutions and discuss the global behavior of solutions of the difference equation

$$x_{n+1} = \frac{a x_n x_{n-2}}{b x_n + c x_{n-3}}, \quad n = 0, 1, \dots, \tag{1.1}$$

where a, b, c are positive real numbers and the initial conditions $x_{-3}, x_{-2}, x_{-1}, x_0$ are real numbers.

2. Forbidden set and solutions of equation (1.1)

In this section we derive the forbidden set and introduce an explicit formula for the solutions of the difference equation (1.1).

Proposition 2.1. *The forbidden set F of equation* (1.1) *is*

$$\begin{aligned} F = \bigcup_{n=0}^{\infty} &\left\{ (v_0, v_{-1}, v_{-2}, v_{-3}) : v_0 = v_{-3} \left(-\frac{c}{b \sum_{l=0}^{n} (\frac{a}{c})^i} \right) \right\} \\ &\bigcup \{(v_0, v_{-1}, v_{-2}, v_{-3}) : v_0 = 0\} \bigcup \{(v_0, v_{-1}, v_{-2}, v_{-3}) : v_{-1} = 0\} \\ &\bigcup \{(v_0, v_{-1}, v_{-2}, v_{-3}) : v_{-2} = 0\}. \end{aligned}$$

Proof. Suppose that $x_0 x_{-1} x_{-2} = 0$. Then we have the following. If $x_0 = 0$ and $x_{-1} x_{-2} \neq 0$, then x_4 is undefined. If $x_{-1} = 0$ and $x_0 x_{-2} \neq 0$, then x_3 is undefined. If $x_{-2} = 0$ and $x_0 x_{-1} \neq 0$, then x_2 is undefined.

Now, if $x_{-3} = 0$ and $x_0 x_{-1} x_{-2} \neq 0$, then $x_1 = \frac{a}{b} x_{-2} \neq 0$. Therefore, we can start with the nonzero initial conditions x_{-2}, x_{-1}, x_0, x_1, which we shall discuss. Suppose that $x_{-i} \neq 0$ for all $i \in \{0, 1, 2, 3\}$. From equation (1.1), using the substitution $l_n = \frac{x_{n-3}}{x_n}$, we can obtain the first order difference equation

$$l_{n+1} = \frac{c}{a} l_n + \frac{b}{a}, \quad l_0 = \frac{x_{-3}}{x_0}. \tag{2.1}$$

We shall deduce the forbidden set of equation (1.1). For, consider the function

$$h(x) = \frac{c}{a} x + \frac{b}{a}$$

and suppose that we start from an initial point $(x_0, x_{-1}, x_{-2}, x_{-3})$ such that

$$\frac{x_{-3}}{x_0} = -\frac{b}{c}.$$

If we define

$$u_n = h^{-1}(u_{n-1}) = \frac{a}{c} u_{n-1} - \frac{b}{c} \quad \text{with} \quad u_0 = \frac{x_{-3}}{x_0} = -\frac{b}{c},$$

then we obtain

$$u_n = \frac{x_{n-3}}{x_n} = h^{-n}(u_0) = -\frac{b}{c}\sum_{i=0}^{n}\left(\frac{a}{c}\right)^i.$$

Therefore,

$$x_n = x_{n-3}\left(-\frac{c}{b\sum_{i=0}^{n}(\frac{a}{c})^i}\right).$$

On the other hand, we can observe that if we start from an initial point $(x_0, x_{-1}, x_{-2}, x_{-3})$ such that

$$l_0 = \frac{x_{-3}}{x_0} = -\frac{b}{c}\sum_{i=0}^{n_0}\left(\frac{a}{c}\right)^i$$

for a certain $n_0 \in \mathbb{N}$, then according to equation (2.1) we obtain

$$l_{n_0} = \frac{x_{n_0-3}}{x_{n_0}} = -\frac{b}{c}.$$

This implies that $bx_{n_0} + cx_{n_0-3} = 0$. Therefore, x_{n_0+1} is undefined. This completes the proof. $\square$

Theorem 2.2. *Let x_{-3}, x_{-2}, x_{-1} and x_0 be real numbers such that*

$$(x_0, x_{-1}, x_{-2}, x_{-3}) \notin F. \tag{2.2}$$

If $a \neq c$, then the solution $\{x_n\}_{n=-3}^{\infty}$ of equation (1.1) is

$$x_n = \begin{cases} x_{-2}\prod_{j=0}^{\frac{n-1}{3}} \dfrac{a-c}{\theta(\frac{c}{a})^{3j+1}+b}, & n = 1, 4, 7, \ldots, \\ x_{-1}\prod_{j=0}^{\frac{n-2}{3}} \dfrac{a-c}{\theta(\frac{c}{a})^{3j+2}+b}, & n = 2, 5, 8, \ldots, \\ x_0\prod_{j=0}^{\frac{n-3}{3}} \dfrac{a-c}{\theta(\frac{c}{a})^{3j+3}+b}, & n = 3, 6, 9, \ldots, \end{cases} \tag{2.3}$$

where $\theta = \frac{a-c-b\alpha}{\alpha}$ and $\alpha = \frac{x_0}{x_{-3}}$.

Proof. We can write the given solution (2.3) as

$$x_{3m+i} = x_{-3+i}\prod_{j=0}^{m}\beta_i(j), \quad i = 1, 2, 3 \quad \text{and} \quad m = 0, 1, \ldots, \tag{2.4}$$

where

$$\beta_i(j) = \frac{a-c}{\theta\left(\frac{c}{a}\right)^{3j+i}+b}, \quad i = 1, 2, 3.$$

Hence, we can see that

$$x_1 = x_{-2}\frac{a-c}{\frac{c}{a}\theta + b} = x_{-2}\frac{(a-c)a\alpha}{c(a-c-b\alpha)+ba\alpha} = x_{-2}\frac{a\alpha}{c+b\alpha} = \frac{ax_0x_{-2}}{bx_0+cx_{-3}},$$

$$\begin{aligned} x_2 &= x_{-1}\frac{a-c}{(\frac{c}{a})^2\theta + b} = x_{-1}\frac{(a-c)a^2\alpha}{c^2(a-c-b\alpha)+ba^2\alpha} = x_{-1}\frac{a^2\alpha}{c^2+b\alpha(c+a)} \\ &= \frac{a^2x_{-1}x_0}{c(cx_{-3}+bx_0)+bx_0a} = \frac{ax_{-1}\frac{ax_0}{bx_0+cx_{-3}}}{c+\frac{bx_0a}{bx_0+cx_{-3}}} = \frac{ax_{-1}\frac{x_1}{x_{-2}}}{c+b\frac{x_1}{x_{-2}}} = \frac{ax_1x_{-1}}{bx_1+cx_{-2}} \end{aligned}$$

and

$$\begin{aligned} x_3 &= x_0\frac{a-c}{(\frac{c}{a})^3\theta + b} = x_0\frac{(a-c)a^3\alpha}{c^3(a-c-b\alpha)+ba^3\alpha} = x_0\frac{a^3\alpha}{c^3+b\alpha(a^2+ac+c^2)} \\ &= x_0\frac{a^3x_0}{c^2(cx_{-3}+bx_0)+bx_0a(a+c)} = x_0a^2\frac{\frac{ax_0}{bx_0+cx_{-3}}}{c^2+\frac{bx_0a(a+c)}{bx_0+cx_{-3}}} \\ &= x_0a^2\frac{\frac{x_1}{x_{-1}}}{c^2+b(a+c)\frac{x_1}{x_{-2}}} = x_0a^2\frac{x_1}{c(cx_{-2}+bx_1)+bax_1} \\ &= x_0a\frac{\frac{ax_1}{cx_{-2}+bx_1}}{c+b\frac{ax_1}{cx_{-2}+bx_1}x_1} = x_0a\frac{\frac{x_2}{x_{-1}}}{c+b\frac{x_2}{x_{-1}}} = \frac{ax_2x_0}{bx_1+cx_{-2}}. \end{aligned}$$

Now assume that $m > 1$. Then

$$\begin{aligned} \frac{ax_{3m+1}x_{3m-1}}{bx_{3m+1}+cx_{3m-2}} &= \frac{ax_{-2}\prod_{j=0}^{m}\beta_1(j)x_{-1}\prod_{j=0}^{m-1}\beta_2(j)}{bx_{-2}\prod_{j=0}^{m}\beta_1(j)+cx_{-2}\prod_{j=0}^{m-1}\beta_1(j)} \\ &= \frac{ax_{-2}\prod_{j=0}^{m}\beta_1(j)x_{-1}\prod_{j=0}^{m-1}\beta_2(j)}{x_{-2}\prod_{j=0}^{m-1}\beta_1(j)(b\beta_1(m)+c)} = \frac{a\beta_1(m)x_{-1}\prod_{j=0}^{m-1}\beta_2(j)}{b\beta_1(m)+c} \\ &= \frac{a\frac{a-c}{\theta(\frac{c}{a})^{3m+1}+b}x_{-1}\prod_{j=0}^{m-1}\beta_2(j)}{b\frac{a-c}{\theta(\frac{c}{a})^{3m+1}+b}+c} = \frac{a(a-c)x_{-1}\prod_{j=0}^{m-1}\beta_2(j)}{b(a-c)+c(\theta(\frac{c}{a})^{3m+1}+b)} \\ &= \frac{a(a-c)x_{-1}\prod_{j=0}^{m-1}\beta_2(j)}{c\theta(\frac{c}{a})^{3m+1}+ab} = x_{-1}\frac{a-c}{\theta(\frac{c}{a})^{3m+2}+b}\prod_{j=0}^{m-1}\beta_2(j) \\ &= x_{-1}\beta_2(m)\prod_{j=0}^{m-1}\beta_2(j) = x_{-1}\prod_{j=0}^{m}\beta_2(j) = x_{3m+2}. \end{aligned}$$

This completes the proof. $\square$

3. Global behavior of equation (1.1)

In this section, we investigate the global behavior of equation (1.1) with $a \neq c$, using the explicit formula of its solution.

Theorem 3.1. *Let* $\{x_n\}_{n=-3}^{\infty}$ *be a solution of equation* (1.1) *such that* (2.2) *holds. Then the following statements are true.*

(1) *If* $a < c$, *then* $\{x_n\}_{n=-3}^{\infty}$ *converges to* 0.
(2) *If* $a > c$, *then we have the following:*
  (a) *If* $\frac{a-c}{b} < 1$, *then* $\{x_n\}_{n=-3}^{\infty}$ *converges to* 0.
  (b) *If* $\frac{a-c}{b} > 1$, *then the subsequences* $\{x_{3m+i}\}_{m=-1}^{\infty}$, $i = 1, 2, 3$, *are unbounded.*

Proof. (1) If $a < c$, then $\beta_i(j)$ converges to 0 as $j \to \infty$, $i = 1, 2, 3$. It follows that, for a given $0 < \epsilon < 1$, there exists $j_0 \in \mathbb{N}$ such that, $\mid \beta_i(j) \mid< \epsilon$ for all $j \geq j_0$. Therefore,

$$
\begin{aligned}
\mid x_{3m+i} \mid &=\mid x_{-3+i} \mid\mid \prod_{j=0}^{m} \beta_i(j) \mid \\
&=\mid x_{-3+i} \mid\mid \prod_{j=0}^{j_0-1} \beta_i(j) \mid\mid \prod_{j=j_0}^{m} \beta_i(j) \mid \\
&<\mid x_{-3+i} \mid\mid \prod_{j=0}^{j_0-1} \beta_i(j) \mid \epsilon^{m-j_0+1}.
\end{aligned}
$$

As m tends to infinity, the solution $\{x_n\}_{n=-3}^{\infty}$ converges to 0.

(2) Suppose that $a > c$. Then we have the following.

(i) If $\frac{a-c}{b} < 1$, then $\beta_i(j)$ converges to $\frac{a-c}{b} < 1$ as $j \to \infty$, $i = 1, 2, 3$. This implies that, there exists $j_1 \in \mathbb{N}$ such that, $\beta_i(j) < \mu_1$, where $0 < \mu_1 < 1$ for all $j \geq j_1$ and $i = 1, 2, 3$. Therefore, the solution $\{x_n\}_{n=-3}^{\infty}$ converges to 0 as in the proof of (1).
(ii) If $\frac{a-c}{b} > 1$, then $\beta_i(j)$ converges to $\frac{a-c}{b} > 1$ as $j \to \infty$, $i = 1, 2, 3$. Then for a given $\mu_2 > 1$ there exists $j_2 \in \mathbb{N}$ such that, $\beta_i(j) > \mu_2 > 1$, for all $j \geq j_2$ and $i = 1, 2, 3$.

For large values of m we have

$$
\begin{aligned}
\mid x_{3m+i} \mid &=\mid x_{-3+i} \mid\mid \prod_{j=0}^{m} \beta_i(j) \mid \\
&=\mid x_{-3+i} \mid\mid \prod_{j=0}^{j_2-1} \beta_i(j) \mid\mid \prod_{j=j_2}^{m} \beta_i(j) \mid
\end{aligned}
$$

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$$>| x_{-3+i} || \prod_{j=0}^{j_2-1} \beta_i(j) | \mu_2^{m-j_2+1}.$$

Therefore, the subsequences $\{x_{3m+i}\}_{m=-1}^{\infty}$, $i = 1, 2, 3$, are unbounded. $\square$

Example 1. Figure 1 shows that if $a = 1.5$, $b = 1$, $c = 1$ ($a - c < b$), then the solution $\{x_n\}_{n=-3}^{\infty}$ with initial conditions $x_{-3} = 2$, $x_{-2} = -2$, $x_{-1} = 2$ and $x_0 = -7$ converges to zero.

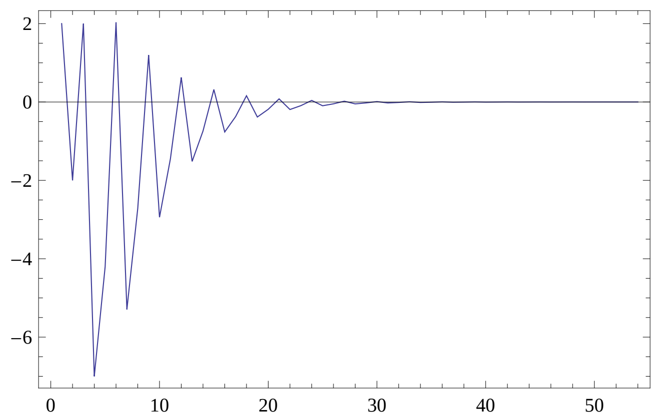

FIGURE 1. The difference equation $x_{n+1} = \frac{1.5x_n x_{n-2}}{x_{n-1}+x_{n-3}}$.

Example 2. Figure 2 shows that if $a = 2$, $b = 0.5$, $c = 1$ ($a - c > b$), then the solution $\{x_n\}_{n=-3}^{\infty}$ with initial conditions $x_{-3} = 2$, $x_{-2} = -2$, $x_{-1} = 2$ and $x_0 = 7$ is unbounded.

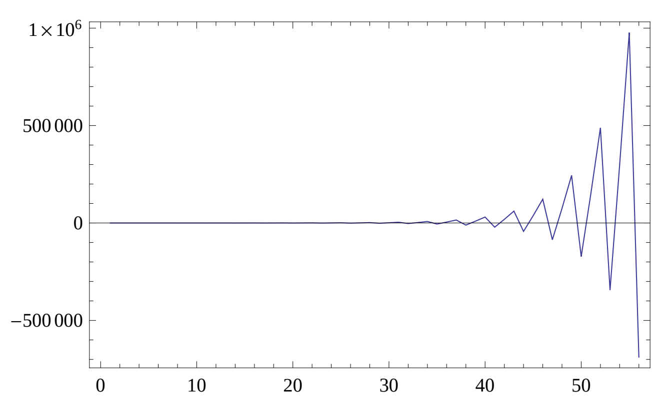

FIGURE 2. The difference equation $x_{n+1} = \frac{2x_n x_{n-2}}{0.5x_{n-1}+x_{n-3}}$.

4. Case $a - c = b$

In this section, we study the case when $a - c = b$.

Theorem 4.1. *Assume that* $\{x_n\}_{n=-3}^{\infty}$ *is a solution of equation* (1.1) *such that* (2.2) *holds and let* $a - c = b$. *If* $\alpha = 1$, *then* $\{x_n\}_{n=-3}^{\infty}$ *is a periodic solution with period* 3.

Proof. Assume that $a - c = b$. If $\alpha = 1$, then $\theta = 0$. Therefore,

$$x_{3m+i} = x_{-3+i} \prod_{j=0}^{m} \frac{a-c}{\theta(\frac{c}{a})^{3j+i} + b} = x_{-3+i}, \quad i = 1, 2, 3 \quad \text{and} \quad m = 0, 1, \ldots.$$

□

Theorem 4.2. *Assume that* $\{x_n\}_{n=-3}^{\infty}$ *is a solution of equation* (1.1) *such that* (2.2) *holds and let* $a - c = b$. *If* $\alpha \neq 1$, *then* $\{x_n\}_{n=-3}^{\infty}$ *converges to a solution with period* 3.

Proof. Suppose that $\{x_n\}_{n=-3}^{\infty}$ is a solution of equation (1.1) such that (2.2) holds and let $a - c = b$. As

$$\lim_{j\to\infty} \beta_i(j) = \lim_{j\to\infty} \frac{a-c}{\theta(\frac{c}{a})^{3j+i} + b} = 1, \quad i = 1, 2, 3,$$

there exists $j_0 \in \mathbb{N}$ such that, $\beta_i(j) > 0$, for all $i = 1, 2, 3$ and $j \geq j_0$. Hence,

$$\begin{aligned} x_{3m+i} &= x_{-3+i} \prod_{j=0}^{m} \beta_i(j) = x_{-3+i} \prod_{j=0}^{j_0-1} \beta_i(j) \prod_{j=j_0}^{m} \beta_i(j) \\ &= x_{-3+i} \prod_{j=0}^{j_0-1} \beta_i(j) \exp\Big(\sum_{j=j_0}^{m} \ln(\beta_i(j))\Big). \end{aligned}$$

We shall test the convergence of the series $\sum_{j=j_0}^{\infty} |\ln(\beta_i(j))|$. Since

$$\lim_{j\to\infty} \left| \frac{\ln(\beta_i(j+1))}{\ln(\beta_i(j))} \right| = \frac{0}{0},$$

using l'Hospital's rule we obtain

$$\lim_{j\to\infty} \left| \frac{\ln(\beta_i(j+1))}{\ln(\beta_i(j))} \right| = \left(\frac{c}{a}\right)^3 < 1.$$

It follows from d'Alembert's test that the series $\sum_{j=j_0}^{\infty} |\ln(\beta_i(j))|$ is convergent.

This ensures that there are three positive real numbers ν_1, ν_2, ν_3 such that

$$\lim_{m\to\infty} x_{3m+i} = \nu_i, \quad i = 1, 2, 3,$$

where

$$\nu_i = x_{-3+i} \prod_{j=0}^{\infty} \frac{b}{\theta(\frac{c}{a})^{3j+i} + b}, \quad i = 1, 2, 3.$$

$\square$

Example 3. Figure 3 shows that if $a = 2$, $b = 1$, $c = 1$ ($a - c = b$), then the solution $\{x_n\}_{n=-3}^{\infty}$ with initial conditions $x_{-3} = 1$, $x_{-2} = 2$, $x_{-1} = -2$ and $x_0 = -3$ converges to a 3-period solution.

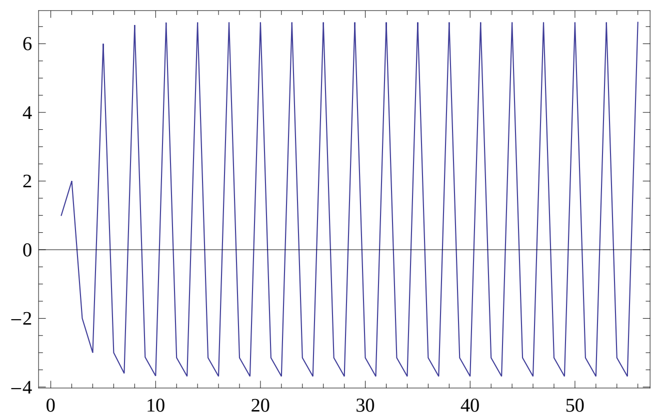

FIGURE 3. The difference equation $x_{n+1} = \frac{2x_n x_{n-2}}{x_{n-1}+x_{n-3}}$.

5. Case $a = c$

We end this work by introducing the main results when $a = c$.

Proposition 5.1. *Assume that $a = c$. Then the forbidden set G of equation* (1.1) *is*

$$G = \bigcup_{n=0}^{\infty} \left\{ (u_0, u_{-1}, u_{-2}, u_{-3}) : u_0 = u_{-3} \left(-\frac{a}{b(n+1)} \right) \right\}$$
$$\bigcup \{(u_0, u_{-1}, u_{-2}, u_{-3}) : u_0 = 0\} \bigcup \{(u_0, u_{-1}, u_{-2}, u_{-3}) : u_{-1} = 0\}$$
$$\bigcup \{(u_0, u_{-1}, u_{-2}, u_{-3}) : u_{-2} = 0\}.$$

Theorem 5.2. *Let x_{-3}, x_{-2}, x_{-1} and x_0 be real numbers such that*

$$(x_0, x_{-1}, x_{-2}, x_{-3}) \notin G. \tag{5.1}$$

If $a = c$, then the solution $\{x_n\}_{n=-3}^{\infty}$ of equation (1.1) *is*

$$x_n = \begin{cases} x_{-2} \displaystyle\prod_{j=0}^{\frac{n-1}{3}} \frac{a\alpha}{a + b\alpha(3j+1)}, & n = 1, 4, 7, \ldots, \\ x_{-1} \displaystyle\prod_{j=0}^{\frac{n-2}{3}} \frac{a\alpha}{a + b\alpha(3j+2)}, & n = 2, 5, 8, \ldots, \\ x_0 \displaystyle\prod_{j=0}^{\frac{n-3}{3}} \frac{a\alpha}{a + b\alpha(3j+3)}, & n = 3, 6, 9, \ldots, \end{cases}$$

where $\alpha = \frac{x_0}{x_{-3}}$.

Theorem 5.3. *Let* $\{x_n\}_{n=-3}^{\infty}$ *be a solution of equation* (1.1) *such that* (5.1) *holds. If* $a = c$, *then* $\{x_n\}_{n=-3}^{\infty}$ *converges to* 0.

Example 4. Figure 4 shows that if $a = 1.5$, $b = 1$, $c = 1.5$ $(a = c)$, then the solution $\{x_n\}_{n=-3}^{\infty}$ with initial conditions $x_{-3} = 1.4$, $x_{-2} = 2$, $x_{-1} = 1$ and $x_0 = -7$ converges to a 3-period solution.

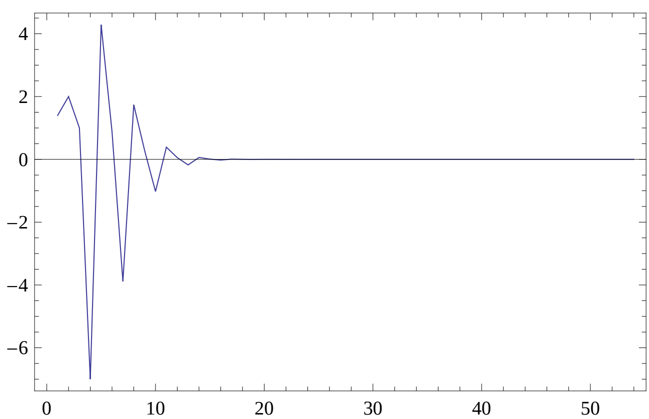

FIGURE 4. The difference equation $x_{n+1} = \frac{1.5x_n x_{n-2}}{x_{n-1}+1.5x_{n-3}}$.

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