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Cumulant-moment relation in free probability theory

Jolanta Pielaszkiewicz, Dietrich von Rosen, and Martin Singull

Abstract. The goal of this paper is to present and prove a cumulant-moment recurrent relation formula in free probability theory. It is convenient tool to determine underlying compactly supported distribution function. The existing recurrent relations between these objects require the combinatorial understanding of the idea of non-crossing partitions, which has been considered by Speicher and Nica. Furthermore, some formulations are given with additional use of the Möbius function. The recursive result derived in this paper does not require introducing any of those concepts. Similarly like the non-recursive formulation of Mottelson our formula demands only summing over partitions of the set. The proof of non-recurrent result is given with use of Lagrange inversion formula, while in our proof the calculations of the Stieltjes transform of the underlying measure are essential.

1. Introduction and background

Free moments and free cumulants are functionals defined within free probability theory. The theory was established in the middle of the 80's by Voiculescu in [14] and together with the result published in [15] regarding asymptotic freeness of random matrices it has established new branches of theories and tools, among others free cumulants and moments.

It is of great importance to understand the behavior of free cumulants, or related free moments, as they give us essentially the full information about a particular probability measure such as the measure connected to the spectral distribution.

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We will consider a general formulation, but in the last section a particular example is given. In order to state the results of the article we fix notation and recall the basic definitions and properties. Let us consider a non-commutative $*$-probability space $(\mathcal{A}, \tau)$, where $\mathcal{A}$ is a unitary algebra over the field of real numbers and τ is a functional such that $\tau : \mathcal{A} \to \mathbb{R}$ is linear, $\tau(1_{\mathcal{A}}) = 1$ and $\tau(a^*a) \geq 0$ for all $a \in \mathcal{A}$. The algebra is equipped with a $*$-operation such that $* : \mathcal{A} \to \mathcal{A}$, $(a^*)^* = a$ and $(ab)^* = b^*a^*$ for all $a, b \in \mathcal{A}$. For more details, see [9]. Then the free k-th moment of a self-adjoint element $a \in \mathcal{A}$ is defined as

$$m_k := \tau(a^k) := \int_{\mathbb{R}} x^k d\mu(x), \tag{1}$$

where μ is a compactly supported $*$-distribution of element $a \in \mathcal{A}$ characterized by moments m_k, $k = 1, \ldots$. The form of the chosen functional τ determines the $*$-distribution of the element a.

To introduce the concept of free cumulants as well as to obtain the relation formula between free cumulants and moments we use the Stieltjes transform. It appears among others in formulations of a number of results published within Random matrix theory, see for example, [6, 2, 11, 4].

Definition 1.1. Let μ be a probability measure on $\mathbb{R}$. Then, the Stieltjes (Cauchy–Stieltjes) transform of μ is given by

$$\mathrm{G}_\mu(z) = \int_{\mathbb{R}} \frac{1}{z - x} d\mu(x),$$

for all $z \in \mathbb{C}$, $\Im(z) > 0$, where $\Im(z)$ denotes the imaginary part of a complex number z.

Defined in such a way the Stieltjes transform can be inverted on any interval. It can also be given as a series of free moments $\{m_i\}_{i=1}^{\infty}$.

Theorem 1.1. *Let the free moments $m_k = \int_{\mathbb{R}} x^k d\mu(x)$, $k = 1, 2, \ldots$. Then, a formal power series representing the Stieltjes transform is given by*

$$\mathrm{G}_\mu(z) \quad = \quad \frac{1}{z}\left(1 + \sum_{i=1}^{\infty} z^{-i} m_i\right).$$

Proof. We have

$$\begin{aligned} \mathrm{G}_\mu(z) &= \int_{\mathbb{R}} \frac{1}{z - x} d\mu(x) = \frac{1}{z}\int_{\mathbb{R}} \frac{1}{1 - \frac{x}{z}} d\mu(x) = \frac{1}{z}\int_{\mathbb{R}} \sum_{i=0}^{\infty} \left(\frac{x}{z}\right)^i d\mu(x) \\ &= \frac{1}{z}\sum_{i=0}^{\infty} z^{-i} \int_{\mathbb{R}} x^i d\mu(x) = \frac{1}{z}\left(1 + \sum_{i=1}^{\infty} z^{-i} m_i\right), \end{aligned}$$

which completes the proof. □

Although the Stieltjes transform G_μ is a convenient tool, even better suited for studying convolution of measure μ (see [9, 1]) on a non-commutative $*$-probability spaces is the R-transform. The R-transform linearizes free convolution and plays the same role as the log of the Fourier transform in classical probability theory. The relation between the R- and Stieltjes transform G_μ, or more precisely G_μ^{-1}, which is the inverse with respect to composition, is often considered as a definition of the R-transform.

Definition 1.2. Let μ be a probability measure and $\mathrm{G}_\mu(z)$ the related Stieltjes transform. Then

$$\mathrm{R}_\mu(z) = \mathrm{G}_\mu^{-1}(z) - \frac{1}{z} \qquad \text{or, equivalently,} \qquad \mathrm{R}_\mu(\mathrm{G}_\mu(z)) = z - \frac{1}{\mathrm{G}_\mu(z)}$$

defines the R-transform $\mathrm{R}_\mu(z)$ for the underlying measure μ.

The free cumulants $\{k_i\}_{i=1}^{\infty}$ are given as the coefficients of a power series expansion of the R-transform.

Definition 1.3. Let μ be a probability measure and $\mathrm{R}_\mu(z)$ be the related R-transform. Then for a, which is an element of a non-commutative $*$-algebra $\mathcal{A}$, the free cumulants of a, $\{k_i\}_{i=1}^{\infty}$, are defined by

$$\mathrm{R}_\mu(z) = \sum_{i=0}^{\infty} k_{i+1}(a) z^i.$$

To put our result in relation to the other cumulant-moment formulas in free probability theory we recall that a combinatorial branch of free probability theory points out that free cumulants defined by the R-transform, as in Definition 1.3, following [7] and [9], can be defined via non-crossing partitions using the following recursive relation

$$k_1(a) = \tau(a), \qquad \tau(a_1 \cdot \ldots \cdot a_k) = \sum_{\pi \in NC(k)} k_\pi[a_1, \ldots, a_k], \tag{2}$$

where $\tau(a_1 \cdot \ldots \cdot a_k)$ describes mixed free moments of $a_1, \ldots, a_k$, the sum is taken over all non-crossing partitions $NC(k)$ of the set $\{1, 2, \ldots, k\}$, $a_i \in \mathcal{A}$ for all $i = 1, 2, \ldots, k$ and $k_\pi[a_1, \ldots, a_k] = \prod_{i=1}^{r} k_{V(i)}[a_1, \ldots, a_k]$, where $\pi = \{V(1), \ldots, V(r)\}$ and $k_V[a_1, \ldots, a_k] = k_s(a_{v(1)}, \ldots, a_{v(s)})$, where $V = (v(1), \ldots, v(s))$. Then, for $a \in \mathcal{A}$ the cumulant of a is defined as $k_n = k_n(a, \ldots, a)$. The calculations with use of (2) come after the proof of Corollary 2.1.

Another way to look at free cumulants, see [9], is with use of the Möbius function as well as non-crossing partitions

$$k_\pi[a_1,\dots,a_k] = \sum_{\sigma\in NC(k),\sigma\leq\pi} \tau_\sigma[a_1,\dots,a_k]\mu(\sigma,\pi),$$

where $\tau_k(a_1,\dots,a_k) := \tau(a_1,\dots,a_k)$, $\tau_\pi[a_1,\dots,a_k] := \prod_{V\in\pi}\tau_V[a_1,\dots,a_k]$ and μ is the Möbius function on $NC(k)$. For more details about above formulations see [9] and [12]. In the next section we will compare our recursive formula with the result given by equation (2).

Furthermore, the following non-recursive relation between free moment and free cumulant has been shown in [8] together with proof which is based on Lagrange inversion formula and is inspired by the work of Haagerup [3]:

$$\begin{aligned} k_p &= m_p + \sum_{j=2}^{p} \frac{(-1)^{j-1}}{j}\binom{p+j-2}{j-1}\sum_{Q_j} m_{q_1}\cdots m_{q_j}, \\ m_p &= k_p + \sum_{j=2}^{p} \frac{1}{j}\binom{p}{j-1}\sum_{Q_j} k_{q_1}\cdots k_{q_j}, \end{aligned}$$

where $Q_j = \{(q_1,q_2,\dots,q_j)\in\mathbb{N}^j \,|\, \sum_{i=1}^{j} q_i = p\}$.

For a better understanding of the idea with free cumulants we would like to mention that the free and classical cumulants for the $*$-distribution differ by the elements associated with crossing partitions. In the classical case we consider all partitions while in the free cumulant case only non-crossing ones are of interest. Then, obviously, the first three cumulants are the same in free and classical sense, since the sets $\{1\}$, $\{1,2\}$, $\{1,2,3\}$ have no crossing partitions. However, for the fourth cumulant and cumulants of the higher order the free and classical cumulants differ.

2. Main result

The purpose of this paper is to present a recursive formula which is not based on non-crossing partitions.

First introduce a shortened notation for the sum of products of h moments, where each of moments has degree given by index i_k, $k=1,\dots,h$, the sum of indexes $i_1+i_2+\ldots+i_h = t$ and each index $i_k \succ 0$, where $\succ$ reflects the ordering relation

$$\binom{\mathbf{m}, h, \succ}{t} = \sum_{\substack{i_1+i_2+\ldots+i_h=t \\ \forall_k i_k\succ 0}} m_{i_1}m_{i_2}\cdot\ldots\cdot m_{i_h}.$$

Theorem 2.1. *Let $\{k_i\}_{i=1}^{\infty}$ be the free cumulants and $\{m_i\}_{i=1}^{\infty}$ be the free moments for an element of a non-commutative probability space. Then $k_1 =$*

m_1 and the following recursive formula holds:

$$k_t = \sum_{i=1}^{t}(-1)^{i+1}\binom{\mathbf{m}, i, >}{t} - \sum_{h=2}^{t-1} k_h \binom{\mathbf{m}, h-1, \geq}{t-h}, \quad t = 2, 3, \ldots. \quad (3)$$

Proof. Let us consider a non-commutative $*$-probability space $(\mathcal{A}, \tau)$, where $\mathcal{A}$ is a unitary $*$-algebra equipped with the functional $\tau(\cdot)$. Then, the $m_i = \tau(a^i)$ describes the i-th free moment of the element $a \in \mathcal{A}$ as in (1). By Theorem 1.1 the Stieltjes transform $\mathrm{G}_\mu(z)$ is given as

$$\mathrm{G}_\mu(z) = \frac{1}{z}\left(1 + \sum_{i=1}^{\infty} z^{-i} m_i\right).$$

Suppose

$$\mathrm{G}_\mu^{-1}(z) = \frac{1}{z} + \sum_{i=0}^{\infty} k_{i+1} z^i,$$

then it will be shown that k_i can be determined by a recursive formula depending on m_j, $j = 1, 2, \ldots, i$. In this case Definition 1.2 and 1.3 imply that the free cumulants have been found. Now, combining formulas for $\mathrm{G}_\mu(z)$ and $\mathrm{G}_\mu^{-1}(z)$ the following relation will be utilized:

$$\begin{aligned} z &= \mathrm{G}_\mu^{-1}(\mathrm{G}_\mu(z)) = \frac{1}{\mathrm{G}_\mu(z)} + \sum_{i=0}^{\infty} k_{i+1}\, \mathrm{G}_\mu(z)^i \\ &= \frac{z}{1 + \sum_{i=1}^{\infty} z^{-i} m_i} + \sum_{i=0}^{\infty} k_{i+1}\left(\frac{1}{z}\left(1 + \sum_{j=1}^{\infty} z^{-j} m_j\right)\right)^i \\ &= z\sum_{j=0}^{\infty}\left(-\sum_{i=1}^{\infty} z^{-i} m_i\right)^j + \sum_{i=0}^{\infty}\frac{k_{i+1}}{z^i}\left(1 + \sum_{j=1}^{\infty} z^{-j} m_j\right)^i \\ &= z + z\sum_{j=1}^{\infty}\left(-\sum_{i=1}^{\infty} z^{-i} m_i\right)^j + \sum_{i=0}^{\infty}\frac{k_{i+1}}{z^i}\left(\sum_{j=0}^{\infty} z^{-j} m_j\right)^i. \end{aligned}$$

By simple arithmetic calculations this relation leads to the equation

$$z\sum_{j=0}^{\infty}\sum_{l=0}^{j+1}\binom{j+1}{l}(-1)^{l+1}\left(\sum_{i=0}^{\infty} z^{-i} m_i\right)^l = \sum_{i=0}^{\infty}\frac{k_{i+1}}{z^i}\left(\sum_{j=0}^{\infty} z^{-j} m_j\right)^i.$$

The next step will be to apply a formula for the powers of a power series (see [5])

$$\left(\sum_{i=0}^{\infty} m_i z^i\right)^k = \sum_{n=0}^{\infty}\binom{\mathbf{m}, k, \geq}{n} z^n.$$

Therefore,

$$\sum_{j=0}^{\infty}\left(-1+\sum_{l=1}^{j+1}\binom{j+1}{l}(-1)^{l+1}\sum_{t=0}^{\infty}\binom{\mathbf{m},l,\geq}{t}z^{-t}\right)$$
$$=\frac{k_1}{z}+\sum_{i=1}^{\infty}k_{i+1}\sum_{t=0}^{\infty}\binom{\mathbf{m},i,\geq}{t}z^{-t-i-1}.$$

By the identification of coefficients of z^{-t} the cumulants are obtained. Let us denote left hand side and right hand side of the equation by corresponding LHS and RHS. Let $t=0$, then

$$\begin{aligned}LHS &= \sum_{j=0}^{\infty}\left(-1+\sum_{l=1}^{j+1}\binom{j+1}{l}(-1)^{l+1}\binom{\mathbf{m},l,\geq}{0}\right)\\ &= \sum_{j=1}^{\infty}\sum_{l=0}^{j}\binom{j}{l}(-1)^{l+1}=0=RHS.\end{aligned}$$

For $t=1$ we get $k_1=m_1$ since $RHS=k_1$ and

$$LHS=\sum_{j=0}^{\infty}\sum_{l=1}^{j+1}\binom{j+1}{l}(-1)^{l+1}\binom{\mathbf{m},l,\geq}{1}=\sum_{j=1}^{\infty}\sum_{l=1}^{j}\binom{j}{l}(-1)^{l+1}lm_1=m_1.$$

For $t\geq 2$,

$$\begin{aligned}\sum_{j=0}^{\infty}\sum_{l=1}^{j+1}\binom{j+1}{l}(-1)^{l+1}\binom{\mathbf{m},l,\geq}{t} &= \sum_{i=1}^{t-1}k_{i+1}\binom{\mathbf{m},i,\geq}{t-i-1}\\ &= k_t\binom{\mathbf{m},t-1,\geq}{0}+\sum_{i=1}^{t-2}k_{i+1}\binom{\mathbf{m},i,\geq}{t-i-1}\\ &= k_t+\sum_{i=1}^{t-2}k_{i+1}\binom{\mathbf{m},i,\geq}{t-i-1},\end{aligned}$$

$$k_t=\sum_{j=0}^{\infty}\sum_{l=1}^{j+1}\binom{j+1}{l}(-1)^{l+1}\binom{\mathbf{m},l,\geq}{t}-\sum_{i=1}^{t-2}k_{i+1}\binom{\mathbf{m},i,\geq}{t-i-1}.$$

Let us now show that

$$\sum_{j=t}^{\infty}\sum_{l=1}^{j+1}\binom{j+1}{l}(-1)^{l+1}\binom{\mathbf{m},l,\geq}{t}=0. \tag{4}$$

Using the fact that $\binom{\mathbf{m},l}{t}$ is a polynomial of maximally t-th order of l it is enough to show that $\sum_{j=t}^{\infty}\sum_{l=1}^{j+1}\binom{j+1}{l}(-1)^{l+1}l^W=0$ for all $W=1,2,\ldots,t$. We prove the above equation by showing that each element of the sum is zero, i.e., that for any fixed L such that $L \geq t$ and for all $W = 1, 2, \dots, t$ we have $\sum_{l=1}^{L+1} \binom{L+1}{l}(-1)^{l+1} l^W = 0$. Furthermore, the sum can be expressed as

$$\sum_{l=1}^{L+1} \binom{L+1}{l}(-1)^{l+1} l^W = (L+1) \sum_{h=0}^{L} \binom{L}{h}(-1)^h (h+1)^{W-1}.$$

We will prove using mathematical induction with respect to L that for all L and all W such that $L \geq t \geq W$, $L, W \in \mathbb{N} \setminus \{0\}$,

$$\sum_{h=0}^{L} \binom{L}{h}(-1)^h (h+1)^{W-1} = 0.$$

Let $L = 1$, then $\sum_{h=0}^{1} \binom{1}{h}(-1)^h (h+1)^{W-1} = 1 - 2^{W-1} = 0$ as $W \leq L = 1$ and $W \in \mathbb{N} \setminus \{0\}$. If $L = 2$, then

$$\sum_{h=0}^{2} \binom{2}{h}(-1)^h (h+1)^{W-1} = 1 - 2^W + 3^{W-1} \overset{W \in \{1,2\}}{=} 0.$$

Let assume that the equation holds for L. Then

$$\sum_{h=0}^{L+1} \binom{L+1}{h}(-1)^h (h+1)^{W-1} = \underbrace{\sum_{h=0}^{L} \binom{L}{h}(-1)^h (h+1)^{W-1}}_{=0}$$
$$+ \underbrace{\sum_{h=0}^{L-1} \binom{L}{h}(-1)^{h+1} (h+2)^{W-1} + (-1)^{L+1}(L+2)^{W-1}}_{=0} = 0$$

and (4) is proved. Then finally $k_1 = m_1$ and for $t = 2, 3, \dots$

$$k_t = \sum_{i=0}^{t-1} \sum_{h=1}^{i+1} (-1)^{h+1} \binom{i+1}{h} \binom{\mathbf{m}, h, \geq}{t} - \sum_{h=2}^{t-1} k_h \binom{\mathbf{m}, h-1, \geq}{t-h}.$$

Now it is left to show that

$$\sum_{i=0}^{t-1} \sum_{h=1}^{i+1} (-1)^{h+1} \binom{i+1}{h} \binom{\mathbf{m}, h, \geq}{t} = \sum_{i=0}^{t-1} (-1)^{i+2} \binom{\mathbf{m}, i+1, >}{t}. \tag{5}$$

Indeed, the equality

$$\sum_{h=1}^{i+1} (-1)^{h+1} \binom{i+1}{h} \binom{\mathbf{m}, h, \geq}{t} = (-1)^{i+2} \binom{\mathbf{m}, i+1, >}{t}$$

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holds elementwise for all $i = 0, \ldots, t-1$. Then

$$\begin{aligned} LHS &= \underbrace{\sum_{h=1}^{i}(-1)^{h+1}\binom{i+1}{h}\binom{\mathbf{m}, h, \geq}{t}}_{U} + (-1)^{i+2}\binom{i+1}{i+1}\binom{\mathbf{m}, i+1, \geq}{t} \\ &= U + (-1)^{i+2}\binom{\mathbf{m}, i+1, >}{t} + (-1)^{i+2}\sum_{\substack{j_1+j_2+\ldots+j_{i+1}=t \\ \exists k\; j_k=0}} m_{j_1}\cdot\ldots\cdot m_{j_{i+1}}. \end{aligned}$$

So equation (5) is equivalent to

$$\sum_{h=1}^{i}(-1)^{h-i}\binom{i+1}{h}\binom{\mathbf{m}, h, \geq}{t} = \sum_{\substack{j_1+j_2+\ldots+j_{i+1}=t \\ \exists k\; j_k=0}} m_{j_1}\cdot\ldots\cdot m_{j_{i+1}}.$$

Then

$$\begin{aligned} RHS &= \sum_{h=1}^{i}\binom{i+1}{h}\binom{\mathbf{m}, i-h+1, >}{t} = \sum_{h=1}^{i}\binom{i+1}{h}\binom{\mathbf{m}, h, >}{t}, \\ LHS &= \sum_{h=1}^{i}(-1)^{h-i}\binom{i+1}{h}\sum_{k=0}^{h-1}\binom{h}{k}\binom{\mathbf{m}, h-k, >}{t} \\ &= \sum_{h=1}^{i}(-1)^{h-i}\binom{i+1}{h}\sum_{H=1}^{h}\binom{h}{H}\binom{\mathbf{m}, H, >}{t} \\ &= \sum_{H=1}^{i}\underbrace{\sum_{h=H}^{i}(-1)^{h-i}\binom{h}{H}\binom{i+1}{h}}_{=\frac{\Gamma(2+i)}{\Gamma(H+1)\Gamma(2-H+i)}=\binom{i+1}{H}}\binom{\mathbf{m}, H, >}{t} = RHS, \end{aligned}$$

where $\Gamma(k) := (k-1)!$ denotes the Gamma function. Equation (5) holds. Hence, $k_1 = m_1$ and

$$k_t = \sum_{i=1}^{t}(-1)^{i+1}\binom{\mathbf{m}, i, >}{t} - \sum_{h=2}^{t-1}k_h\binom{\mathbf{m}, h-1, \geq}{t-h}, \tag{6}$$

which completes the proof of the theorem. □

The first five free cumulants k_i, $i = 1, \ldots, 5$, given as a function of m_j, $j = 1, \ldots, i$, are stated in Corollary 2.1.

Corollary 2.1. *Let $(\mathcal{A}, \tau)$ be a non-commutative $*$-probability space and $m_i = \tau(a^i)$ denotes the i-th free moment of an element $a \in \mathcal{A}$. Then, the* *first five free cumulants k_i of a are given by*

$$\begin{aligned}
k_1 &= m_1,\\
k_2 &= m_2 - m_1^2,\\
k_3 &= m_3 - 3m_2m_1 + 2m_1^3,\\
k_4 &= m_4 - 4m_3m_1 - 2m_2^2 + 10m_2m_1^2 - 5m_1^4,\\
k_5 &= m_5 - 5m_4m_1 + 15m_3m_1^2 + 15m_2^2m_1 - 35m_2m_1^3 - 5m_3m_2 + 14m_1^5.
\end{aligned}$$

Proof. By definition $k_1 = m_1$. Using relation (3) we obtain

$$\begin{aligned}
k_2 &= \sum_{i=1}^{2}(-1)^{i+1} \sum_{\substack{j_1+\ldots+j_i=2\\ \forall k\ j_k>0}} m_{j_1}\cdot\ldots\cdot m_{j_i}\\
&= (-1)^2 m_2 + (-1)^3 m_1^2 = m_2 - m_1^2,\\
k_3 &= \sum_{i=1}^{3}(-1)^{i+1} \sum_{\substack{j_1+\ldots+j_i=3\\ \forall k\ j_k>0}} m_{j_1}\cdot\ldots\cdot m_{j_i}\\
&\quad -\sum_{h=2}^{3-1} k_h \sum_{\substack{j_1+\ldots+j_{h-1}=3-h\\ \forall k\ j_k\geq 0}} m_{j_1}\cdot\ldots\cdot m_{j_{h-1}}\\
&= (-1)^2 m_3 + (-1)^3 2m_1m_2 + (-1)^4 m_1^3 - k_2m_1\\
&= m_3 - 3m_2m_1 + 2m_1^3,\\
k_4 &= \sum_{i=1}^{4}(-1)^{i+1} \sum_{\substack{j_1+\ldots+j_i=4\\ \forall k\ j_k>0}} m_{j_1}\cdot\ldots\cdot m_{j_i}\\
&\quad -\sum_{h=2}^{3} k_h \sum_{\substack{j_1+\ldots+j_{h-1}=4-h\\ \forall k\ j_k\geq 0}} m_{j_1}\cdot\ldots\cdot m_{j_{h-1}}\\
&= m_4 - 2m_3m_1 - m_2^2 + 3m_2m_1^2 - m_1^4 - k_2m_2 - 2k_3m_1\\
&= m_4 - 4m_3m_1 - 2m_2^2 + 10m_2m_1^2 - 5m_1^4,\\
k_5 &= \sum_{i=1}^{5}(-1)^{i+1} \sum_{\substack{j_1+\ldots+j_i=5\\ \forall k\ j_k>0}} m_{j_1}\cdot\ldots\cdot m_{j_i}\\
&\quad -\sum_{h=2}^{4} k_h \sum_{\substack{j_1+\ldots+j_{h-1}=5-h\\ \forall k\ j_k\geq 0}} m_{j_1}\cdot\ldots\cdot m_{j_{h-1}}\\
&= m_5 - 5m_4m_1 - 5m_3m_2 + 15m_3m_1^2 + 15m_2^2m_1 - 35m_2m_1^3 + 14m_1^5,
\end{aligned}$$

which completes the proof. More details in the proof are given in [10]. □

The above-presented proof of Corollary 2.1 gives examples of direct calculations of free cumulants using Theorem 2.1. Now consider equation (2), which was used to obtain the free cumulants of degree 1 to 5 by the combinatorial approach. The equality $k_1 = m_1$ is again assumed to hold. Then

$$m_2 := \tau(a, a) = \sum_{\pi \in NC(2)} k_\pi[a, a].$$

If $\pi \in NC(2)$, then $\pi = \{1, 2\}$ or $\pi = \{\{1\}, \{2\}\}$, hence $m_2 = k_1 k_1 + k_2$ and $k_2 = m_2 - k_1^2 = m_2 - m_1^2$.

To obtain the third free cumulant the sum is taken over all non-crossing partitions of the three elements set $NC(3)$. Then

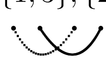
$$\pi \in \{\{1, 2, 3\}, \{\{1, 2\}, \{3\}\}, \{\{1, 3\}, \{2\}\}, \{\{1\}, \{2, 3\}\}, \{\{1\}, \{2\}, \{3\}\}\}.$$

Each of the sets is illustrated with a simple graph. The elements belonging to the same subset are connected with a line. The crossing partition is indicated by the cross of at least two lines from two distinct subsets. Hence,

$$\begin{aligned} m_3 &= \sum_{\pi \in NC(3)} k_\pi[a, a, a] = k_3 + k_2 k_1 + k_2 k_1 + k_1 k_2 + k_1^3 \\ &= k_3 + 3k_1 k_2 + k_1^3, \\ k_3 &= m_3 - 3k_1 k_2 - k_1^3 = m_3 - 3m_1(m_2 - m_1^2) - m_1^3 \\ &= m_3 - 3m_1 m_2 + 2m_1^3. \end{aligned}$$

While calculating the fourth free cumulant we notice that there is only one crossing partition indicated by cross of line illustrating subsets $\{1, 3\}$ and $\{2, 4\}$, i.e.,

$$NC(4) \not\ni \{\{1, 3\}, \{2, 4\}\}.$$



Hence,

$$\begin{aligned} m_4 &= \sum_{\pi \in NC(4)} k_\pi[a, a, a, a] = k_4 + 4k_3 k_1 + 2k_2^2 + 6k_2 k_1^2 + k_1^4, \\ k_4 &= m_4 - 4k_1 k_3 - 2k_2^2 - 6k_2 k_1^2 - k_1^4 = m_4 - 4m_1(m_3 - 3m_1 m_2 + 2m_1^3) \\ &\quad -2(m_2 - m_1^2)^2 - 6m_1^2(m_2 - m_1^2) - m_1^4 \\ &= m_4 - 4m_1 m_3 - 2m_2^2 + 10m_2 m_1^2 - 5m_1^4. \end{aligned}$$

The calculations of the fifth free cumulant, by use of (2), demand the summing over $NC(5)$. Consider the crossing partitions of the set $\{1,2,3,4,5\}$:

$NC(5) \not\ni \{\{1,2,4\},\{3,5\}\}$, $NC(5) \not\ni \{\{1,4\},\{2,3,5\}\}$,

$NC(5) \not\ni \{\{1,3,4\},\{2,5\}\}$, $NC(5) \not\ni \{\{1,3\},\{2,4,5\}\}$,

$NC(5) \not\ni \{\{2,4\},\{1,3,5\}\}$, $NC(5) \not\ni \{\{1,3\},\{2,4\},\{5\}\}$,

$NC(5) \not\ni \{\{1\},\{2,4\},\{3,5\}\}$, $NC(5) \not\ni \{\{1,4\},\{2\},\{3,5\}\}$,

$NC(5) \not\ni \{\{1,3\},\{2,5\},\{4\}\}$, $NC(5) \not\ni \{\{1,4\},\{2,5\},\{3\}\}$.

Then

$$\begin{aligned}
m_5 &= \sum_{\pi \in NC(5)} k_\pi[a,a,a,a,a] = k_5 + 5k_4k_1 + \left(\binom{5}{2} - 5\right)k_3k_2 \\
&\quad + \binom{5}{3}k_3k_1^2 + \left(\binom{5}{1}\frac{1}{2}\binom{4}{2} - 5\right)k_2^2k_1 + \binom{5}{2}k_2k_1^3 + k_1^5 \\
&= k_5 + 5k_4k_1 + 5k_3k_2 + 10k_3k_1^2 + 10k_2^2k_1 + 10k_2k_1^3 + k_1^5, \\
k_5 &= m_5 - 5k_4k_1 - 5k_3k_2 - 10k_3k_1^2 - 10k_2^2k_1 - 10k_2k_1^3 - k_1^5 \\
&= m_5 - 5(m_4 - 4m_3m_1 - 2m_2^2 + 10m_2m_1^2 - 5m_1^4)m_1 \\
&\quad -5(m_3 - 3m_2m_1 + 2m_1^3)(m_2 - m_1^2) - 10(m_3 - 3m_2m_1 + 2m_1^3)m_1^2 \\
&\quad -10(m_2 - m_1^2)^2m_1 - 10(m_2 - m_1^2)m_1^3 - m_1^5 \\
&= m_5 - 5m_4m_1 + 15m_3m_1^2 + 15m_2^2m_1 - 35m_2m_1^3 - 5m_3m_2 + 14m_1^5.
\end{aligned}$$

The calculations with use of both methods are presented. To some extent we find that summing over the $i_1,\ldots,i_h$, such that $i_1+\ldots+i_h = k$ is simpler than summing over non-crossing partitions.

3. Example of calculations for free cumulants and moments

It is important to mention a particular example of a non-commutative $*$-probability space $(\mathrm{RM}_p(\mathbb{R}),\tau)$ as an illustration and due to the extended engineering applications. Here, $\mathcal{A} = \mathrm{RM}_p(\mathbb{R})$ denotes set of all $p\times p$ random matrices with entries being real random variables on a probability space $(\Omega,\mathcal{F},P)$ with finite moments of any order. Defined in this way $\mathrm{RM}_p(\mathbb{R})$ is a $*$-algebra, with the classical matrix product as multiplication and the transpose as $*$–operation. The $*$-algebra is equipped with tracial functional τ defined as expectation of the normalized trace by

$$\tau(\mathbf{X}) := \mathbb{E}\left(\frac{1}{p}\operatorname{Tr}\mathbf{X}\right) = \frac{1}{p}\mathbb{E}\sum_{i=1}^{p}\lambda_i = \int_{\mathbb{R}} x\frac{1}{p}\sum_{i=1}^{p}\delta_{\{\lambda_i\le x\}}dx = \int_{\mathbb{R}} x d\mu(x),$$

where $\mathbf{X} = (X_{ij})_{i,j=1}^{p} \in \mathrm{RM}_p(\mathbb{R})$, δ_B denotes Dirac delta function on set B, λ_i are eigenvalues of matrix $\mathbf{X}$ and $\mu = \frac{1}{p}\sum_{i=1}^{p}\delta_{\{\lambda_i\le x\}}$ is $*$-distribution, usually called the eigenvalue distribution (spectral density) of the matrix $\mathbf{X}$. This set up is of common use, when studying the spectral measure of random matrices. Often related research problems arise within, e.g., theoretical physics and wireless communication, see [1] and [13].

Let us consider a matrix $\mathbf{M}_p = \frac{1}{p}\mathbf{X}\mathbf{X}'$, where $X_{ij} \sim \mathcal{N}(0,1)$, which also belongs to $(\mathrm{RM}_p(\mathbb{R}),\tau)$. A matrix $\mathbf{W} = p\mathbf{M}_p = \mathbf{X}\mathbf{X}' \sim W_p(I,p)$. For the Wishart matrix $\mathbf{W}$ the relation

$$\mathbb{E}(\operatorname{Tr}\mathbf{W}^{k+1}) = k\mathbb{E}(\operatorname{Tr}\mathbf{W}^{k}) + \sum_{\substack{i+j=k\\ i,j\ge 0}}\mathbb{E}(\operatorname{Tr}\mathbf{W}^{i}\operatorname{Tr}\mathbf{W}^{j})$$

holds. Then

$$\begin{aligned}\tau(\mathbf{M}_p^{k+1}) &= \frac{1}{p^{k+2}}\mathbb{E}(\operatorname{Tr}\mathbf{W}^{k+1})\\ &= \frac{k}{p^{k+2}}\mathbb{E}(\operatorname{Tr}\mathbf{W}^{k}) + \frac{1}{p^{k+2}}\sum_{\substack{i+j=k\\ i,j\ge 0}}\mathbb{E}(\operatorname{Tr}\mathbf{W}^{i}\operatorname{Tr}\mathbf{W}^{j}).\end{aligned}$$

And the first free moments $m_k = \tau(\mathbf{M}_p^k)$ for the matrix $\mathbf{M}_p$ are given by

$$\begin{aligned}m_1 &= \frac{1}{p^2}\mathbb{E}(\operatorname{Tr}\mathbf{W}^{1}) = \frac{1}{p^2}\mathbb{E}(\operatorname{Tr}\mathbf{W}^{0}\operatorname{Tr}\mathbf{W}^{0}) = \frac{p^2}{p^2} = 1,\\ m_2 &= \frac{1}{p^3}\mathbb{E}(\operatorname{Tr}\mathbf{W}^{2}) = \frac{1}{p^3}\left(\mathbb{E}(\operatorname{Tr}\mathbf{W}) + \sum_{\substack{i+j=1\\ i,j\ge 0}}\mathbb{E}(\operatorname{Tr}\mathbf{W}^{i}\operatorname{Tr}\mathbf{W}^{j})\right) = 2 + \frac{1}{p}.\end{aligned}$$

Similarly,

$$\begin{aligned}m_3 &= \frac{4+6p+5p^2}{p^2},\\ m_4 &= \frac{20+42p+29p^2+14p^3}{p^3}.\end{aligned}$$

Then, using Corollary 2.1, we get the free cumulants for the $p \times p$ matrix $\mathbf{M}_p$ as follows:

$$k_1 = m_1 = 1,$$
$$k_2 = m_2 - m_1^2 = 1 + \frac{1}{p},$$
$$k_3 = m_3 - 3m_2m_1 + 2m_1^3 = \frac{4p^2 + 3p^3 + p^4}{p^4},$$
$$k_4 = \frac{20 + 24p + 7p^2 + p^3}{p^3}.$$

The free cumulants for the fixed p give us the R-transform for the desired matrices $\mathbf{M}_p$. While $p \to \infty$ the matrix $\mathbf{M}_{p\to\infty}$, which is an "infinite matrix" realized by a sequence of matrices of increasing size, has the R-transform $R_{\mathbf{M}_{p\to\infty}}(z) = \sum_{j=0}^{\infty} k_{j+1} z^j = 1+z+z^2+z^3+\ldots$, which by the inverse Stieltjes formula corresponds to the spectral distribution given by the Marčenko–Pastur law [6], i.e., $\mu'_{p\to\infty}(x) = \frac{1}{2\pi x}\sqrt{4x - x^2}$.

4. Conclusions

In this article we prove a new recursive relation formula between free cumulants and moments using the concepts of Stieltjes and R-transforms. The demonstrated results are not based on the combinatorial idea of non-crossing partitions as in the previous studies. This implies that the relation can be obtained with use of, in our opinion, simpler computations. There is a strong believe that the result can successfully complete already existing knowledge regarding cumulant-moment relations in free probability and in some particular cases replace previously used formulas in order to provide easier calculations or avoid introducing crossing partition related concepts.

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Linköping University, 581 83 Linköping, Sweden
E-mail address: Jolanta.Pielaszkiewicz@liu.se

Swedish University of Agricultural Sciences, 750 07 Uppsala, Sweden
E-mail address: Dietrich.von.Rosen@slu.se

Linköping University, 581 83 Linköping, Sweden
E-mail address: Martin.Singull@liu.se