ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 18, Number 2, December 2014
Available online at http://acutm.math.ut.ee

Approximating the Riemann–Stieltjes integral via a Chebyshev type functional

S. S. Dragomir

Abstract. Some new sharp upper bounds for the absolute value of the error functional $D(f,u)$ in approximating the Riemann–Stieltjes integral $\int_a^b f(t)\,du(t)$ by the quantity $[u(b)-u(a)]\cdot\frac{1}{b-a}\int_a^b f(t)\,dt$ are given.

1. Introduction

In order to approximate the Riemann–Stieltjes integral $\int_a^b f(t)\,du(t)$ by the simpler quantity

$$[u(b)-u(a)]\cdot\frac{1}{b-a}\int_a^b f(t)\,dt,$$

provided that both integrals exist, Dragomir and Fedotov introduced in [11] the following *error functional of Chebyshev type*

$$D(f;u)=\int_a^b f(t)\,du(t)-[u(b)-u(a)]\cdot\frac{1}{b-a}\int_a^b f(t)\,dt,$$

and pointed out the following sharp upper bound for $|D(f;u)|$, namely

$$|D(f;u)|\le\frac{1}{2}L(M-m)(b-a), \tag{1.1}$$

provided the *integrator* $u:[a,b]\to\mathbb{R}$ is L-Lipschitzian on $[a,b]$, i.e.,

$$|u(x)-u(y)|\le L|x-y|$$

for any $x,y\in[a,b]$ and the *integrand* $f:[a,b]\to\mathbb{R}$ is Riemann integrable on $[a,b]$ and satisfies the boundedness condition

$$-\infty<m\le f(x)\le M<\infty\quad\text{for a.e. } x\in[a,b].$$

Received December 18. 2013.

2010 *Mathematics Subject Classification.* 26D15, 41A55.

Key words and phrases. Riemann–Stieltjes integral, quadrature rules, Grüss type inequalities, Chebyshev functional, error bounds.

http://dx.doi.org/10.12097/ACUTM.2014.18.20

14

The multiplicative constant $\frac{1}{2}$ in (1.1) is the best possible in the sense that it cannot be replaced by a smaller constant.

In the follow-up paper [12], the authors provided a different bound, namely

$$|D(f;u)| \leq \frac{1}{2}K(b-a)\bigvee_a^b(u), \tag{1.2}$$

provided that f is K-Lipschitzian and u is of bounded variation on $[a,b]$.

The result (1.2) was improved in [10] for the case of monotonic non-decreasing functions. We have shown in this case that

$$|D(f;u)| \leq \frac{1}{2}K(b-a)[u(b)-u(a)-S(u)] \tag{1.3}$$
$$\left(\leq \frac{1}{2}K(b-a)[u(b)-u(a)]\right),$$

where

$$S(u) := \frac{4}{(b-a)^2}\int_a^b u(t)\left(t-\frac{a+b}{2}\right)dt \geq 0.$$

In (1.3) the constant $\frac{1}{2}$ is the best possible in both inequalities.

For other sharp bounds on the error functional $D(f;u)$, see the recent papers [9], [7] and [14]. For other inequalities for the Riemann–Stieltjes integral, see [1], [2], [3], [4], [6] and [13].

The main aim of this paper is to further investigate the error functional $D(f;u)$. Two representations are given. These are applied to obtain some inequalities for $D(f;u)$ which improve earlier results.

Applications for the classical *Chebyshev functional* $C(f,g)$, where

$$C(f,g) := \frac{1}{b-a}\int_a^b f(t)g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \cdot \frac{1}{b-a}\int_a^b g(t)\,dt, \tag{1.4}$$

and f,g are integrable and belonging to different classes of functions, are also provided.

2. Representation results

For a function $g:[a,b]\to\mathbb{R}$, consider the *generalised trapezoid error transform* $\Phi_g:[a,b]\to\mathbb{R}$ given by

$$\Phi_g(t) := \frac{1}{b-a}[(b-t)g(a)+(t-a)g(b)] - g(t), \quad t\in[a,b]$$

and if g is Lebesgue integrable, the *Ostrowski transform*, which is the error of approximating the function by its integral mean, defined by

$$\Theta_g(t) := g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds, \quad t\in[a,b].$$

We also define the kernel $Q : [a, b]^2 \to \mathbb{R}$,

$$Q(t, s) := \begin{cases} t - b & \text{if } a \le s \le t \le b, \\ \\ t - a & \text{if } a \le t < s \le b. \end{cases} \tag{2.1}$$

The following representation result in terms of Θ_g and Q may be stated.

Lemma 1. *If $f, u : [a, b] \to \mathbb{R}$ are bounded functions and such that the Riemann–Stieltjes integral $\int_a^b f(t)\, du(t)$ and the Riemann integral $\int_a^b f(t)\, dt$ exist, then we have the representation*

$$D(f; u) = \int_a^b \Theta_f(s)\, du(s) = \frac{1}{b-a} \int_a^b \left(\int_a^b Q(t, s)\, df(t) \right) du(s). \tag{2.2}$$

Proof. We have by the definition of Q and integrating by parts in the Riemann–Stieltjes integral that

$$\begin{aligned} &\frac{1}{b-a} \int_a^b \left(\int_a^b Q(t, s)\, df(t) \right) du(s) \\ &= \frac{1}{b-a} \int_a^b \left[\int_a^s (t-a)\, df(t) + \int_s^b (t-b)\, df(t) \right] du(s) \\ &= \frac{1}{b-a} \int_a^b \left[f(t)(t-a) \Big|_a^s - \int_a^s f(t)\, dt + (t-b) f(t) \Big|_s^b - \int_s^b f(t)\, dt \right] du(s) \\ &= \frac{1}{b-a} \int_a^b \left[f(s)(s-a) - \int_a^s f(t)\, dt + (b-s) f(s) - \int_s^b f(t)\, dt \right] du(s) \\ &= \int_a^b \Theta_f(s)\, du(s), \end{aligned}$$

and the second equality is proved.

The first identity is obvious by the definition of $D(f; u)$. □

The following corollary can be stated about the representation of the *Chebyshev functional* $C(f, g)$ defined in (1.4).

Corollary 1. *Assume that $f, g : [a, b] \to \mathbb{R}$ are Riemann integrable on $[a, b]$, then*

$$\begin{aligned} C(f, g) &= \frac{1}{b-a} \int_a^b \Theta_f(s)\, g(s)\, ds \\ &= \frac{1}{(b-a)^2} \int_a^b \left(\int_a^b Q(t, s)\, df(t) \right) g(s)\, ds. \end{aligned}$$

Proof. It is well known (see for instance [5, Theorem 7.33, p. 162] that if g is Riemann integrable and $u(t) = \int_a^t g(s)\, ds$, then for any Riemann integrable function f we have that the Riemann–Stieltjes integral $\int_a^b f(t)\, du(t)$

exists and $\int_a^b f(t)\,du(t) = \int_a^b f(t)\,g(t)\,dt$. Therefore, we have $D(f;u) = (b-a)\,C(f,g)$ and

$$\int_a^b \left(\int_a^b Q(t,s)\,df(t)\right) du(s) = \int_a^b \left(\int_a^b Q(t,s)\,df(t)\right) g(s)\,ds.$$

□

The second representation of $D(f;u)$ is incorporated in

Lemma 2. *With the assumptions in Lemma 1, we have*

$$D(f;u) = \int_a^b \Phi_u(t)\,df(t) = \frac{1}{b-a}\int_a^b \left(\int_a^b Q(t,s)\,du(s)\right) df(t), \qquad (2.3)$$

where Q *is defined by* (2.1).

Proof. By the Fubini type theorem for the Riemann–Stieltjes integral (see for instance [5, Theorem 7.41, p. 167]) we have that

$$\int_a^b \left(\int_a^b Q(t,s)\,du(s)\right) df(t) = \int_a^b \left(\int_a^b Q(t,s)\,df(t)\right) du(s),$$

and the equality between the first and the last term in (2.3) is proved.

Now, observe that

$$\begin{aligned}\int_a^b Q(t,s)\,du(s) &= \int_a^t (t-b)\,du(s) + \int_t^b (t-a)\,du(s)\\ &= (t-b)\,[u(t)-u(a)] + (t-a)\,[u(b)-u(t)]\\ &= (b-a)\,\Phi_u(t),\end{aligned}$$

for any $t \in [a,b]$, and then integrating over $f(t)$, we deduce the second equality in (2.3). □

Corollary 2. *Assume that* f *and* g *are Riemann integrable on* $[a,b]$. *Then*

$$\begin{aligned}C(f,g) &= \frac{1}{b-a}\int_a^b \tilde{\Phi}_g(t)\,df(t)\\ &= \frac{1}{(b-a)^2}\int_a^b \left(\int_a^b Q(t,s)\,g(s)\,ds\right) df(t),\end{aligned}$$

where

$$\tilde{\Phi}_g(t) = \Phi_{\int_a^{\cdot} g}(t) = \frac{t-a}{b-a}\int_a^b g(s)\,ds - \int_a^t g(s)\,ds, \quad t \in [a,b].$$

3. Bounds in the case when u is of bounded variation

The following lemma is of interest in itself.

Lemma 3. *If $p:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$ and $v:[a,b]\to\mathbb{R}$ is of bounded variation on $[a,b]$, then*

$$\begin{aligned}\left|\int_a^b p(t)\,dv(t)\right| &\le \int_a^b |p(t)|\,d\bigvee_a^t(v)\\ &\le \left[\bigvee_a^b(v)\right]^{\frac{1}{q}}\left\{\int_a^b |p(t)|^r\,d\left[\bigvee_a^t(v)\right]\right\}^{\frac{1}{r}}\\ &\le \max_{t\in[a,b]}|p(t)|\bigvee_a^b(v),\end{aligned}\tag{3.1}$$

where $q>1,\ 1/q+1/r=1$.

Proof. Since the Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists, for any division $I_n: a=t_0<t_1<\cdots<t_{n-1}<t_n=b$ with the norm $v(I_n):=\max_{i\in\{0,\dots,n-1\}}(t_{i+1}-t_i)\to 0$ and for any intermediate points $\xi_i\in[t_i,t_{i+1}]$, $i\in\{0,\dots,n-1\}$, we have

$$\begin{aligned}\left|\int_a^b p(t)\,dv(t)\right| &= \left|\lim_{v(I_n)\to 0}\sum_{i=0}^{n-1} p(\xi_i)\left[v(t_{i+1})-v(t_i)\right]\right|\\ &\le \lim_{v(I_n)\to 0}\sum_{i=0}^{n-1}|p(\xi_i)|\left|v(t_{i+1})-v(t_i)\right|.\end{aligned}$$

However,

$$|v(t_{i+1})-v(t_i)|\le \bigvee_{t_i}^{t_{i+1}}(v)=\bigvee_a^{t_{i+1}}(v)-\bigvee_a^{t_i}(v),\tag{3.2}$$

for any $i\in\{0,\dots,n-1\}$ and by (3.2) we have

$$\begin{aligned}\left|\int_a^b p(t)\,dv(t)\right| &\le \lim_{v(I_n)\to 0}\sum_{i=0}^{n-1}|p(\xi_i)|\left[\bigvee_a^{t_{i+1}}(v)-\bigvee_a^{t_i}(v)\right]\\ &= \int_a^b |p(t)|\,d\left[\bigvee_a^t(v)\right],\end{aligned}$$

and the last Riemann–Stieltjes integral exists since $|p|$ is continuous and $\bigvee_a^{\cdot}(v)$ is monotonic nondecreasing.

The last part follows from the following Hölder type inequality

$$\left|\int_a^b g(t)\,dv(t)\right| \le [v(b)-v(a)]^{\frac{1}{q}}\left[\int_a^b |g(t)|^r\,dv(t)\right]^{\frac{1}{r}},\quad q>1,\ \frac{1}{q}+\frac{1}{r}=1,$$

that holds for any continuous function $g:[a,b]\to\mathbb{R}$ and any monotonic nondecreasing function $v:[a,b]\to\mathbb{R}$. The details are omitted. □

The following result holds.

Theorem 1. *Assume that $f,u:[a,b]\to\mathbb{R}$ are of bounded variation and such that the Riemann–Stieltjes integral $\int_a^b f(t)\,du(t)$ exists. Then*

$$\begin{aligned}
|D(f;u)| &\le \frac{1}{b-a}\left[\int_a^b \bigvee_a^s (f)(2s-a-b)\,d\left(\bigvee_a^s (u)\right)\right.\\
&\quad \left.+2\int_a^b \left(\bigvee_a^s (u)\cdot\bigvee_a^s (f)\right)ds-\bigvee_a^b (u)\int_a^b\left(\bigvee_a^s (f)\right)ds\right]\\
&\le \frac{1}{b-a}\int_a^b \bigvee_a^s (f)(2s-a-b)\,d\left(\bigvee_a^s (u)\right)\\
&\quad +\frac{1}{b-a}\int_a^b\left(\bigvee_a^s (u)\cdot\bigvee_a^s (f)\right)ds\\
&\le \frac{1}{b-a}\int_a^b \bigvee_a^s (f)(2s-a-b)\,d\left(\bigvee_a^s (u)\right)\\
&\quad +\bigvee_a^b (u)\cdot\bigvee_a^b (f).
\end{aligned}\tag{3.3}$$

Proof. Utilising the identity (2.2) and the first inequality in (3.1) we have

$$\begin{aligned}
|D(f;u)| &\le \frac{1}{b-a}\int_a^b\left|\int_a^b Q(t,s)\,df(t)\right|d\left(\bigvee_a^s (u)\right)\\
&= \frac{1}{b-a}\int_a^b\left|\int_a^s (t-a)\,df(t)+\int_s^b (t-b)\,df(t)\right|d\left(\bigvee_a^s (u)\right)\\
&\le \frac{1}{b-a}\int_a^b\left[\left|\int_a^s (t-a)\,df(t)\right|+\left|\int_s^b (t-b)\,df(t)\right|\right]d\left(\bigvee_a^s (u)\right)\\
&=: I.
\end{aligned}\tag{3.4}$$

Since f is of bounded variation, by the same inequality in (3.1) we have

$$\begin{aligned}\left|\int_a^s (t-a)\,df(t)\right| &\le \int_a^s (t-a)\,d\left(\bigvee_a^t (f)\right)\\ &= \bigvee_a^s (f)\cdot(s-a) - \int_a^s \left(\bigvee_a^t (f)\right)dt\end{aligned}$$

and

$$\begin{aligned}\left|\int_s^b (t-b)\,df(t)\right| &\le \int_s^b (t-b)\,d\bigvee_s^t (f) = \int_s^b \left(\bigvee_s^t (f)\right)dt\\ &= \int_s^b \left[\bigvee_a^t (f) - \bigvee_a^s (f)\right]dt = \int_s^b \left(\bigvee_a^t (f)\right)dt - (b-s)\bigvee_a^s (f).\end{aligned}$$

This gives that

$$\begin{aligned}I &\le \frac{1}{b-a}\int_a^b \left[\bigvee_a^s (f)(s-a) - \int_a^s \left(\bigvee_a^t (f)\right)dt\right.\\ &\quad \left.+ \int_s^b \left(\bigvee_a^t (f)\right)dt - (b-s)\bigvee_a^s (f)\right] d\left(\bigvee_a^s (u)\right)\\ &= \frac{1}{b-a}\int_a^b \left[\bigvee_a^s (f)(2s-a-b) - \int_a^s \left(\bigvee_a^t (f)\right)dt\right.\\ &\quad \left.+ \int_s^b \left(\bigvee_a^t (f)\right)dt\right] d\left(\bigvee_a^s (u)\right)\\ &= \frac{1}{b-a}\int_a^b (2s-a-b)\bigvee_a^s (f)\,d\left(\bigvee_a^s (u)\right)\\ &\quad + \frac{1}{b-a}\int_a^b \left[\int_a^b \left(\bigvee_a^t (f)\right)dt - 2\int_a^s \left(\bigvee_a^t (f)\right)dt\right] d\left(\bigvee_a^s (u)\right)\\ &= \frac{1}{b-a}\int_a^b (2s-a-b)\bigvee_a^s (f)\,d\left(\bigvee_a^s (u)\right)\\ &\quad + \frac{1}{b-a}\int_a^b \left(\bigvee_a^t (f)\right)dt\cdot\bigvee_a^b (u)\\ &\quad - \frac{2}{b-a}\int_a^b \left(\int_a^s \left(\bigvee_a^t (f)\right)dt\right) d\left(\bigvee_a^s (u)\right). \qquad (3.5)\end{aligned}$$

However, integrating by parts in the Riemann–Stieltjes integral we have

$$\begin{aligned}
&\int_a^b \left(\int_a^s \left(\bigvee_a^t (f)\right) dt\right) d\left(\bigvee_a^s (u)\right) \\
&\qquad = \int_a^s \left(\bigvee_a^t (f)\right) dt \cdot \bigvee_a^s (u)\Bigg|_a^b - \int_a^b \bigvee_a^s (u) \cdot \bigvee_a^s (f)\, ds \\
&\qquad = \bigvee_a^b (u) \cdot \int_a^b \left(\bigvee_a^t (f)\right) dt - \int_a^b \bigvee_a^s (u) \cdot \bigvee_a^s (f)\, ds.
\end{aligned}$$

Inserting this value in the expression of I from (3.5) we deduce the first inequality in (3.3).

The other inequalities are obvious. □

The following result may be stated as well.

Theorem 2. *If $u : [a, b] \to \mathbb{R}$ is of bounded variation and $f : [a, b] \to \mathbb{R}$ is L-Lipschitzian, then*

$$\begin{aligned}
|D(f; u)| &\leq L \left[\frac{1}{2}(b-a) \bigvee_a^b (u) \right. \\
&\qquad \left. - \frac{2}{b-a} \int_a^b \left(\bigvee_a^s (u)\right) \left(s - \frac{a+b}{2}\right) ds\right] \qquad (3.6) \\
&\leq \frac{1}{2} L (b-a) \bigvee_a^b (u).
\end{aligned}$$

The constant $\frac{1}{2}$ is sharp in both inequalities.

Proof. It is well known that if $p : [\alpha, \beta] \to \mathbb{R}$ is L-Lipschitzian and $v : [\alpha, \beta] \to \mathbb{R}$ is Riemann integrable, then the Riemann–Stieltjes integral $\int_\alpha^\beta p(s)\, dv(s)$ exists and $\left|\int_\alpha^\beta p(s)\, dv(s)\right| \leq L \int_\alpha^\beta |p(s)|\, ds$. Utilising this property, we then have

$$\begin{aligned}
\left|\int_a^s (t-a)\, df(t)\right| &\leq L \int_a^s (t-a)\, dt = \frac{L}{2}(s-a)^2, \\
\left|\int_s^b (t-b)\, df(t)\right| &\leq L \int_s^b (b-t)\, dt = \frac{L}{2}(b-s)^2.
\end{aligned}$$

Therefore, by relation (3.4) we have

$$I \leq \frac{L}{2(b-a)} \int_a^b \left[(b-s)^2 + (s-a)^2\right] d\left(\bigvee_a^s (u)\right)$$

$$
\begin{aligned}
&= \frac{L}{2(b-a)}\left[\left[(b-s)^2+(s-a)^2\right]\bigvee_a^s(u)\Big|_a^b - 2\int_a^b \bigvee_a^s(u)(2s-a-b)\,ds\right] \\
&= \frac{L}{2(b-a)}\left[(b-a)^2\bigvee_a^b(u) - 4\int_a^b \bigvee_a^s(u)\left(s-\frac{a+b}{2}\right)ds\right]
\end{aligned}
$$

and the first inequality in (3.6) is proved.

To prove the last part, we use the Chebyshev inequality which states that for two nondecreasing functions g and h,

$$
\frac{1}{b-a}\int_a^b g(s)\,h(s)\,ds \geq \frac{1}{b-a}\int_a^b g(s)\,ds\cdot\frac{1}{b-a}\int_a^b h(s)\,ds.
$$

Then

$$
\begin{aligned}
&\frac{1}{b-a}\int_a^b \bigvee_a^s(u)\left(s-\frac{a+b}{2}\right)ds \\
\geq\ &\frac{1}{b-a}\int_a^b\left(\bigvee_a^s(u)\right)ds\cdot\frac{1}{b-a}\int_a^b\left(s-\frac{a+b}{2}\right)ds
\end{aligned}
$$

and since $\int_a^b\left(s-\frac{a+b}{2}\right)ds=0$, the inequality is proved.

For the sharpness of the constant, we consider the functions $f(t)=t-\frac{a+b}{2}$, $t\in[a,b]$, and $u:[a,b]\to\mathbb{R}$ defined by

$$
u(t) := \begin{cases} 1 & \text{if } t=a, \\ 0 & \text{if } t\in(a,b), \\ 1 & \text{if } t=b. \end{cases}
$$

Then f is Lipschitzian with $L=1$ and u is of bounded variation on $[a,b]$. We have $\bigvee_a^s(u)=1,\ s\in(a,b)$, and $\bigvee_a^b(u)=2$. Also,

$$
\begin{aligned}
D(f;u) &= \int_a^b f(t)\,du(t) - \frac{u(b)-u(a)}{b-a}\int_a^b f(t)\,dt \\
&= \int_a^b f(t)\,du(t) \\
&= f(t)\,u(t)\Big|_a^b - \int_a^b u(t)\,df(t) = b-a
\end{aligned}
$$

and

$$
\int_a^b\left(\bigvee_a^s(u)\right)\left(s-\frac{a+b}{2}\right)ds = \int_a^b\left(s-\frac{a+b}{2}\right)ds = 0.
$$

Replacing the values in (3.6) we get in all sides the same quantity $b-a$. This shows that the constant $\frac{1}{2}$ is the best possible in both inequalities. □

Remark 1. The inequality between the first and last term in (3.6) was firstly discovered by Dragomir and Fedotov in [12] where they also showed the sharpness of the constant $\frac{1}{2}$.

The following result may be stated as well.

Theorem 3. *Assume that* $u : [a,b] \to \mathbb{R}$ *is of bounded variation and* $f : [a,b] \to \mathbb{R}$ *is monotonic nondecreasing and such that the Riemann–Stieltjes integral* $\int_a^b f(t)\,du(t)$ *exists. Then,*

$$|D(f;u)| \le \frac{1}{b-a}\left[\int_a^b (2s-a-b)\, f(s)\, d\left(\bigvee_a^s (u)\right) + 2\int_a^b \left(\bigvee_a^s (u)\right) f(s)\, ds - \int_a^b f(s)\, ds \cdot \bigvee_a^b (u)\right]. \tag{3.7}$$

Proof. It is well known that if the Stieltjes integrals $\int_\alpha^\beta p(t)\,dv(t)$ and $\int_\alpha^\beta |p(t)|\,dv(t)$ exist and v is monotonic nondecreasing on $[\alpha,\beta]$, then

$$\left|\int_\alpha^\beta p(t)\,dv(t)\right| \le \int_\alpha^\beta |p(t)|\,dv(t).$$

Utilising this property we then have

$$\left|\int_a^s (t-a)\,df(t)\right| \le \int_a^s (t-a)\,df(t) = (s-a)\,f(s) - \int_a^s f(t)\,dt$$

and

$$\left|\int_s^b (t-b)\,df(t)\right| \le \int_s^b (t-b)\,df(t) = \int_s^b f(t)\,dt - (b-s)\,f(s)$$

for any $s \in [a,b]$.

Utilising relation (3.4), we obtain

$$\begin{aligned} I &\le \frac{1}{b-a}\left[\int_a^b \left\{(s-a)\,f(s) - \int_a^s f(t)\,dt + \int_s^b f(t)\,dt - (b-s)\,f(s)\right\} d\left(\bigvee_a^s (u)\right)\right] \\ &= \frac{1}{b-a}\left[\int_a^b (2s-a-b)\,f(s)\,d\left(\bigvee_a^s (u)\right) + \int_a^b \left(\int_s^b f(t)\,dt\right) d\left(\bigvee_a^s (u)\right) - \int_a^b \left(\int_a^s f(t)\,dt\right) d\left(\bigvee_a^s (u)\right)\right] =: J. \end{aligned} \tag{3.8}$$

However, integrating by parts in the Riemann–Stieltjes integral, we have

$$\begin{aligned}
&\int_a^b \left(\int_s^b f(t)\,dt\right) d\left(\bigvee_a^s (u)\right) \\
&\quad= \left(\int_s^b f(t)\,dt\right)\cdot \bigvee_a^s (u)\Bigg|_a^b - \int_a^b \left(\bigvee_a^s (u)\right) d\left(\int_s^b f(t)\,dt\right) \\
&\quad= \int_a^b \left(\bigvee_a^s (u)\right) f(s)\,ds
\end{aligned}$$

and

$$\begin{aligned}
&\int_a^b \left(\int_a^s f(t)\,dt\right) d\left(\bigvee_a^s (u)\right) \\
&\quad= \left(\int_a^s f(t)\,dt\right)\cdot \bigvee_a^s (u)\Bigg|_a^b - \int_a^b \left(\bigvee_a^s (u)\right) d\left(\int_a^s f(t)\,dt\right) \\
&\quad= \int_a^b f(t)\,dt\cdot \bigvee_a^b (u) - \int_a^b \left(\bigvee_a^s (u)\right) f(s)\,ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
J &= \frac{1}{b-a}\left[\int_a^b (2s-a-b)\,f(s)\,d\left(\bigvee_a^s (u)\right)\right. \\
&\quad \left. + \int_a^b \left(\bigvee_a^s (u)\right) f(s)\,ds - \int_a^b f(t)\,dt\cdot \bigvee_a^b (u) + \int_a^b \left(\bigvee_a^s (u)\right) f(s)\,ds\right] \\
&= \frac{1}{b-a}\left[\int_a^b (2s-a-b)\,f(s)\,d\left(\bigvee_a^s (u)\right)\right. \\
&\quad \left. + 2\int_a^b \left(\bigvee_a^s (u)\right) f(s)\,ds - \int_a^b f(s)\,ds\cdot \bigvee_a^b (u)\right].
\end{aligned}$$

This together with inequalities (3.4) and (3.8) produces the desired result (3.7). □

4. Bounds in the case when f is of bounded variation

We can state the following result as well.

Theorem 4. *Assume that $f:[a,b]\to\mathbb{R}$ is of bounded variation on $[a,b]$. If $u:[a,b]\to\mathbb{R}$ is continuous and such that there exists constants $L_a, L_b > 0$ and $\alpha,\beta > 0$ with the properties*

$$|u(t)-u(a)| \le L_a (t-a)^{\alpha}, \qquad |u(t)-u(b)| \le L_b (b-t)^{\beta} \tag{4.1}$$

for any $t \in [a,b]$, *then*

$$\begin{aligned}
|D(f;u)| \leq \frac{1}{b-a} L_a & \left[\int_a^b \left(\bigvee_a^t (f) \right) (t-a)^{\alpha} \, dt \right. \\
& \left. - \alpha \int_a^b \left(\bigvee_a^t (f) \right) (b-t)(t-a)^{\alpha-1} \, dt \right] \\
& + \frac{1}{b-a} L \left[{}_b\beta \int_a^b \left(\bigvee_a^t (f) \right) (t-a)(b-t)^{\beta-1} \, dt \right. \\
& \left. - \int_a^b \left(\bigvee_a^t (f) \right) (b-t)^{\beta} \, dt \right] .
\end{aligned} \tag{4.2}$$

Proof. Utilising the identity (2.3) and the first inequality in (3.1), we have successively,

$$\begin{aligned}
|D(f;u)| & \leq \frac{1}{b-a} \int_a^b \left| \int_a^b Q(t,s) \, du(s) \right| d\left(\bigvee_a^t (f) \right) \\
& = \frac{1}{b-a} \int_a^b \left| \int_a^t Q(t,s) \, du(s) + \int_t^b Q(t,s) \, du(s) \right| d\left(\bigvee_a^t (f) \right) \\
& \leq \frac{1}{b-a} \int_a^b \left[\left| \int_a^t Q(t,s) \, du(s) \right| + \left| \int_t^b Q(t,s) \, du(s) \right| \right] d\left(\bigvee_a^t (f) \right) \\
& = \frac{1}{b-a} \int_a^b \left[(b-t) |u(t)-u(a)| + (t-a) |u(b)-u(t)| \right] d\left(\bigvee_a^t (f) \right) \\
& =: P.
\end{aligned}$$

Now, on making use of condition (4.1), we can state that

$$\begin{aligned}
P & \leq \frac{1}{b-a} \int_a^b \left[L_a (b-t)(t-a)^{\alpha} + L_b (t-a)(b-t)^{\beta} \right] d\left(\bigvee_a^t (f) \right) \\
& = \frac{1}{b-a} \left[L_a \int_a^b (b-t)(t-a)^{\alpha} \, d\left(\bigvee_a^t (f) \right) \right. \\
& \qquad \left. + L_b \int_a^b (t-a)(b-t)^{\beta} \, d\left(\bigvee_a^t (f) \right) \right] .
\end{aligned} \tag{4.3}$$

However,

$$
\begin{aligned}
&\int_a^b (b-t)(t-a)^{\alpha}\, d\left(\bigvee_a^t (f)\right) \\
&= (b-t)(t-a)^{\alpha} \bigvee_a^t (f)\Big|_a^b - \int_a^b \left(\bigvee_a^t (f)\right) d\left[(b-t)(t-a)^{\alpha}\right] \\
&= -\int_a^b \left(\bigvee_a^t (f)\right)\left[-(t-a)^{\alpha} + \alpha (b-t)(t-a)^{\alpha-1}\right] dt \\
&= \int_a^b \left(\bigvee_a^t (f)\right)(t-a)^{\alpha}\, dt - \alpha \int_a^b \left(\bigvee_a^t (f)\right)(b-t)(t-a)^{\alpha-1}\, dt
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_a^b (t-a)(b-t)^{\beta}\, d\left(\bigvee_a^t (f)\right) \\
&= (t-a)(b-t)^{\beta} \bigvee_a^t (f)\Big|_a^b - \int_a^b \left(\bigvee_a^t (f)\right) d\left[(t-a)(b-t)^{\beta}\right] \\
&= -\int_a^b \left(\bigvee_a^t (f)\right)\left[(b-t)^{\beta} - \beta (t-a)(b-t)^{\beta-1}\right] dt \\
&= \beta \int_a^b \left(\bigvee_a^t (f)\right)(t-a)(b-t)^{\beta-1}\, dt - \int_a^b \left(\bigvee_a^t (f)\right)(b-t)^{\beta}\, dt,
\end{aligned}
$$

and from (4.3) we deduce the desired inequality (4.2). □

Corollary 3. *If f is as in Theorem 4 and u is of r-H-Hölder type, i.e.,*

$$|u(t) - u(s)| \le H|u-t|^{r} \quad \textit{for any} \;\; t, s \in [a,b],$$

where $H > 0$ and $r \in (0,1)$ are given, then

$$
\begin{aligned}
|D(f;u)| \le \frac{1}{b-a} H \int_a^b \left(\bigvee_a^t (f)\right) \Big\{ &(t-a)^{r} - (b-t)^{r} \\
&+ r(b-t)^{r-1}(t-a)^{r-1}\left[(t-a)^{1-r} - (b-t)^{1-r}\right]\Big\}\, dt.
\end{aligned}
$$

Remark 2. If $r = \frac{1}{2}$ in Corollary 3, then we obtain the inequality

$$|D(f;u)| \leq \frac{1}{b-a} H \int_a^b \left(\bigvee_a^t (f) \right) \left(\sqrt{t-a} - \sqrt{b-t} \right) \times \left(1 + \frac{1}{2\sqrt{(b-t)(t-a)}} \right) dt.$$

The following particular result may be useful for applications.

Corollary 4. *If f is as in Theorem 4 and $u : [a,b] \to \mathbb{R}$ is Lipschitzian with the constant $K > 0$, then*

$$|D(f;u)| \leq \frac{4}{b-a} \cdot K \int_a^b \left(t - \frac{a+b}{2} \right) \cdot \bigvee_a^t (f)\, dt \leq \begin{cases} K(b-a) \bigvee_a^b (f); \\ \frac{2(b-a)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} K \left(\int_a^b \left[\bigvee_a^t (f) \right]^p dt \right)^{\frac{1}{p}}, \quad p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ 2K \int_a^b \left(\bigvee_a^t (f) \right) dt. \end{cases} \tag{4.4}$$

The multiplication constant 4 is the best possible.

Proof. The first inequality follows by Theorem 4 on choosing $L_a = L_b = K$ and $\alpha = \beta = 1$.

Now, on utilising Hölder's inequality, we have

$$\int_a^b \left(t - \frac{a+b}{2} \right) \cdot \left(\bigvee_a^t (f) \right) dt \leq \begin{cases} \sup_{t \in [a,b]} \left(\bigvee_a^t (f) \right) \int_a^b \left| t - \frac{a+b}{2} \right| dt; \\ \left(\int_a^b \left[\bigvee_a^t (f) \right]^p dt \right)^{\frac{1}{p}} \left(\int_a^b \left| t - \frac{a+b}{2} \right|^q dt \right)^{\frac{1}{q}}, \quad p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \int_a^b \left(\bigvee_a^t (f) \right) dt \sup_{t \in [a,b]} \left| t - \frac{a+b}{2} \right|. \end{cases} \tag{4.5}$$

However, $\sup_{t\in[a,b]}\left|t-\frac{a+b}{2}\right|=\frac{b-a}{2}$ and

$$\int_a^b\left|t-\frac{a+b}{2}\right|^q dt=2\int_{\frac{a+b}{2}}^b\left(t-\frac{a+b}{2}\right)^q dt=\frac{(b-a)^{q+1}}{2^q(q+1)},\quad q\geq 1,$$

and by (4.5) we deduce

$$\int_a^b\left(t-\frac{a+b}{2}\right)\cdot\bigvee_a^t(f)\,dt\leq\begin{cases}\frac{(b-a)^2}{4}\bigvee_a^t(f)\,;\\[2ex]\frac{(b-a)^{1+\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}}\left(\int_a^b\left[\bigvee_a^t(f)\right]^p dt\right)^{\frac{1}{p}},\quad p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex]\frac{b-a}{2}\int_a^b\left(\bigvee_a^t(f)\right)dt,\end{cases}$$

and the second part is proved.

To prove the sharpness of the constant 4 in the first inequality in (4.4) assume that there exists $A>0$ such that

$$|D(f;u)|\leq\frac{A}{b-a}\cdot K\int_a^b\left(t-\frac{a+b}{2}\right)\cdot\bigvee_a^t(f)\,dt,\tag{4.6}$$

provided that f is of bounded variation and u is K-Lipschitzian.

Let $f:[a,b]\to\mathbb{R}$,

$$f(t)=\begin{cases}0 & \text{if } t\in\left[a,\frac{a+b}{2}\right],\\[1ex] k & \text{if } t\in\left(\frac{a+b}{2},b\right],\end{cases}$$

with $k>0$. Then

$$\bigvee_a^t(f)=\begin{cases}0 & \text{if } t\in\left[a,\frac{a+b}{2}\right],\\[1ex] k & \text{if } t\in\left(\frac{a+b}{2},b\right].\end{cases}$$

Also, we have

$$\int_a^b\left(t-\frac{a+b}{2}\right)\cdot\bigvee_a^t(f)\,dt=\int_{\frac{a+b}{2}}^b\left(t-\frac{a+b}{2}\right)k\,dt=\frac{k(b-a)^2}{8}.$$

Consider $u : [a,b] \to \mathbb{R}$, $u(t) = \left|t - \frac{a+b}{2}\right|$. Then u is K-Lipschitzian with $K = 1$. Also,

$$D(f;u) = \int_a^b f(t)\,du(t) - \frac{u(b)-u(a)}{b-a}\int_a^b f(t)\,dt$$
$$= k\int_{\frac{a+b}{2}}^b du(t) = k\left[u(b) - u\left(\frac{a+b}{2}\right)\right]$$
$$= \frac{(b-a)\,k}{2}.$$

Substituting these values into (4.6) produces the inequality

$$\frac{(b-a)\,k}{2} \le \frac{A}{b-a}\cdot\frac{k\,(b-a)^2}{8},$$

which implies that $A \ge 4$. □

5. Inequalities for (l, L)-Lipschitzian functions

The following simple lemma holds.

Lemma 4. *Let $u : [a,b] \to \mathbb{R}$ and $l, L \in \mathbb{R}$ with $L > l$. The following statements are equivalent:*

(i) *The function $u - \frac{l+L}{2}\cdot e$, where $e(t) = t$, $t \in [a,b]$ is $\frac{1}{2}(L-l)$-Lipschitzian;*

(ii) *We have the inequalities*

$$l \le \frac{u(t)-u(s)}{t-s} \le L \quad \textit{for each} \;\; t,s \in [a,b],\; t \ne s;$$

(iii) *We have the inequalities*

$$l\,(t-s) \le u(t) - u(s) \le L\,(t-s) \quad \textit{for each} \;\; t,s \in [a,b] \quad \textit{with} \;\; t > s.$$

The proof is obvious and we omit the details.

Definition 1 (see also [14])**.** The function $u : [a,b] \to \mathbb{R}$ which satisfies one of the equivalent conditions (i) − (iii) from Lemma 4 is said to be (l, L)-*Lipschitzian* on $[a,b]$. If $L > 0$ and $l = -L$, then $(-L, L)$-Lipschitzian means L-Lipschitzian in the classical sense.

The following result can be stated.

Theorem 5. *Let $f : [a,b] \to \mathbb{R}$ be a function of bounded variation and $u : [a,b] \to \mathbb{R}$ an $(l,L)-$Lipschitzian function. Then*

$$|D(f;u)| \leq \frac{2}{b-a}(L-l)\int_a^b \left(t-\frac{a+b}{2}\right)\cdot \bigvee_a^t (f)\,dt$$

$$\leq \begin{cases} \frac{1}{2}(L-l)(b-a)\bigvee_a^b (f); \\ \frac{(b-a)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}(L-l)\left(\int_a^b \left[\bigvee_a^t (f)\right]^p dt\right)^{\frac{1}{p}}, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ (L-l)\int_a^b \left(\bigvee_a^t (f)\right) dt. \end{cases}$$

The constant 2 *in the first inequality is sharp.*

Proof. Observe that

$$\begin{aligned} & D\left(f;u-\frac{l+L}{2}\cdot e\right) \\ &= \int_a^b \left(f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right) d\left[u(t)-\frac{l+L}{2}\cdot t\right] \\ &= \int_a^b \left(f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right) du(t) \\ &\quad -\frac{l+L}{2}\int_a^b \left(f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right) dt \\ &= D(f;u). \end{aligned}$$

Now, applying Corollary 4 for the function $u-\frac{l+L}{2}e$, which is $\frac{1}{2}(L-l)$-Lipschitzian, we get

$$\begin{aligned} \left|D\left(f;u-\frac{l+L}{2}e\right)\right| &\leq \frac{4}{b-a}\cdot\frac{1}{2}(L-l)\int_a^b \left(t-\frac{a+b}{2}\right)\cdot\bigvee_a^t (f)\,dt \\ &= \frac{2}{b-a}(L-l)\int_a^b \left(t-\frac{a+b}{2}\right)\cdot\bigvee_a^t (f)\,dt \end{aligned}$$

and the theorem is proved. □

The second result may be stated as follows.

Theorem 6. *Let $u : [a,b] \to \mathbb{R}$ be a function of bounded variation. If $f : [a,b] \to \mathbb{R}$ is (ϕ, Φ)-Lipschitzian with $\Phi > \phi$, then*

$$\begin{aligned}
&\left| D(f;u) - \frac{\phi+\Phi}{2}\left[\frac{u(b)+u(a)}{2}(b-a) - \int_a^b u(t)\,dt\right]\right| \\
&\le \frac{1}{2}(\Phi-\phi)\cdot\left[\frac{1}{2}(b-a)\bigvee_a^b(u) - \frac{2}{b-a}\int_a^b\left(\bigvee_a^s(u)\right)\left(s-\frac{a+b}{2}\right)ds\right] \\
&\le \frac{1}{4}\cdot(\Phi-\phi)(b-a)\bigvee_a^b(u).
\end{aligned} \tag{5.1}$$

The constant $\frac{1}{2}$ in front of $(\Phi-\phi)$ and $\frac{1}{4}$ are the best possible.

Proof. Observe that

$$\begin{aligned}
&D\left(f - \frac{\phi+\Phi}{2}\cdot e; u\right) \\
&= \int_a^b \left[f(t) - \frac{\phi+\Phi}{2}\cdot t - \frac{1}{b-a}\int_a^b\left(f(s) - \frac{\phi+\Phi}{2}\cdot s\right)ds\right]du(t) \\
&= \int_a^b \left[f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds - \left(\frac{\phi+\Phi}{2}t - \frac{1}{b-a}\int_a^b \frac{\phi+\Phi}{2}\cdot sds\right)\right]du(t) \\
&= D(f;u) - \frac{\phi+\Phi}{2}\int_a^b\left(t - \frac{1}{b-a}\int_a^b sds\right)du(t) \\
&= D(f;u) - \frac{\phi+\Phi}{2}\int_a^b\left(t-\frac{a+b}{2}\right)du(t).
\end{aligned}$$

Integrating by parts in the Riemann–Stieltjes integral we have

$$\int_a^b\left(t-\frac{a+b}{2}\right)du(t) = \frac{u(b)+u(a)}{2}(b-a) - \int_a^b u(t)\,dt.$$

Then

$$D\left(f-\frac{\phi+\Phi}{2}e;u\right) = D(f;u) - \frac{\phi+\Phi}{2}\left[\frac{u(b)+u(a)}{2}(b-a) - \int_a^b u(t)\,dt\right].$$

Now, on applying Theorem 2 for the function $f - \frac{\phi+\Phi}{2}e$ which is $\frac{1}{2}(L-l)$-Lipschitzian, we deduce the desired result (5.1). □

6. Applications for the Chebyshev functional

If we choose $u(t) := \int_a^t g(\tau)\,d\tau$, $t \in [a,b]$, where $g : [a,b] \to \mathbb{R}$ is Lebesgue integrable on $[a,b]$, then we have the equality

$$C(f;g) = \frac{1}{b-a} D(f;u).$$

Also, u is of bounded variation on any subinterval $[a,s]$, $s \in [a,b]$, and if g is continuous on $[a,b]$, then

$$\bigvee_a^s (u) = \int_a^s |g(\tau)|\,d\tau, \quad s \in [a,b].$$

If f is of bounded variation on $[a,b]$, then on utilising the inequality (3.3) we have

$$\begin{aligned}
|C(f;g)| &\le \frac{1}{(b-a)^2}\left[\int_a^b (2s-a-b)\,|g(s)| \bigvee_a^s (f)\,ds \right.\\
&\quad \left. + 2\int_a^b \left(\int_a^s |g(\tau)|\,d\tau\right) \bigvee_a^s (f)\,ds - \int_a^b |g(\tau)|\,d\tau \cdot \int_a^b \left(\bigvee_a^s (f)\right) ds\right]\\
&\le \frac{1}{(b-a)^2}\int_a^b (2s-a-b)\,|g(s)| \bigvee_a^s (f)\,ds\\
&\quad + \frac{1}{(b-a)^2}\int_a^b \left(\int_a^s |g(\tau)|\,d\tau\right) \bigvee_a^s (f)\,ds\\
&\le \frac{1}{(b-a)^2}\int_a^b (2s-a-b)|g(s)| \bigvee_a^s (f) ds + \frac{1}{b-a}\int_a^b |g(\tau)|\,d\tau \cdot \bigvee_a^b (f).
\end{aligned}$$

Now, if f is monotonic nondecreasing, then by (3.7) we have

$$\begin{aligned}
|C(f;g)| &\le \frac{1}{(b-a)^2}\left[\int_a^b (2s-a-b) f(s)\,|g(s)|\,ds \right.\\
&\quad \left. + 2\int_a^b \left(\int_a^s |g(\tau)|\,d\tau\right) f(s)\,ds - \int_a^b f(s)\,ds \cdot \int_a^b |g(\tau)|\,d\tau\right].
\end{aligned}$$

The case where f is L-Lipschitzian provides via (3.6) a simpler inequality

$$\begin{aligned}
|C(f;g)| &\le L\left[\frac{1}{2}\int_a^b |g(\tau)|\,d\tau \right.\\
&\qquad \left. - \frac{2}{(b-a)^2}\int_a^b \left(\int_a^s |g(\tau)|\,d\tau\right)\left(s - \frac{a+b}{2}\right) ds\right]\\
&\le \frac{1}{2} L \int_a^b |g(s)|\,ds.
\end{aligned}$$

Now, if f is of bounded variation and $|g|$ is bounded above by M, i.e., $|g(t)| \le M$ for a.e. $t \in [a,b]$, then by (4.4) we have

$$|C(f;g)| \le \frac{4}{(b-a)^2} M \int_a^b \left(t - \frac{a+b}{2}\right) \bigvee_a^t (f)\, dt \le \begin{cases} M \bigvee_a^b (f); \\ \frac{2(b-a)^{\frac{1}{q}-1}}{(q+1)^{\frac{1}{q}}} M \left(\int_a^b \left[\bigvee_a^t (f)\right]^p dt\right)^{\frac{1}{p}}, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{2M}{b-a} \int_a^b \left(\bigvee_a^t (f)\right) dt. \end{cases} \tag{6.1}$$

The constant 4 in (6.1) is the best possible.

Finally, if $-\infty < \phi \le g(t) \le \Phi$ for a.e. $t \in [a,b]$, then $\left|g(t) - \frac{\phi+\Phi}{2}\right| \le \frac{1}{2}(\Phi - \phi)$, and since

$$C\left(f; g - \frac{\phi+\Phi}{2}\right) = C(f;g),$$

by (6.1) we deduce the inequalities

$$|C(f;g)| \le \frac{2}{(b-a)^2} (\Phi - \phi) \int_a^b \left(t - \frac{a+b}{2}\right) \bigvee_a^t (f)\, dt \le \begin{cases} \frac{1}{2}(\Phi-\phi) \bigvee_a^b (f); \\ \frac{(b-a)^{\frac{1}{q}-1}}{(q+1)^{\frac{1}{q}}} (\Phi-\phi) \left(\int_a^b \left[\bigvee_a^t (f)\right]^p dt\right)^{\frac{1}{p}}, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{\Phi-\phi}{b-a} \int_a^b \left(\bigvee_a^t (f)\right) dt. \end{cases}$$

The constant 2 in the first inequality is the best possible.

References

[1] M. W. Alomari, *Some Grüss type inequalities for Riemann-Stieltjes integral and applications*, Acta Math. Univ. Comenian. (N.S.) **81** (2012), No. 2, 211–220.

[2] M. W. Alomari, *A companion of Ostrowski's inequality for the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$, where f is of bounded variation and u is of r-H-Hölder type and applications*, Appl. Math. Comput. **219** (2013), No. 9, 4792–4799.

[3] G. A. Anastassiou, *Univariate Ostrowski inequalities, revisited.* Monatsh. Math. **135** (2002), No. 3, 175–189.

[4] G. A. Anastassiou, *Chebyshev-Grüss type and comparison of integral means inequalities for the Stieltjes integral*, Panamer. Math. J. **17** (2007), No. 3, 91–109.

[5] T. Apostol, *Mathematical Analysis*, Addison-Wesley Publishing Co., Mass. – London – Don Mills, 1974.

[6] S. S. Dragomir, *Inequalities of Grüss type for the Stieltjes integral*, Kragujevac J. Math. **26** (2004), 89–122.

[7] S. S. Dragomir, *Approximation of the Riemann-Stieltjes integral with* (l, L)*-Lipschitzian integrators*, in: Numerical Analysis and Applied Mathematics (Edited by T. E. Simos et al.), AIP Conf. Proc. **936**, (2007), pp. 686–690.

[8] S. S. Dragomir, *Inequalities for Stieltjes integrals with convex integrators and applications*, Appl. Math. Lett., **20** (2007), 123–130.

[9] S. S. Dragomir, *A sharp bound of the Čebyšev functional for the Riemann-Stieltjes integral and applications*, J. Inequal. Appl. **2008**, Art. ID 824610, 11 pp.

[10] S. S. Dragomir, *Accurate approximations for the Riemann-Stieltjes integral via theory of inequalities*, J. Math. Inequal. **3** (2009), No. 4, 663–681.

[11] S. S. Dragomir and I. Fedotov, *An inequality of Grüss type for the Riemann-Stieltjes integral and applications for special means*, Tamkang J. Math., **29**(4) (1998), 287–292.

[12] S. S. Dragomir and I. Fedotov, *A Grüss type inequality for mappings of bounded variation and applications for numerical analysis*, Nonlinear Funct. Anal. Appl. **6**(3) (2001), 425–433.

[13] S. S. Dragomir and C. E. M. Pearce, *Some inequalities relating to upper and lower bounds for the Riemann-Stieltjes integral*, J. Math. Inequal. **3** (2009), No. 4, 607–616.

[14] Z. Liu, *Refinement of an inequality of Grüss type for Riemann-Stieltjes integral*, Soochow J. Math., **30**(4) (2004), 483–489.

Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia; School of Computational & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir