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The n-exponential convexity for majorization inequality for functions of two variables and related results

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Abstract. We apply the refined method of producing n-exponential convex functions of J. Pečarić and J. Perić to extend some known results on majorization type and related inequalities.

1. Introduction

In [11], J. Pečarić and J. Perić introduced the notion of n-exponentially convex functions. In this paper, we give extensions of some results given in [1] – [4]. For several results concerning exponential convexity, see [5, 8, 11].

In order to obtain our main results, let us recall some known results.

Matrix majorization. The notion of majorization concerns a partial ordering of the diversity of the components of two vectors $\mathbf{x}$ and $\mathbf{y}$ such that $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$. A natural problem of interest is the extension of this notion from m-tuples (vectors) to $n \times m$ matrices. For example, let

$$X = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)' \quad \text{and} \quad Y = (\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n)'$$

be two $n \times m$ real matrices, where $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$; $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n$ are the corresponding row vectors.

Definition 1 (see [12])**.** Let X, Y be two $n \times m$ real matrices for $n \geq 2$, $m \geq 2$. X is said to *row-wise majorize* Y ($X \succ^r Y$) if $\mathbf{x}_i \succ \mathbf{y}_i$ holds for $i = 1, 2, \ldots, n$.

In the following result, the inner product on $\mathbb{R}^m$ is defined in the usual way. Furthermore, $e = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m\}$ is a basis in $\mathbb{R}^m$, and $d = \{\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_m\}$ is the dual basis of e, that is $\langle \mathbf{e}_i, \mathbf{d}_j \rangle = \delta_{ij}$ (Kronecker delta). One denotes

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$J = \{1, 2, \ldots, m\}$. Let J_1 and J_2 be two sets of indices such that $J_1 \cup J_2 = J$. Let $\mathbf{v} \in \mathbb{R}^m$ and $\mu \in \mathbb{R}$. A vector $\mathbf{z} \in \mathbb{R}^m$ is said to be $\mu, \mathbf{v}$-*separable* on J_1 and J_2 (with respect to the basis e) if

$$\langle \mathbf{e}_i, \mathbf{z} - \mu\mathbf{v} \rangle \geq 0 \ \text{ for } i \in J_1, \ \text{ and } \ \langle \mathbf{e}_j, \mathbf{z} - \mu\mathbf{v} \rangle \leq 0 \ \text{ for } j \in J_2.$$

A vector $\mathbf{z} \in \mathbb{R}^m$ is said to be $\mathbf{v}$-*separable* on J_1 and J_2 (w.r.t. e) if $\mathbf{z}$ is $\mu, \mathbf{v}$-separable on J_1 and J_2 for some μ. For an interval I, one says that a function $\varphi\colon I \to \mathbb{R}$ *preserves* $\mathbf{v}$-*separability* on J_1 and J_2 w.r.t. e, if $(\varphi(z_1), \varphi(z_2), \ldots, \varphi(z_m))$ is $\mathbf{v}$-separable on J_1 and J_2 w.r.t. e for each $\mathbf{z} = (z_1, z_2, \ldots, z_m) \in I^m$ such that $\mathbf{z}$ is $\mathbf{v}$-separable on J_1 and J_2 w.r.t. e.

Theorem 1 (see [2]). *Let $\phi\colon (a, b) \to \mathbb{R}$ be a convex function and $X = [x_{ij}]$, $Y = [y_{ij}]$ and $W = [w_{ij}]$ be matrices, where $x_{ij}, y_{ij} \in (a, b)$ and $w_{ij} \in \mathbb{R}$ $(i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m)$.*

(a) *If $X \succ^r Y$, then*

$$\sum_{i=1}^{n} \sum_{j=1}^{m} \phi(x_{ij}) \geq \sum_{i=1}^{n} \sum_{j=1}^{m} \phi(y_{ij}). \tag{1}$$

If ϕ is strictly convex on (a, b), then the strict inequality holds in (1) if and only if $X \neq Y$.

(b) *If $(x_{ij})_{j=\overline{1,m}}$, $(y_{ij})_{j=\overline{1,m}}$ $(i = 1, 2, \ldots, n)$ are decreasing and satisfy the conditions*

$$\sum_{j=1}^{k} w_{ij} x_{ij} \geq \sum_{j=1}^{k} w_{ij} y_{ij}, \quad k = 1, 2, \ldots, m-1, \tag{2}$$

and

$$\sum_{j=1}^{m} w_{ij} x_{ij} = \sum_{j=1}^{m} w_{ij} y_{ij}, \tag{3}$$

then

$$\sum_{i=1}^{n} \sum_{j=1}^{m} w_{ij} \phi(x_{ij}) \geq \sum_{i=1}^{n} \sum_{j=1}^{m} w_{ij} \phi(y_{ij}). \tag{4}$$

(c) *If $(y_{ij})_{j=\overline{1,m}}$ $(i = 1, 2, \ldots, n)$ is decreasing with $w_{ij} > 0$ $(i = 1, 2, \ldots, n,$ $j = 1, 2, \ldots, m)$ and satisfying conditions (2) and (3), then (4) holds. If ϕ is strictly convex on I, then the strict inequality holds in (4) if and only if $X \neq Y$.*

(d) *If $(x_{ij})_{j=\overline{1,m}}$ $(i = 1, 2, \ldots, n)$ is increasing with $w_{ij} > 0$ $(i = 1, 2, \ldots, n,$ $j = 1, 2, \ldots, m)$ and satisfying conditions (2) and (3), then the reverse inequality in (4) holds. If ϕ is strictly convex on (a, b), then the reverse strict inequality holds in (4) if and only if $X \neq Y$.*

(e) *If $(x_{ij} - y_{ij})_{j=\overline{1,m}}$ and $(y_{ij})_{j=\overline{1,m}}$ $(i = 1, 2, \ldots, n)$ are nondecreasing (nonincreasing) with $w_{ij} \geq 0$ $(i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m)$ and satisfying* *condition* (3)*, then* (4) *holds. If ϕ is strictly convex on (a,b) and $w_{ij} > 0$, then the strict inequality holds in* (4) *if and only if $X \neq Y$.*

(f) *Let $w_{ij} > 0$ $(i = 1,2,\dots,n,\ j = 1,2,\dots,m)$ and $\mathbf{u},\mathbf{v} \in \mathbb{R}^m$ with $\langle \mathbf{u},\mathbf{v}\rangle > 0$. If there exist index sets J_1 and J_2 with $J_1 \cup J_2 = J$ such that, for each $i = 1,2,\dots,n$,*

(i) $(y_{ij})_{j=\overline{1,m}}$ *is* $\mathbf{v}$*-separable on J_1 and J_2 w.r.t. e,*

(ii) $(x_{ij} - y_{ij})_{j=\overline{1,m}}$ *is* $\lambda,\mathbf{u}$*-separable on J_1 and J_2 w.r.t. d, where* $\lambda = \langle (x_{ij} - y_{ij})_{j=\overline{1,m}}, \mathbf{v}\rangle / \langle \mathbf{u},\mathbf{v}\rangle$,

(iii) $\langle (x_{ij} - y_{ij})_{j=\overline{1,m}}, \mathbf{v}\rangle = 0$, *or* $\langle (x_{ij} - y_{ij})_{j=\overline{1,m}}, \mathbf{v}\rangle \langle (z_{ij})_{j=\overline{1,m}}, \mathbf{u}\rangle \geq 0$, *where* $(z_{ij})_{j=\overline{1,m}} = (\varphi(y_{i1}),\dots,\varphi(y_{im}))$,

(iv) φ *preserves* $\mathbf{v}$*-separability on J_1 and J_2 w.r.t. e,*

then (4) *holds.*

The following theorem is an integral analogue of the above theorem.

Theorem 2 (see [1])**.** (a) *Let $\phi\colon I \to \mathbb{R}$ be a continuous convex function on an interval I, let $w,x,y\colon [a,b]\times[c,d] \to I$ be continuous functions such that $x(t,s), y(t,s)$ are decreasing in $t \in [a,b]$, and let $\mu\colon [a,b] \to \mathbb{R}$ be a function of bounded variation and $u\colon [c,d] \to \mathbb{R}$ an increasing function.*

(a$_1$) *If, for each $s \in [c,d]$,*

$$\int_a^\nu w(t,s)y(t,s)\,d\mu(t) \leq \int_a^\nu w(t,s)x(t,s)\,d\mu(t), \quad \nu \in [a,b], \tag{5}$$

and

$$\int_a^b w(t,s)x(t,s)\,d\mu(t) = \int_a^b w(t,s)y(t,s)\,d\mu(t), \tag{6}$$

then

$$\begin{aligned}\int_c^d \int_a^b w(t,s)\phi\left(y(t,s)\right)\,d\mu(t)du(s)&\\ \leq \int_c^d \int_a^b w(t,s)\phi\left(x(t,s)\right)d\mu(t)du(s).&\end{aligned} \tag{7}$$

(a$_2$) *If, for each $s \in [c,d]$, inequality* (5) *holds, then, for any continuous increasing convex function $\phi\colon I \to \mathbb{R}$, inequality* (7) *holds.*

(b) *Suppose that $\phi\colon [0,\infty) \to \mathbb{R}$ is a convex function and $w,x,y\colon [a,b]\times[c,d] \to \mathbb{R}^+$ are integrable functions. Let $\mu\colon [a,b] \to \mathbb{R}$, $u\colon [c,d] \to \mathbb{R}$ be increasing functions and satisfying conditions* (5) *and* (6)*.*

(b$_1$) *If, for each $s \in [c,d]$, the function $y(t,s)$ is decreasing in $t \in [a,b]$, then* (7) *holds.*

(b$_2$) *If, for each $s \in [c,d]$, the function $x(t,s)$ is increasing in $t \in [a,b]$, then the reverse inequality in* (7) *holds.*

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(c) *Let $\phi\colon I \to \mathbb{R}$ be a continuous convex function, let $w, x, y\colon [a,b] \times [c,d] \to I$ be continuous functions with $w(t,s) > 0$ being of bounded variation, and let $\mu\colon [a,b] \to \mathbb{R}$ and $u\colon [c,d] \to \mathbb{R}$ be increasing functions. If $y(t,s)$ and $x(t,s) - y(t,s)$ are increasing (decreasing) in $t \in [a,b]$ and satisfying condition (6), then (7) is true.*

(d) *Let $\phi\colon I \to \mathbb{R}$ be a continuous convex function, $\varphi \in \partial\phi$ ($\partial\phi$ is the subdifferential of ϕ), let $w, x, y, g, h\colon [a,b] \times [c,d] \to \mathbb{R}$ be continuous functions with $x(t,s), y(t,s) \in I$, $w(t,s), g(t,s), h(t,s) > 0$, and let $\mu\colon [a,b] \to \mathbb{R}$ and $u\colon [c,d] \to \mathbb{R}$ be increasing functions. Denote*

$$\lambda = \frac{\int_a^b w(t,s)(x(t,s) - y(t,s))\, d\mu(t)}{\int_a^b w(t,s) g(t,s) h(t,s)\, d\mu(t)}$$

and suppose that there exist two intervals I_1 and I_2 with $I_1 \cup I_2 = [a,b]$ such that, for each $s \in [c,d]$, $t_1 \in I_1$ and $t_2 \in I_2$, we have

$$\frac{\varphi(y(t_2,s))}{h(t_2,s)} \leq \frac{\varphi(y(t_1,s))}{h(t_1,s)}$$

and

$$\frac{x(t_2,s) - y(t_2,s)}{g(t_2,s)} \leq \lambda \leq \frac{x(t_1,s) - y(t_1,s)}{g(t_1,s)}.$$

If

$$\int_a^b w(t,s)(x(t,s) - y(t,s)) h(t,s)\, d\mu(t) \int_a^b w(t,s)\varphi(y(t,s)) w(t,s)\, d\mu(t) \geq 0,$$

then (7) holds.

Consider the Green function G defined on $[\alpha,\beta] \times [\alpha,\beta]$ by

$$G(t,s) = \begin{cases} \frac{(t-\beta)(s-\alpha)}{\beta-\alpha}, & \alpha \leq s \leq t, \\ \frac{(s-\beta)(t-\alpha)}{\beta-\alpha}, & t \leq s \leq \beta. \end{cases} \tag{8}$$

The function G is convex in s, it is symmetric, so it is also convex in t. The function G is continuous in s and continuous in t.

For any function $\phi\colon [\alpha,\beta] \to \mathbb{R}$, $\phi \in C^2([\alpha,\beta])$, we can easily show by using integration by parts that

$$\phi(x) = \frac{\beta - x}{\beta - \alpha}\phi(\alpha) + \frac{x - \alpha}{\beta - \alpha}\phi(\beta) + \int_\alpha^\beta G(x,s)\phi''(s)\, ds,$$

where the function G is defined by (8) (see [14]).

The following theorem is given in [2].

Theorem 3 (see [2])**.** *Let $X = [x_{ij}]$, $Y = [y_{ij}]$ and $W = [w_{ij}]$ be matrices, where $x_{ij}, y_{ij} \in [\alpha,\beta]$ and $w_{ij} \in \mathbb{R}$ ($i = 1,2,\ldots,n$, $j = 1,2,\ldots,m$) such that condition (3) is satisfied. Then the following two statements are equivalent.*

(i) *For every continuous convex function* $\phi\colon [\alpha,\beta]\to\mathbb{R}$*, inequality* (4) *holds.*

(ii) *For all* $\tau\in[\alpha,\beta]$*,*

$$\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}G(x_{ij},\tau)\geq\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}G\left(y_{ij},\tau\right). \tag{9}$$

Moreover, the statements (i) *and* (ii) *are also equivalent if we reverse the inequalities in* (4) *and* (9).

Theorem 4 (see [1])**.** *Let* $w,x,y\colon [a,b]\times[c,d]\to\mathbb{R}$*,* $\mu\colon [a,b]\to\mathbb{R}$ *and* $u\colon [c,d]\to\mathbb{R}$ *be continuous functions, and let* $[\alpha,\beta]$ *be an interval such that* $x(t,s),y(t,s)\in[\alpha,\beta]$ *for* $(t,s)\in[a,b]\times[c,d]$*. Also, let* (6) *hold.*

Then the following statements are equivalent.

(i) *For every continuous convex function* $\phi\colon [\alpha,\beta]\to\mathbb{R}$*, the inequality* (7) *holds.*

(ii) *For all* $\tau\in[\alpha,\beta]$*,*

$$\begin{aligned}\int_c^d\int_a^b & w(t,s)G(y(t,s),\tau)d\mu(t)du(s)\\ &\leq\int_c^d\int_a^b w(t,s)G(x(t,s),\tau)d\mu(t)du(s).\end{aligned}\tag{10}$$

Moreover, the statements (i) *and* (ii) *are also equivalent if we reverse the inequalities in* (7) *and* (10).

Theorem 5 (see [9])**.** *Let* $\phi\colon (a,b)\to\mathbb{R}$ *be a differentiable convex function and let* $x_i\in(a,b)$*,* $i=1,2,\ldots,n$ $(n\geq 2)$*. Define*

$$\bar{y}=\frac{1}{W_n}\sum_{i=1}^{n} w_i\phi(x_i),$$

where $w_i\geq 0$ $(i=1,2,\ldots,n)$ *are such that* $W_n=\sum_{i=1}^{n} w_i>0$*. If* $d\in(a,b)$*, then we have*

$$\bar{y}\leq\phi(d)+\frac{1}{W_n}\sum_{i=1}^{n} w_i(x_i-d)\phi'(x_i). \tag{11}$$

Also, when ϕ *is strictly convex, we have the equality in* (11) *if and only if* $x_i=d$ *for all* i *with* $w_i>0$*.*

Inequality (11) of Matić and Pečarić is the best in the sense that if

$$\sum_{i=1}^{n} w_i\phi'(x_i)\neq 0 \text{ and } \bar{\bar{x}}=\frac{\sum_{i=1}^{n} w_i x_i\phi'(x_i)}{\sum_{i=1}^{n} w_i\phi'(x_i)}\in(a,b),$$

then, by setting $d = \bar{\bar{x}}$ in (11), we immediately obtain Slater's inequality

$$\frac{1}{W_n}\sum_{i=1}^{n} w_i\phi(x_i) \leq \phi\left(\frac{\sum_{i=1}^{n} w_i\phi'(x_i)x_i}{\sum_{i=1}^{n} w_i\phi'(x_i)}\right).$$

This is more general than the inequality obtained by Slater for monotone convex functions in [13]. For a multidimensional version of Slater's inequality, see [10]. Also, by setting $d = \bar{x}$ in (11), we get the converse of Jensen's inequality given in [7]. For further refinements of Jensen's inequality, see [6].

The following theorem is the integral analogue of Theorem 5.

Theorem 6 (see [9]). *Let (Ω, A, μ) be a measure space with $0 < \mu(\Omega) < \infty$ and let $\phi\colon (a,b) \to \mathbb{R}$ be a differentiable convex function. If $f\colon \Omega \to (a,b)$ is such that f, $\phi(f), \phi'(f)$ and $\phi'(f)f$ are in $L^1(\mu)$, then, for any $d \in (a,b)$, we have*

$$\frac{1}{\mu(\Omega)}\int_{\Omega}\phi(f)\,d\mu \leq \phi(d) + \frac{1}{\mu(\Omega)}\int_{\Omega}(f-d)\phi'(f)\,d\mu. \tag{12}$$

Also, when ϕ is strictly convex, we have the equality in (12) *if and only if $f = d$ almost everywhere on Ω.*

From (12), we can obtain Slater's integral inequality and a converse of Jensen's inequality.

In [1] – [4], the authors used some families of convex functions and proved exponential convexity and log-convexity of the functionals associated with majorization type inequalities and the inequalities (11) and (12). They established improvements and reversions of Slater's and related inequalities. In this paper, we give all these results for some general family of functions having some special property. In this way the results given in [1] – [4] become special cases of our results (see Examples 1 and 2).

2. Main results

Definition 2 (see, e.g., [12, p. 2]). A function $\phi : I \to \mathbb{R}$ is *convex* on an interval I if

$$\phi(x_1)(x_3 - x_2) + \phi(x_2)(x_1 - x_3) + \phi(x_3)(x_2 - x_1) \geq 0 \tag{13}$$

holds for all $x_1, x_2, x_3 \in I$ such that $x_1 < x_2 < x_3$.

Definition 3 (see [11]). A function $\phi\colon I \to \mathbb{R}$ is *n-exponentially convex in the Jensen sense* on I if

$$\sum_{k,l=1}^{n} \alpha_k\alpha_l\phi\left(\frac{x_k + x_l}{2}\right) \geq 0$$

holds for $\alpha_k \in \mathbb{R}$ and $x_k \in I$, $k = 1, 2, \ldots, n$.

Definition 4 (see [11]). A function $\phi\colon\ I \to \mathbb{R}$ is *n-exponentially convex* on I if it is n-exponentially convex in the Jensen sense and continuous on I.

Remark 1. From the definition it is clear that 1-exponentially convex functions in the Jensen sense are in fact non-negative functions. Also, n-exponentially convex functions in the Jensen sense are m-exponentially convex in the Jensen sense for every $m \in \mathbb{N},\ m \le n$.

Proposition 1. *If $\phi\colon\ I \to \mathbb{R}$ is an n-exponentially convex in the Jensen sense function, then the matrix $\left[\phi\left(\frac{x_k+x_l}{2}\right)\right]_{k,l=1}^{m}$ is a positive semi-definite matrix for all $m \in \mathbb{N},\ m \le n$. In particular,*

$$\det\left[\phi\left(\frac{x_k+x_l}{2}\right)\right]_{k,l=1}^{m} \ge 0$$

for all $m \in \mathbb{N}$, $m = 1, 2, \dots, n$.

Definition 5. A function $\phi\colon\ I \to \mathbb{R}$ is *exponentially convex in the Jensen sense* on I if it is n-exponentially convex in the Jensen sense for all $n \in \mathbb{N}$.

Definition 6. A function $\phi\colon\ I \to \mathbb{R}$ is *exponentially convex* if it is exponentially convex in the Jensen sense and continuous.

Remark 2. It is easy to show that $\phi\colon\ I \to \mathbb{R}^+$ is log-convex in the Jensen sense if and only if

$$\alpha^2\phi(x) + 2\alpha\beta\phi\left(\frac{x+y}{2}\right) + \beta^2\phi(y) \ge 0$$

holds for every $\alpha, \beta \in \mathbb{R}$ and $x, y \in I$. It follows that a function is log-convex in the Jensen sense if and only if it is 2-exponentially convex in the Jensen sense.

Also, using basic convexity theory it follows that a function is log-convex if and only if it is 2-exponentially convex.

When dealing with functions with different degree of smoothness, divided differences are found to be very useful.

Definition 7. The *second order divided difference* of a function $\phi\colon\ I \to \mathbb{R}$ at mutually different points $y_0, y_1, y_2 \in I$ is defined recursively by

$$[y_i; \phi] = \phi(y_i),\ \ i = 0, 1, 2,$$

$$[y_i, y_{i+1}; \phi] = \frac{\phi(y_{i+1}) - \phi(y_i)}{y_{i+1} - y_i},\ \ i = 0, 1,$$

$$[y_0, y_1, y_2; \phi] = \frac{[y_1, y_2; \phi] - [y_0, y_1; \phi]}{y_2 - y_0}. \tag{14}$$

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Remark 3. The value $[y_0, y_1, y_2; \phi]$ is independent of the order of the points y_0, y_1, and y_2. By taking limits, this definition may be extended to include the cases in which any two or all three points coincide as follows: for all $y_0,\ y_1,\ y_2 \in I$,

$$\lim_{y_1 \to y_0} [y_0, y_1, y_2; \phi] = [y_0, y_0, y_2; \phi] = \frac{f(y_2) - f(y_0) - \phi'(y_0)(y_2 - y_0)}{(y_2 - y_0)^2}, \quad y_2 \neq y_0,$$

provided that ϕ' exists, and furthermore, taking the limits $y_i \to y_0$ $(i = 1, 2)$ in (14), we get

$$[y_0, y_0, y_0; \phi] \ = \lim_{y_i \to y_0} [y_0, y_1, y_2; \phi] = \frac{\phi''(y_0)}{2} \quad \text{for } i = 1, 2$$

provided that ϕ'' exists.

Let X, Y, W and ϕ be defined as in Theorem 1. We define the functional $\Lambda_1(X, Y, W; \phi)$ by

$$\Lambda_1(X, Y, W; \phi) = \sum_{i=1}^{n} \sum_{j=1}^{m} w_{ij} \phi(x_{ij}) - \sum_{i=1}^{n} \sum_{j=1}^{m} w_{ij} \phi(y_{ij}).$$

Let w, x, y, u, μ, ϕ be defined as in Theorem 4. We define the functional $\Lambda_2(x, y, w; \phi)$ by

$$\begin{aligned}\Lambda_2(x, y, w; \phi) = & \int_c^d \int_a^b w(t, s) \phi(x(t, s)) d\mu(t) du(s) \\ & - \int_c^d \int_a^b w(t, s) \phi(y(t, s))\, d\mu(t) du(s).\end{aligned}$$

Under the assumptions of Theorems 5 and 6, we consider the functionals

$$F_5(\mathbf{x}, d, \phi) = \phi(d) + \frac{1}{W_n} \sum_{i=1}^{n} w_i (x_i - d) \phi'(x_i) - \frac{1}{W_n} \sum_{i=1}^{n} w_i \phi(x_i)$$

and

$$F_6(f, d, \phi) = \phi(d) + \frac{1}{\mu(\Omega)} \int_\Omega (f - d) \phi'(f) d\mu - \frac{1}{\mu(\Omega)} \int_\Omega \phi(f) d\mu,$$

respectively.

We use an idea from [8] to give an elegant method of producing n-exponentially convex functions and exponentially convex functions.

Theorem 7. *Let $\Omega = \{\phi_t \colon\ t \in I \subseteq \mathbb{R}\}$ be a family of differentiable functions defined on (a, b) such that the function $t \mapsto [y_0, y_1, y_2; \phi_t]$ is n-exponentially convex in the Jensen sense on I for every three mutually different points $y_0, y_1, y_2 \in (a, b)$. Consider $F_1 = \Lambda_1(X, Y, W; \phi_t)$ and $F_2 = \Lambda_2(x, y, w; \phi_t)$ if* (9) *and* (10) *hold for every $\tau \in [\alpha, \beta]$, and consider $F_3 = -\Lambda_1(X, Y, W; \phi_t)$ and $F_4 = -\Lambda_2(x, y, w; \phi_t)$ if the reverse inequalities in* (9) *and* (10) *hold for every* $\tau \in [\alpha, \beta]$. *Then, for the linear functionals* $\digamma_j(.,.,\phi_t)$ $(j = 1, 2, .., 6)$, *the following statements hold.*

(i) *The function* $t \mapsto \digamma_j(.,.,\phi_t)$ *is* n*-exponentially convex in the Jensen sense on* I *and the matrix* $[\digamma_j(.,.,\phi_{\frac{t_k+t_l}{2}})]_{k,l=1}^m$ *is a positive semi-definite matrix for all* $m \in \mathbb{N}, m \le n$, $t_1, \ldots, t_m \in I$. *In particular,*

$$\det[\digamma_j(.,.,\phi_{\frac{t_k+t_l}{2}})]_{k,l=1}^m \ge 0 \text{ for all } m \in \mathbb{N},\ m = 1, 2, \ldots, n.$$

(ii) *If the function* $t \mapsto \digamma_j(.,.,\phi_t)$ *is continuous on* I, *then it is* n*-exponentially convex on* I.

Proof. Fix $j = 1, 2, \ldots, 6$. To prove (i), let us define the function

$$\omega(y) = \sum_{k,l=1}^{n} b_k b_l \phi_{t_{kl}}(y),$$

where

$$t_{kl} = \frac{t_k + t_l}{2},\ \ t_k \in I, b_k \in \mathbb{R},\ \ k = 1, 2, \ldots, n.$$

Since the function $t \mapsto [y_0, y_1, y_2; \phi_t]$ is n-exponentially convex in the Jensen sense on I by assumption, it follows that

$$[y_0, y_1, y_2; \omega] = \sum_{k,l=1}^{n} b_k b_l [y_0, y_1, y_2; \phi_{t_{kl}}] \ge 0.$$

This implies that ω is convex on (a, b). Hence $\digamma_j(.,.,\omega) \ge 0$, which is equivalent to

$$\sum_{k,l=1}^{n} b_k b_l \digamma_j(.,.,\phi_{t_{kl}}) \ge 0,$$

and so we conclude that the function $t \mapsto \digamma_j(.,.,\phi_t)$ is n-exponentially convex function in the Jensen sense on I. The remaining part follows from Proposition 1.

(ii) If the function $t \mapsto \digamma_j(.,.,\phi_t)$ is continuous on I, then it is n-exponentially convex on I by definition. □

As a consequence of the above theorem we can give the following corollary.

Corollary 1. *Let* $\Omega = \{\phi_t \colon t \in I \subseteq \mathbb{R}\}$ *be a family of differentiable functions defined on* (a, b) *such that the function* $t \mapsto [y_0, y_1, y_2; \phi_t]$ *is exponentially convex in the Jensen sense on* I *for every three mutually different points* $y_0, y_1, y_2 \in (a, b)$. *Consider* $\digamma_1 = \Lambda_1(X, Y, W; \phi_t)$ *and* $\digamma_2 = \Lambda_2(x, y, w; \phi_t)$ *if* (9) *and* (10) *hold for every* $\tau \in [\alpha, \beta]$, *and consider* $\digamma_3 = -\Lambda_1(X, Y, W; \phi_t)$ *and* $\digamma_4 = -\Lambda_2(x, y, w; \phi_t)$ *if the reverse inequalities in* (9) *and* (10) *hold for every* $\tau \in [\alpha, \beta]$. *Then, for the linear functionals* $\digamma_j(.,.,\phi_t)$ $(j = 1, 2, \ldots, 6)$, *the following statements hold.*

(i) *The function* $t \mapsto \digamma_j(.,.,\phi_t)$ *is exponentially convex in the Jensen sense on* I *and the matrix* $[\digamma_j(.,.,\phi_{\frac{t_k+t_l}{2}})]_{k,l=1}^m$ *is a positive semi-definite matrix for all* $m \in \mathbb{N}$, $t_1, \ldots, t_m \in I$. *In particular,*

$$\det[\digamma_j(.,.,\phi_{\frac{t_k+t_l}{2}})]_{k,l=1}^m \geq 0.$$

(ii) *If the function* $t \mapsto \digamma_j(.,.,\phi_t)$ *is continuous on* I, *then it is exponentially convex on* I.

Corollary 2. *Let* $\Omega = \{\phi_t\colon t \in I \subseteq \mathbb{R}\}$ *be a family of differentiable functions defined on* (a,b) *such that the function* $t \mapsto [y_0, y_1, y_2; \phi_t]$ *is* 2-*exponentially convex in the Jensen sense on* I *for every three mutually different points* $y_0, y_1, y_2 \in (a,b)$. *Consider* $\digamma_1 = \Lambda_1(X,Y,W;\phi_t)$ *and* $\digamma_2 = \Lambda_2(x,y,w;\phi_t)$ *if* (9) *and* (10) *hold for every* $\tau \in [\alpha,\beta]$, *and consider* $\digamma_3 = -\Lambda_1(X,Y,W;\phi_t)$ *and* $\digamma_4 = -\Lambda_2(x,y,w;\phi_t)$ *if the reverse inequalities in* (9) *and* (10) *hold for every* $\tau \in [\alpha,\beta]$. *Suppose that* $\digamma_j(.,.,\phi_t)$ $(j = 1,2,\ldots,6)$ *is strictly positive for* $\phi_t \in \Omega$. *Then, for the linear functionals* $\digamma_j(.,.,\phi_t)$ $(j = 1,2,\ldots,6)$, *the following statements hold.*

(i) *If the function* $t \mapsto \digamma_j(.,.,\phi_t)$ *is continuous on* I, *then it is* log-*convex on* I *and, for* $r,s,t \in I$ *such that* $r < s < t$, *we have*

$$(\digamma_j(.,.,\phi_s))^{t-r} \leq (\digamma_j(.,.,\phi_r))^{t-s}(\digamma_j(.,.,\phi_t))^{s-r}. \tag{15}$$

If $r < t < s$ *or* $s < r < t$, *then the reverse inequality in* (15) *holds.*

(ii) *If the function* $t \mapsto \digamma_j(.,.,\phi_t)$ *is differentiable on* I, *then, for every* $s,t,u,v \in I$, *such that* $s \leq u$ *and* $t \leq v$, *we have*

$$\mathfrak{B}_{s,t}(.,.,\digamma_j,\Omega) \leq \mathfrak{B}_{u,v}(.,.,\digamma_j,\Omega) \tag{16}$$

where

$$\mathfrak{B}^j_{s,t}(\Omega) = \mathfrak{B}_{s,t}(.,.,\digamma_j,\Omega) = \begin{cases} \left(\frac{\digamma_j(.,.,\phi_s)}{\digamma_j(.,.,\phi_t)}\right)^{\frac{1}{s-t}}, & s \neq t, \\ \exp\left(\frac{\frac{d}{ds}\digamma_j(.,.,\phi_s)}{\digamma_j(.,.,\phi_s)}\right), & s = t, \end{cases} \tag{17}$$

for $\phi_s, \phi_t \in \Omega$.

Proof. (i) By Remark 2 and Theorem 7, we have log-convexity of $\digamma_j(.,.,\phi_t)$, and, by using $\phi(x) = \log \digamma_j(.,.,\phi_x)$ in (13), we get (15).

(ii) For a convex function ϕ, the inequality

$$\frac{\phi(s) - \phi(t)}{s - t} \leq \frac{\phi(u) - \phi(v)}{u - v}, \tag{18}$$

holds for all $s,t,u,v \in I$ such that $s \leq u$, $t \leq v$, $s \neq t$, $u \neq v$ (see [12, p. 2]). Since, because of (i), the function $\digamma_j(.,.,\phi_s)$ is log-convex, by setting $\phi(s) = \ln \digamma_j(.,.,\phi_s)$ in (18), we have

$$\frac{\ln \digamma_j(.,.,\phi_s) - \ln \digamma_j(.,d,\phi_t)}{s-t} \leq \frac{\ln \digamma_j(.,.,\phi_u) - \ln \digamma_j(.,.,\phi_v)}{u-v}$$

for $s \leq u$, $t \leq v$, $s \neq t$, $u \neq v$, which is equivalent to (16). The cases $s = t$ and $u = v$ can be treated similarly. □

The following improvement and reversion of Slater's inequality are valid.

Theorem 8. *Let $\Lambda = \{\phi_t\colon\ t \in I \subseteq \mathbb{R}\}$ be a family of differentiable functions defined on (a,b) such that the function $t \mapsto [y_0, y_1, y_2; \phi_t]$ is 2-exponentially convex in the Jensen sense on I for every three mutually different points $y_0, y_1, y_2 \in (a,b)$. Let the function $t \mapsto \digamma_5(\mathbf{x}, d, \phi_t)$ be strictly positive and continuous on I, $\sum_{i=1}^{n} w_i \phi_t'(x_i) \neq 0$, and*

$$d_t = \frac{\sum_{i=1}^{n} w_i x_i \phi_t'(x_i)}{\sum_{i=1}^{n} w_i \phi_t'(x_i)} \in (a,b).$$

Consider the function F_t defined by

$$F_t = \phi_t\left(d_t\right) - \frac{1}{W_n}\sum_{i=1}^{n} w_i \phi_t(x_i).$$

Then

$$F_t \geq \left[\digamma_5(\mathbf{x}, d_t, \phi_s)\right]^{\frac{t-r}{s-r}} \left[\digamma_5(\mathbf{x}, d_t, \phi_r)\right]^{\frac{s-t}{s-r}} \tag{19}$$

for $r, s, t \in I$ such that $r < s < t$ or $t < r < s$, and

$$F_t \leq \left[\digamma_5(\mathbf{x}, d_t, \phi_s)\right]^{\frac{t-r}{s-r}} \left[\digamma_5(\mathbf{x}, d_t, \phi_r)\right]^{\frac{s-t}{s-r}} \tag{20}$$

for $r, s, t \in I$ with $r < t < s$.

Proof. For $j = 5$ and $d = d_t$ in (15), we have

$$\left(\digamma_5(\mathbf{x}, d_t, \phi_s)\right)^{t-r} \leq \left(\digamma_5(\mathbf{x}, d_t, \phi_r)\right)^{t-s} (F_t)^{s-r},$$

where $r, s, t \in I$ are such that $r < s < t$, which is equivalent to

$$(F_t)^{s-r} \geq \left(\digamma_5(\mathbf{x}, d_t, \phi_s)\right)^{t-r} \left(\digamma_5(\mathbf{x}, d_t, \phi_r)\right)^{s-t}. \tag{21}$$

From (21), we have (19) and, similarly, by setting $d = d_t$ in (15) for $r, s, t \in I$ such that $t < r < s$, we get

$$\left(\digamma_5(\mathbf{x}, d_t, \phi_r)\right)^{s-t} \leq (F_t)^{s-r} \left(\digamma_5(\mathbf{x}, d_t, \phi_s)\right)^{r-t},$$

which is equivalent to (19).

As, for $r < t < s$, the reverse inequality holds in (15), by taking $d = d_t$ for $j = 5$ in the reverse of (15), we get (20). □

Theorem 9. *Let $\Lambda = \{\phi_t\colon\ t \in I \subseteq \mathbb{R}\}$ be a family of differentiable functions defined on an interval (a,b) such that the function $t \mapsto [y_0, y_1, y_2; \phi_t]$ is 2-exponentially convex in the Jensen sense on I for every three mutually different points $y_0, y_1, y_2 \in (a,b)$. Then, for every $m \in \mathbb{N}$ and for any $t_k \in I$, $k \in \{1, 2, \ldots, m\}$, the matrix $[\digamma_5(\mathbf{x}, d_t, \phi_{\frac{t_k+t_l}{2}})]_{k,l=1}^{m}$ is positive semi-definite. In particular,*

$$\det[\digamma_5(\mathbf{x}, d_t, \phi_{\frac{t_k+t_l}{2}})]_{k,l=1}^{m} \geq 0. \tag{22}$$

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Proof. By taking $j = 5$ and $d = d_t$ in Corollary 1 (i), we get the required result. □

Remark 4. We note that $\digamma_5(\mathbf{x}, d_t, \phi_t) = F_t$. So, by setting $m = 2$, $t = t_1$ in (22), we have the special case of (19) for $t = t_1$, $s = t_2$, $r = \frac{t_1+t_2}{2}$ if $t_1 < t_2$, and for $t = t_1$, $r = t_2$, $s = \frac{t_1+t_2}{2}$ if $t_2 < t_1$. Similarly, for the case $m = 2$, $t = \frac{t_1+t_2}{2}$ in (22), we have the special case of (20) for $r = t_1$, $s = t_2$, $t = \frac{t_1+t_2}{2}$ if $t_1 < t_2$, and for $r = t_2$, $s = t_1$, $t = \frac{t_1+t_2}{2}$ if $t_2 < t_1$.

The following improvement and reversion of the right inequality in [9, (3.6)] are also valid.

Theorem 10. *Let $\Lambda = \{\phi_t\colon\ t \in I \subseteq \mathbb{R}\}$ be a family of differentiable functions defined on an interval (a, b) such that the function $t \mapsto [y_0, y_1, y_2; \phi_t]$ is 2-exponentially convex in the Jensen sense on I for every three mutually different points $y_0, y_1, y_2 \in (a, b)$. Let the function $t \mapsto \digamma_5(\mathbf{x}, d, \phi_t)$ be strictly positive and continuous on I, and*

$$\bar{d}_t = (\phi_t')^{-1}\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i\phi_t'(x_i)\right) \in (a, b). \tag{23}$$

Consider the function G_t defined by

$$G_t = \phi_t(\bar{d}_t) + \frac{1}{W_n}\sum_{i=1}^{n} w_i(x_i - \bar{d}_t)\phi_t'(x_i) - \frac{1}{W_n}\sum_{i=1}^{n} w_i\phi_t(x_i).$$

Then

$$G_t \geq [\digamma_5(\mathbf{x}, \bar{d}_t, \phi_s)]^{\frac{t-r}{s-r}}[\digamma_5(\mathbf{x}, \bar{d}_t, \phi_r)]^{\frac{s-t}{s-r}}$$

for $r, s, t \in I$ such that $r < s < t$ or $t < r < s$, and

$$G_t \leq [\digamma_5(\mathbf{x}, \bar{d}_t, \phi_s)]^{\frac{t-r}{s-r}}[\digamma_5(\mathbf{x}, \bar{d}_t, \phi_r)]^{\frac{s-t}{s-r}}$$

for $r, s, t \in I$ such that $r < t < s$.

Proof. The proof is similar to the proof of Theorem 8, but uses $\bar{d}_t$ instead of d_t. □

Theorem 11. *Let $\Lambda = \{\phi_t\colon\ t \in I \subseteq \mathbb{R}\}$ be a family of differentiable functions defined on (a, b) such that the function $t \mapsto [y_0, y_1, y_2; \phi_t]$ is 2-exponentially convex in the Jensen sense on I for every three mutually different points $y_0, y_1, y_2 \in (a, b)$. If (23) holds, then for every $m \in \mathbb{N}$ and for any $t_k \in I$, $k \in \{1, 2, \ldots, m\}$, the matrix $[\digamma_5(\mathbf{x}, \bar{d}_t, \phi_{\frac{t_k+t_l}{2}})]_{k,l=1}^{m}$ is positive semi-definite. In particular,*

$$\det[\digamma_5(\mathbf{x}, \bar{d}_t, \phi_{\frac{t_k+t_l}{2}})]_{k,l=1}^{m} \geq 0. \tag{24}$$

Proof. For $j = 5$ and $d = \bar{d}_t$ in Corollary 1 (i), we get the required results. □

Remark 5. We note that $\digamma_5(\mathbf{x}, \bar{d}_t, \phi_t) = G_t$. So, by setting $m = 2$, $t = t_1$ in (24), we have the special case of (10) for $t = t_1$, $s = t_2$, $r = \frac{t_1+t_2}{2}$ if $t_1 < t_2$, and for $t = t_1$, $r = t_2$, $s = \frac{t_1+t_2}{2}$ if $t_2 < t_1$. Similarly, by setting $m = 2, t = \frac{t_1+t_2}{2}$ in (24), we have the special case of (10) for $r = t_1$, $s = t_2$, $t = \frac{t_1+t_2}{2}$ if $t_1 < t_2$, and for $r = t_2$, $s = t_1$, $t = \frac{t_1+t_2}{2}$ if $t_2 < t_1$.

Remark 6. Analogously, we can give the integral versions of Theorems 8 – 11.

3. Examples

In this section, we present several families of functions which fulfil the conditions of the results proved in the previous section. This enables us to construct large families of functions which are exponentially convex.

Example 1. Let

$$\Xi_1 = \{\varphi_t\colon\ (0, \infty) \mapsto \mathbb{R}\colon\ t \in \mathbb{R}\}$$

be a family of functions defined by

$$\varphi_t(x) = \begin{cases} \frac{x^t}{t(t-1)}, & t \neq 0, 1, \\ -\ln x, & t = 0, \\ x \ln x, & t = 1. \end{cases}$$

Since φ_t is convex on $\mathbb{R}^+$ and $t \mapsto \frac{d^2}{dx^2}\varphi_t(x)$ is exponentially convex (see [8]), we have that $g(y) = \sum_{k,l=1}^{n} b_k b_l \varphi_{t_{kl}}(y)$ is convex on $\mathbb{R}^+$, implying that $t \mapsto [y_0, y_1, y_2; \varphi_t]$ is exponentially convex (and thus exponentially convex in the Jensen sense). By using Corollary 1, we conclude that $t \mapsto \digamma_j(., ., \varphi_t)$ ($j = 1, 2, \ldots, 6$) are exponentially convex in the Jensen sense. These mappings are continuous (see [3] – [4]), thus $t \mapsto \digamma_j(., ., \varphi_t)$ are exponentially convex. Therefore, the results proved in the previous section can be applied to this family.

In [3] – [4], the authors proved all the results of the previous section for Ξ_1 and also constructed a class of Cauchy means.

Example 2. Let

$$\Xi_2 = \{\psi_t\colon\ \mathbb{R} \to [0, \infty)\colon\ t \in \mathbb{R}\}$$

be a family of functions defined by

$$\psi_t(x) = \begin{cases} \frac{1}{t^2}\, e^{tx}, & t \neq 0, \\ \frac{1}{2}\, x^2, & t = 0. \end{cases}$$

We have $\frac{d^2}{dx^2}\psi_t(x) = e^{tx} > 0$, which shows that ψ_t is convex on $\mathbb{R}$ for every $t \in \mathbb{R}$ and $t \mapsto \frac{d^2}{dx^2}\psi_t(x)$ is exponentially convex by definition (see [8]). It is easy to prove that the function $t \mapsto [y_0, y_1, y_2; \psi_t]$ is exponentially convex.

Arguing as in Example 1, we have that $t \mapsto \digamma_j(.,.,\psi_t)$ $(j = 1,2,\dots,6)$ are exponentially convex. Therefore, for this family of functions, results of Theorems 8−11 hold. For Ξ_2, means are of the form

$$\ln \mathfrak{B}^j_{s,t} = \begin{cases} \frac{1}{s-t}\ln\left(\frac{\digamma_j(.,.,\psi_s)}{\digamma_j(.,.,\psi_t)}\right), & s \neq t, \\ \frac{\digamma_j(.,.,id.\psi_s)}{\digamma_j(.,.,\psi_s)} - \frac{2}{s}, & s = t \neq 0, \\ \frac{\digamma_j(.,.,id.\psi_0)}{3\digamma_j(.,.,\psi_0)}, & s = t = 0. \end{cases}$$

For $j = 1,2,3,4$, the authors in [2] and [1] proved all the results of the previous section for Ξ_2 and also constructed a class of Cauchy means.

For $j = 5$, we have

$$\ln \mathfrak{B}^5_{s,t} = \frac{1}{s-t}\ln\left(\frac{t^2}{s^2}\cdot\frac{\sum_{i=1}^n w_i[e^{sd} - e^{sx_i} - se^{sx_i}(d-x_i)]}{\sum_{i=1}^n w_i[e^{td} - e^{tx_i} - te^{tx_i}(d-x_i)]}\right), \quad s \neq t,$$

$$\ln \mathfrak{B}^5_{s,s} = \frac{\sum_{i=1}^n w_i[de^{sd} - x_ie^{sx_i} - e^{sx_i}(sx_i+1)(d-x_i)]}{\sum_{i=1}^n w_i[e^{sd} - e^{sx_i} - se^{sx_i}(d-x_i)]} - \frac{2}{s}, \quad s \neq 0,$$

$$\ln \mathfrak{B}^5_{0,0} = \frac{\sum_{i=1}^n w_i[d^3 - x_i^3 - 3x_i^2(d-x_i)]}{3\sum_{i=1}^n w_i[d^2 - x_i^2 - 2x_i(d-x_i)]}.$$

By (16), these means are monotonic.

Example 3. Let

$$\Xi_3 = \{\theta_t\colon\ (0,\infty) \to (0,\infty)\colon\ t \in (0,\infty)\}$$

be a family of functions defined by

$$\theta_t(x) = \frac{e^{-x\sqrt{t}}}{t}.$$

The function $t \mapsto \frac{d^2}{dx^2}\theta_t(x) = e^{-x\sqrt{t}}$ is exponentially convex (see [8]). By the same argument as given in Example 1, the functions $\digamma_j(.,.,\theta_t)$ $(j = 1,2,\dots,6)$ are exponentially convex. Therefore, for this family of functions, Theorems 8−11 hold. Also, $\mathfrak{B}^j_{s,t}(\Xi_3)$ $(j = 1,2,\dots,6)$ from (17) becomes

$$\mathfrak{B}^j_{s,t}(\Xi_3) = \begin{cases} \left(\frac{\digamma_j(.,.,\theta_s)}{\digamma_j(.,.,\theta_t)}\right)^{\frac{1}{s-t}}, & s \neq t, \\ \exp\left(-\frac{\digamma_j(.,.,id.\theta_s)}{2\sqrt{s}(\digamma_j(.,.,\theta_s))} - \frac{1}{s}\right), & s = t. \end{cases}$$

In particular, for $j = 1,2$, using the notation

$$S(z_{ij}) = \sum_{i=1}^n\sum_{j=1}^m z_{ij}, \quad T(z(t,s)) = \int_c^d\int_a^b z(t,s)\,d\mu(t)\,du(s), \tag{25}$$

we have

$$\mathfrak{B}^1_{s,t}(\Xi_3) = \begin{cases} \left(\frac{t}{s} \, \frac{S\left(w_{ij}\left(e^{x_{ij}\sqrt{s}} - e^{y_{ij}\sqrt{s}}\right)\right)}{S\left(w_{ij}\left(e^{x_{ij}\sqrt{t}} - e^{y_{ij}\sqrt{t}}\right)\right)} \right)^{\frac{1}{s-t}}, & s \neq t, \\ \exp\left(\frac{S\left(w_{ij}\left(x_{ij}e^{x_{ij}\sqrt{s}} - y_{ij}e^{y_{ij}\sqrt{s}}\right)\right)}{2\sqrt{s}\, S\left(w_{ij}\left(e^{y_{ij}\sqrt{s}} - e^{x_{ij}\sqrt{s}}\right)\right)} - \frac{1}{s} \right), & s = t, \end{cases}$$

and

$$\mathfrak{B}^2_{s,t}(\Xi_3) = \begin{cases} \left(\frac{m}{l} \, \frac{T\left(w(t,s)\left(e^{-x(t,s)\sqrt{l}} - e^{-y(t,s)\sqrt{l}}\right)\right)}{T\left(w(t,s)\left(e^{-x(t,s)\sqrt{m}} - e^{-y(t,s)\sqrt{m}}\right)\right)} \right)^{\frac{1}{l-m}}, & l \neq m, \\ \exp\left(\frac{T\left(w(t,s)\left(x(t,s)\, e^{-x(t,s)\sqrt{m}} - y(t,s)\, e^{-y(t,s)\sqrt{m}}\right)\right)}{2\sqrt{m}\, T\left(w(t,s)\left(e^{-y(t,s)\sqrt{m}} - e^{-x(t,s)\sqrt{m}}\right)\right)} - \frac{1}{m} \right), & l = m. \end{cases}$$

By (16), these means are monotonic in parameters s and t .

Example 4. Let

$$\Xi_4 = \{\delta_t \colon (0,\infty) \to (0,\infty) \colon t \in (0,\infty)\}$$

be a family of functions defined by

$$\delta_t(x) = \begin{cases} \frac{t^{-x}}{(\ln t)^2}, & t \neq 1, \\ \frac{x^2}{2}, & t = 1. \end{cases}$$

Since $\frac{d^2}{dx^2}\delta_t(x) = t^{-x} = e^{-x\ln t} > 0$ for $x > 0$, by the same argument as given in Example 1, the functions $t \mapsto F_j(.,.,\delta_t)$ $(j = 1,\ldots,6)$ are exponentially convex. Therefore, for this family of functions, Theorems 8 − 11 hold and $\mathfrak{B}^j_{s,t}(\Xi_4)$ $(j = 1,2,\ldots,6)$ from (17) becomes

$$\mathfrak{B}^j_{s,t}(\Xi_4) = \begin{cases} \left(\frac{F_j(.,.,\delta_s)}{F_j(.,.,\delta_t)} \right)^{\frac{1}{s-t}}, & s \neq t, \\ \exp\left(-\frac{F_j(.,.,id.\delta_s)}{sF_j(.,.,\delta_s)} - \frac{2}{s\ln s} \right), & s = t \neq 1, \\ \exp\left(-\frac{1}{3}\frac{F_j(.,.,id.\delta_1)}{F_j(.,.,\delta_1)} \right), & s = t = 1. \end{cases}$$

In particular, for $j = 1, 2$, we have

$$\mathfrak{B}^1_{s,t}(\Xi_4) = \begin{cases} \left(\frac{(\ln t)^2}{(\ln s)^2} \, \frac{S\left(w_{ij}\left(s^{-x_{ij}} - s^{-y_{ij}}\right)\right)}{S\left(w_{ij}\left(t^{-x_{ij}} - t^{-y_{ij}}\right)\right)} \right)^{\frac{1}{s-t}}, & s \neq t, \\ \exp\left(\frac{S\left(w_{ij}\left(y_{ij}\, s^{-y_{ij}} - x_{ij}\, s^{-x_{ij}}\right)\right)}{s\, S\left(w_{ij}\left(s^{-x_{ij}} - s^{-y_{ij}}\right)\right)} - \frac{2}{s\ln s} \right), & s = t \neq 1, \\ \exp\left(\frac{1}{3} \, \frac{S\left(w_{ij}\left(y_{ij}^3 - x_{ij}^3\right)\right)}{S\left(w_{ij}\left(x_{ij}^2 - y_{ij}^2\right)\right)} \right), & s = t = 1, \end{cases}$$

and

$$\mathfrak{B}^2_{l,m}(\Xi_4) = \begin{cases} \left(\frac{(\ln m)^2}{(\ln l)^2} \frac{T\left(w(t,s)\left(l^{-x(t,s)} - l^{-y(t,s)}\right)\right)}{T\left(w(t,s)\left(m^{-x(t,s)} - m^{-y(t,s)}\right)\right)} \right)^{\frac{1}{l-m}}, & l \neq m, \\ \exp\left(\frac{T\left(w(t,s)\left(y(t,s)\, l^{-y(t,s)} - x(t,s)\, l^{-x(t,s)}\right)\right)}{l\, T\left(w(t,s)\left(l^{-x(t,s)} - l^{y(t,s)}\right)\right)} - \frac{2}{l \ln l} \right), & l = m \neq 1, \\ \exp\left(\frac{1}{3} \frac{T(w(t,s)(y^3(t,s) - x^3(t,s)))}{T(w(t,s)(x^2(t,s) - y^2(t,s)))} \right), & l = m = 1, \end{cases}$$

where S and T are defined by (25). The monotonicity of $\mathfrak{B}^j_{s,t}(\Xi_4)$ follows from (16).

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