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A note on embedding of semigroup amalgams

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Abstract. We give necessary conditions for the embedding of completely regular semigroup, Clifford semigroup and commutative regular semigroup amalgams.

A *semigroup amalgam* (cf. [3]) is a list $\mathcal{A} \equiv (U; S_1, S_2; \phi_1, \phi_2)$ comprising three semigroups U, S_1 and S_2, and two monomorphisms $\phi_i : U \longrightarrow S_i$, $i \in \{1, 2\}$ (recall that monomorphisms of semigroups are precisely the injective semigroup homomorphisms). We say that $\mathcal{A}$ is *embeddable* if there exist a semigroup T and monomorphisms $\psi_i : S_i \longrightarrow T$ such that

(1) $\psi_1 \circ \phi_1 = \psi_2 \circ \phi_2$ and

(2) $\psi_1(s_1) = \psi_2(s_2)$, $s_1 \in S_1$, $s_2 \in S_2$ implies that $s_1 = \phi_1(u)$, $s_2 = \phi_2(u)$ for some $u \in U$.

If condition (2) is not necessarily satisfied, we call $\mathcal{A}$ *weakly embeddable*. In [1], Theorem 2.4, Howie proved that a semigroup amalgam $(U; S_1, S_2; \phi_1, \phi_2)$, in which S_1 and S_2 are both groups, is embeddable if and only if U is also a group. The main objective of this note is to generalize the necessity part of this theorem to the unions and semilattices of groups. A semigroup S is called *completely regular* if it is a union of groups. We call S a *Clifford semigroup* if it is a semilattice of groups.

A semigroup S is called *regular* (cf. [3], pp. 50–51) if there exists a unary operation ${}^{-1} : S \longrightarrow S$ given by $s \longmapsto s^{-1}$ such that $ss^{-1}s = s$, $s^{-1}ss^{-1} = s^{-1}$. The element s^{-1} is called an inverse of s. One can show (see [3], Proposition 4.1.1) that S is completely regular if and only if it is regular and $ss^{-1} = s^{-1}s$ for all $s \in S$. Also, S is a Clifford semigroup if and only if it is completely regular and $(ss^{-1})(tt^{-1}) = (tt^{-1})(s^{-1}s)$ for all $s, t \in S$ (refer to Theorem 4.2.1 of [3]). A regular semigroup S is termed an *inverse semigroup* if $(ss^{-1})(tt^{-1}) = (tt^{-1})(s^{-1}s)$ for all $s, t \in S$. Thus Clifford semigroups are

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completely regular inverse semigroups. Moreover, every commutative regular semigroup is a Clifford semigroup.

Theorem 1. *A semigroup amalgam $(U; S_1, S_2; \phi_1, \phi_2)$, in which S_1 and S_2 are completely regular, is embeddable only if U is also completely regular.*

Proof. Consider an amalgam $\mathcal{A} \equiv (U; S_1, S_2; \phi_1, \phi_2)$ in which S_1 and S_2 are both completely regular. Suppose $\mathcal{A}$ is embeddable, say in a semigroup T, and consider an element $u \in U$. Let us denote $\phi_1(u) = s_1$ and $\phi_2(u) = s_2$. Now, because S_1 and S_2 are completely regular, there exist inverses $s_1^{-1} \in S_1$ and $s_2^{-1} \in S_2$ of s_1 and s_2 respectively. Let $\psi_i : S_i \longrightarrow T$, $i \in \{1, 2\}$, be the embedding monomorphisms. First note that $\psi_1(s_1) = \psi_1\phi_1(u) = \psi_2\phi_2(u) = \psi_2(s_2)$. We can calculate in T:

$$\begin{aligned}
\psi_1(s_1^{-1}) &= \psi_1(s_1^{-1}s_1s_1^{-1}) \\
&= \psi_1(s_1^{-1})\psi_1(s_1)\psi_1(s_1^{-1}) = \psi_1(s_1^{-1})\psi_2(s_2)\psi_1(s_1^{-1}) \\
&= \psi_1(s_1^{-1})\psi_2(s_2s_2^{-1}s_2s_2^{-1}s_2)\psi_1(s_1^{-1}) \\
&= \psi_1(s_1^{-1})\psi_2(s_2)\psi_2(s_2^{-1}s_2s_2^{-1}s_2)\psi_1(s_1^{-1}) \\
&= \psi_1(s_1^{-1})\psi_1(s_1)\psi_2(s_2s_2^{-1}s_2^{-1}s_2)\psi_1(s_1^{-1}) \\
&= \psi_1(s_1^{-1}s_1)\psi_2(s_2)\psi_2(s_2^{-1}s_2^{-1})\psi_2(s_2)\psi_1(s_1^{-1}) \\
&= \psi_1(s_1s_1^{-1})\psi_1(s_1)\psi_2(s_2^{-1}s_2^{-1})\psi_1(s_1)\psi_1(s_1^{-1}) \\
&= \psi_1(s_1s_1^{-1}s_1)\psi_2(s_2^{-1}s_2^{-1})\psi_1(s_1s_1^{-1}) \\
&= \psi_1(s_1)\psi_2(s_2^{-1}s_2^{-1})\psi_1(s_1^{-1}s_1) = \psi_2(s_2)\psi_2(s_2^{-1}s_2^{-1})\psi_1(s_1^{-1}s_1) \\
&= \psi_2(s_2s_2^{-1}s_2^{-1})\psi_1(s_1^{-1}s_1) = \psi_2(s_2^{-1}s_2^{-1}s_2)\psi_1(s_1^{-1}s_1) \\
&= \psi_2(s_2^{-1}s_2^{-1})\psi_2(s_2)\psi_1(s_1^{-1}s_1) = \psi_2(s_2^{-1}s_2^{-1})\psi_1(s_1)\psi_1(s_1^{-1}s_1) \\
&= \psi_2(s_2^{-1}s_2^{-1})\psi_1(s_1s_1^{-1}s_1) \\
&= \psi_2(s_2^{-1}s_2^{-1})\psi_1(s_1) = \psi_2(s_2^{-1}s_2^{-1})\psi_2(s_2) \\
&= \psi_2(s_2^{-1}s_2^{-1}s_2) = \psi_2(s_2^{-1}s_2s_2^{-1}) \\
&= \psi_2(s_2^{-1}).
\end{aligned}$$

Now, using condition (2) of embeddability, there exists $u' \in U$ such that $\phi_i(u') = s_i^{-1}$, $i \in \{1, 2\}$. Then, because $s_1s_1^{-1}s_1 = s_1$ implies $\phi_1^{-1}(s_1s_1^{-1}s_1) = \phi_1^{-1}(s_1)$, we have $uu'u = u$ due to injectivity of ϕ_1. Similarly we can conclude that $u'uu' = u'$ and $uu' = u'u$. Thus U is completely regular. □

Corollary 1. *It suffices to show that U is an inverse semigroup. To this end, observe that the niqueness of u' (see Theorem 5.1.1 of* [3]*) in the previous proof follows from the uniqueness of s_1^{-1}.*

Proof. The corollary follows by noting that there exists $u \in U$ such that $u^{-1} \notin U$. □

Corollary 2. *A semigroup amalgam* $(U; S_1, S_2; \phi_1, \phi_2)$*, in which* S_1 *and* S_2 *are commutative regular semigroups, is embeddable if and only if* U *is regular.*

Proof. Because S_1 and S_2 are completely regular, the necessity part follows from the argument employed in the above corollary. The sufficiency part follows from Theorem 3.1 of [2]. $\square$

Conclusion 1. Can we generalize Theorem 1 to the class of inverse semigroups? Also, can embedding be replaced by weak embedding in Theorem 1?

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