First-order random coefficient autoregressive (RCA(1)) model: Joint Whittle estimation and information

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ABSTRACT. Random coefficient autoregressive model, RCA(p), has been discussed widely as a suitable model for nonlinear time series. The conditional least squares and likelihood parameter estimation of RCA(p) model has also been discussed in [3]. The statistical inference of RCA(1) model has been presented in [4] while the conditional least square estimates for nonstationary processes is studied in [7]. The optimal estimation for nonlinear time series using estimating equations has been investigated in [6].

Recently there has been interest in joint prediction based on spectral density of popular nonlinear time series models such as RCA(p) models. Another way of estimating the parameters of the RCA(1) model is to do Whittle's estimation. In this paper the Whittle estimates of the parameters of an RCA(p) model are studied. It is shown that the Whittle information of the autoregressive parameter in an RCA(p) model is larger than the corresponding information in an autoregressive (AR) model.

1. Introduction

As a result of growth of interest in dynamic systems such as financial, environmental and biological science, lots of studies have been done to investigate an appropriate model for such systems. In the statistical literature, the random coefficient autoregressive model (RCA(p)) has been discussed widely as a suitable model for nonlinear time series (see [3]). The RCA(p) model is simply a generalisation of the standard autoregressive (AR) models where the coefficients are allowed to vary over time. Specifically, the RCA(1)

model is defined as

$$Y_t = (\phi + b_t)Y_{t-1} + u_t, \tag{1}$$

where ϕ is constant coefficient, $\{b_t\}$ is a sequence of independent and identically distributed ($i.i.d.$) random variables with mean zero and constant variance σ_b^2, $\{u_t\}$ is a white noise process with mean zero and variance σ_u^2 and $\{b_t\}$ and $\{u_t\}$ are mutually independent. Note that if $\sigma_b^2 = 0$, then the RCA(1) model reduces to the standard AR(1) model. The AR(1) model has a constant coefficient while the RCA(1) model includes the extra random variables $\{b_t\}$ which vary over the time, thus the RCA(p) model is more general and appropriate for nonlinear data sets.

The conditional least squares and likelihood parameter estimation of RCA(p) model is discussed in [3]. The statistical inference of RCA(1) model has been presented in [4] while the conditional least square estimates for nonstationary processes is studied in [7]. The optimal estimation for nonlinear time series using estimating equations has been investigated in [6] while the optimal estimation functions for RCA(p) model have been studied in [1]. The moment properties and the corresponding squared RCA(p) process have been investigated in [2]. It has been illustrated in [2] that the autocorrelation structure of the RCA(p) model is the same as for the AR(p) model, while they have different marginal variances. Furthermore, it is also shown that in [2] the joint prediction of mean and volatility of the RCA model and the spectral density function of the RCA(1) model is given as

$$f(\lambda) = \frac{\sigma_u^2(1 - \phi^2)}{2\pi(1 - \phi^2 - \sigma_b^2)} \left| \frac{1}{1 - \phi e^{-i\lambda}} \right|^2,$$

provided $\phi^2 + \sigma_b^2 < 1$ (see [2]).

Another way of estimating the parameters of the RCA(1) model is to do Whittle's estimation. The Whittle estimates are obtained by minimizing the function (see [5])

$$2n \ln 2\pi + \sum_{j=0}^{n-1} \ln f(\omega_j) + \sum_{j=0}^{n-1} \frac{I(\omega_j)}{f(\omega_j)},$$

where n in the number of observations, $\omega_j = \frac{2\pi j}{n}$, and $I(\omega_j)$ is given by

$$I(\omega_j) = \frac{1}{2\pi n} |\sum_{t=0}^{n-1} Y_t \exp(-i\omega_j t)|^2, \quad j = 0, 1, 2, \ldots, n - 1.$$

The Whittle estimates of the parameters of an RCA(1) model have not been fully investigated and hence in this paper the Whittle estimates of the parameters of an RCA(1) model are studied. It will be shown in Section 2 that the Whittle information of the autoregressive parameter in an RCA(1)

model is larger than the corresponding information in an AR(1) model. Finally, the conclusions are drawn in Section 3.

2. Whittle's information matrix

Proposition. *For the RCA(1) model as defined in (1), the Whittle information matrix is*

$$
n \begin{bmatrix}
\frac{2\phi^2\sigma_b^4}{(1-\phi^2)^2(1-\phi^2-\sigma_b^2)^2} + \frac{1}{(1-\phi^2)} & \frac{\phi\sigma_b^2}{(1-\phi^2)(1-\phi^2-\sigma_b^2)^2} & \frac{\phi\sigma_b^2}{\sigma_u^2(1-\phi^2)(1-\phi^2-\sigma_b^2)} \\
\frac{\phi\sigma_b^2}{(1-\phi^2)(1-\phi^2-\sigma_b^2)^2} & \frac{1}{2(1-\phi^2-\sigma_b^2)^2} & \frac{1}{2\sigma_u^2(1-\phi^2-\sigma_b^2)} \\
\frac{\phi\sigma_b^2}{\sigma_u^2(1-\phi^2)(1-\phi^2-\sigma_b^2)} & \frac{1}{2\sigma_u^2(1-\phi^2-\sigma_b^2)} & \frac{1}{2\sigma_u^4}
\end{bmatrix}.
$$

$$(2)$$

Proof. The spectral density function of RCA(1) model is given as

$$
\begin{aligned}
f(\lambda) &= \frac{\sigma_u^2(1-\phi^2)}{2\pi(1-\phi^2-\sigma_b^2)} \left| \frac{1}{1-\phi e^{-i\lambda}} \right|^2 \\
&= \frac{\sigma_u^2(1-\phi^2)}{2\pi(1-\phi^2-\sigma_b^2)} [1 - 2\phi\cos\lambda + \phi^2]^{-1},
\end{aligned}
$$

while the Whittle information matrix is

$$
n \begin{bmatrix}
W_{11} & W_{12} & W_{13} \\
W_{21} & W_{22} & W_{23} \\
W_{31} & W_{32} & W_{33}
\end{bmatrix},
$$

$$(3)$$

where

$$
W_{11} = \frac{1}{4\pi} \int_{-\pi}^{\pi} \left(\frac{\partial \ln f(\lambda)}{\partial \phi} \right)^2 d\lambda,
$$

$$
W_{22} = \frac{1}{4\pi} \int_{-\pi}^{\pi} \left(\frac{\partial \ln f(\lambda)}{\partial \sigma_b^2} \right)^2 d\lambda,
$$

$$
W_{33} = \frac{1}{4\pi} \int_{-\pi}^{\pi} \left(\frac{\partial \ln f(\lambda)}{\partial \sigma_u^2} \right)^2 d\lambda,
$$

$$
W_{12} = W_{21} = \frac{1}{4\pi} \int_{-\pi}^{\pi} \left(\frac{\partial \ln f(\lambda)}{\partial \phi} \right) \left(\frac{\partial \ln f(\lambda)}{\partial \sigma_b^2} \right) d\lambda,
$$

$$
W_{13} = W_{31} = \frac{1}{4\pi} \int_{-\pi}^{\pi} \left(\frac{\partial \ln f(\lambda)}{\partial \phi} \right) \left(\frac{\partial \ln f(\lambda)}{\partial \sigma_u^2} \right) d\lambda,
$$

and

$$
W_{23} = W_{32} = \frac{1}{4\pi} \int_{-\pi}^{\pi} \left(\frac{\partial \ln f(\lambda)}{\partial \sigma_b^2} \right) \left(\frac{\partial \ln f(\lambda)}{\partial \sigma_u^2} \right) d\lambda.
$$

The natural logarithm of $f(\lambda)$ is

$$
\ln f(\lambda) = \ln \sigma_u^2 + \ln(1-\phi^2) - \ln(2\pi) - \ln(1-\phi^2-\sigma_b^2) - \ln(1-2\phi\cos\lambda+\phi^2)
$$

and the partial derivative of $\ln f(\lambda)$ with respect to ϕ is

$$\frac{\partial \ln f(\lambda)}{\partial \phi} = \frac{-2\phi}{(1-\phi^2)} + \frac{2\phi}{(1-\phi^2-\sigma_b^2)} + \frac{2(\cos\lambda - \phi)}{(1-2\phi\cos\lambda + \phi^2)}$$

$$= \frac{-2\phi + 2\phi^3 + 2\phi\sigma_b^2 + 2\phi - 2\phi^3}{(1-\phi^2)(1-\phi^2-\sigma_b^2)} + \frac{2(\cos\lambda - \phi)}{(1-2\phi\cos\lambda + \phi^2)}$$

$$= \frac{2\phi\sigma_b^2}{(1-\phi^2)(1-\phi^2-\sigma_b^2)} + \frac{2(\cos\lambda - \phi)}{(1-2\phi\cos\lambda + \phi^2)}.$$

Next we evaluate the integral

$$\int_{-\pi}^{\pi}\left[\frac{\partial \ln f(\lambda)}{\partial \phi}\right]^2 d\lambda = \int_{-\pi}^{\pi}\left[\frac{2\phi\sigma_b^2}{(1-\phi^2)(1-\phi^2-\sigma_b^2)} + \frac{2(\cos\lambda - \phi)}{(1-2\phi\cos\lambda + \phi^2)}\right]^2 d\lambda$$

$$= \int_{-\pi}^{\pi}\frac{4\phi^2\sigma_b^4}{(1-\phi^2)^2(1-\phi^2-\sigma_b^2)^2}d\lambda$$

$$+ \int_{-\pi}^{\pi}\frac{8\phi\sigma_b^2}{(1-\phi^2)(1-\phi^2-\sigma_b^2)} \times \frac{(\cos\lambda - \phi)}{(1-2\phi\cos\lambda + \phi^2)}d\lambda$$

$$+ \int_{-\pi}^{\pi}\frac{4(\cos\lambda - \phi)^2}{(1-2\phi\cos\lambda + \phi^2)^2}d\lambda. \tag{4}$$

The above integrals in (4) could be evaluated as follows:

$$\int_{-\pi}^{\pi}\frac{4\phi^2\sigma_b^4}{(1-\phi^2)^2(1-\phi^2-\sigma_b^2)^2}d\lambda = \frac{8\pi\phi^2\sigma_b^4}{(1-\phi^2)^2(1-\phi^2-\sigma_b^2)^2},$$

$$\int_{-\pi}^{\pi}\left[\frac{8\phi\sigma_b^2}{(1-\phi^2)(1-\phi^2-\sigma_b^2)} \times \frac{(\cos\lambda - \phi)}{(1-2\phi\cos\lambda + \phi^2)}\right]d\lambda$$

$$= \frac{8\phi\sigma_b^2}{(1-\phi^2)(1-\phi^2-\sigma_b^2)}$$

$$\times \int_{-\pi}^{\pi}\frac{(\cos\lambda - \phi)}{(1-2\phi\cos\lambda + \phi^2)}d\lambda.$$

However, according to proof of Lemma 5 [5],

$$\int_{-\pi}^{\pi}\frac{(\cos\lambda - \phi)}{(1-2\phi\cos\lambda + \phi^2)}d\lambda = 0.$$

Hence, we have

$$\int_{-\pi}^{\pi}\left[\frac{8\phi\sigma_b^2}{(1-\phi^2)(1-\phi^2-\sigma_b^2)} \times \frac{(\cos\lambda - \phi)}{(1-2\phi\cos\lambda + \phi^2)}\right]d\lambda = 0.$$

In addition, based on the proof of Lemma 1 [5],

$$\int_{-\pi}^{\pi}\frac{4(\cos\lambda - \phi)^2}{(1-2\phi\cos\lambda + \phi^2)^2}d\lambda = \frac{4\pi}{(1-\phi^2)}.$$

Thus we have

$$\int_{-\pi}^{\pi} \left[\frac{\partial \ln f(\lambda)}{\partial \phi}\right]^2 d\lambda \;=\; \frac{8\pi\phi^2\sigma_b^4}{(1-\phi^2)^2(1-\phi^2-\sigma_b^2)^2} + \frac{4\pi}{(1-\phi^2)}.$$

and, consequently, W_{11} is

$$\frac{1}{4\pi}\int_{-\pi}^{\pi} \left[\frac{\partial \ln f(\lambda)}{\partial \phi}\right]^2 d\lambda = \left[\frac{2\phi^2\sigma_b^4}{(1-\phi^2)^2(1-\phi^2-\sigma_b^2)^2} + \frac{1}{(1-\phi^2)}\right].$$

The partial derivative of $\ln f(\lambda)$ with respect to σ_b^2 is

$$\frac{\partial \ln f(\lambda)}{\partial \sigma_b^2} = \frac{1}{(1-\phi^2-\sigma_b^2)}$$

and thus

$$\begin{aligned}
W_{22} &= \frac{1}{4\pi}\int_{-\pi}^{\pi} \left(\frac{\partial \ln f(\lambda)}{\partial \sigma_b^2}\right)^2 d\lambda \\
&= \frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{1}{(1-\phi^2-\sigma_b^2)^2} d\lambda \\
&= \frac{1}{(4\pi)(1-\phi^2-\sigma_b^2)^2}\int_{-\pi}^{\pi} d\lambda \\
&= \frac{2\pi}{(4\pi)(1-\phi^2-\sigma_b^2)^2} \\
&= \frac{1}{2(1-\phi^2-\sigma_b^2)^2}.
\end{aligned}$$

The partial derivative of $\ln f(\lambda)$ with respect to σ_u^2 is

$$\frac{\partial \ln f(\lambda)}{\partial \sigma_u^2} = \frac{1}{\sigma_u^2}$$

and we have

$$\begin{aligned}
W_{33} &= \frac{1}{4\pi}\int_{-\pi}^{\pi} \left(\frac{\partial \ln f(\lambda)}{\partial \sigma_u^2}\right)^2 d\lambda \\
&= \frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{1}{\sigma_u^4} d\lambda \\
&= \frac{1}{4\pi\sigma_u^4}\int_{-\pi}^{\pi} d\lambda \\
&= \frac{2\pi}{4\pi\sigma_u^4} \\
&= \frac{1}{2\sigma_u^4}.
\end{aligned}$$

Next we compute $W_{12} = W_{21}$ as follows:

$$
\begin{aligned}
W_{12} = W_{21} &= \frac{1}{4\pi} \int_{-\pi}^{\pi} \left(\frac{\partial \ln f(\lambda)}{\partial \phi} \right) \left(\frac{\partial \ln f(\lambda)}{\partial \sigma_b^2} \right) d\lambda \\
&= \frac{1}{4\pi} \int_{-\pi}^{\pi} \left(\frac{2\phi\sigma_b^2}{(1-\phi^2)(1-\phi^2-\sigma_b^2)} \right) \left(\frac{1}{1-\phi^2-\sigma_b^2} \right) d\lambda \\
&+ \frac{1}{4\pi} \int_{-\pi}^{\pi} \left(\frac{2(\cos\lambda-\phi)}{1-2\phi\cos\lambda+\phi^2} \right) \left(\frac{1}{1-\phi^2-\sigma_b^2} \right) d\lambda \\
&= \frac{\phi\sigma_b^2}{(1-\phi^2)(1-\phi^2-\sigma_b^2)^2}.
\end{aligned}
$$

Similarly, $W_{13} = W_{31}$ is computed as

$$
\begin{aligned}
W_{13} = W_{31} &= \frac{1}{4\pi} \int_{-\pi}^{\pi} \left(\frac{\partial \ln f(\lambda)}{\partial \phi} \right) \left(\frac{\partial \ln f(\lambda)}{\partial \sigma_u^2} \right) d\lambda \\
&= \frac{1}{4\pi} \int_{-\pi}^{\pi} \left(\frac{2\phi\sigma_b^2}{(1-\phi^2)(1-\phi^2-\sigma_b^2)} \right) \left(\frac{1}{\sigma_u^2} \right) d\lambda \\
&+ \frac{1}{4\pi} \int_{-\pi}^{\pi} \left(\frac{2(\cos\lambda-\phi)}{1-2\phi\cos\lambda+\phi^2} \right) \left(\frac{1}{\sigma_u^2} \right) d\lambda \\
&= \frac{\phi\sigma_b^2}{\sigma_u^2(1-\phi^2)(1-\phi^2-\sigma_b^2)}.
\end{aligned}
$$

Finally, $W_{23} = W_{32}$ is computed as follows:

$$
\begin{aligned}
W_{23} = W_{32} &= \frac{1}{4\pi} \int_{-\pi}^{\pi} \left(\frac{\partial \ln f(\lambda)}{\partial \sigma_b^2} \right) \left(\frac{\partial \ln f(\lambda)}{\partial \sigma_u^2} \right) d\lambda \\
&= \frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{1}{\sigma_u^2(1-\phi^2-\sigma_b^2)} d\lambda \\
&= \frac{1}{2\sigma_u^2(1-\phi^2-\sigma_b^2)}.
\end{aligned}
$$

Upon substituting the expressions of W_{11}, W_{22}, W_{33}, W_{12}, W_{21}, W_{13}, W_{31}, W_{23}, W_{32} into (3) we obtain (2). $\square$

Remark. If we assume that the parameters σ_b^2 and σ_u^2 are known values, then Whittle's information associated with ϕ for the RCA(1) model is

$$
n \left[\frac{2\phi^2\sigma_b^4}{(1-\phi^2)^2(1-\phi^2-\sigma_b^2)^2} + \frac{1}{(1-\phi^2)} \right], \tag{5}
$$

while it is important to note that the information associated with ϕ in the AR(1) model is

$$
n \left[\frac{1}{(1-\phi^2)} \right]. \tag{6}
$$

Some numerical calculations for the information associated with ϕ using equations (5) and (6) are listed down in Table 1.

TABLE 1. Numerical values for the Information associated with ϕ for the RCA(1) and for the AR(1) models

| ϕ | $\sigma_b^2 = 0.3$ | | $\sigma_b^2 = 0.6$ | |
| | RCA(1) | AR(1) | RCA(1) | AR(1) |
	Equation (5)	Equation (6)	Equation (5)	Equation (6)
-0.6	293.1	156.3	39707.0	156.3
-0.5	172.8	133.3	1555.6	133.3
-0.4	133.0	119.0	402.5	119.0
-0.3	115.1	109.9	191.3	109.9
-0.2	106.0	104.2	128.3	104.2
-0.1	101.4	101.0	105.8	101.0
0.0	100.0	100.0	100.0	100.0
0.1	101.4	101.0	105.8	101.0
0.2	106.0	104.2	128.3	104.2
0.3	115.1	109.9	191.3	109.9
0.4	113.0	119.0	402.5	119.0
0.5	172.8	133.3	1555.6	133.3
0.6	293.1	156.3	39707.0	156.3

Clearly we can see that the Whittle information for the RCA(1) model is larger than the AR(1) model when $\sigma_b^2 \neq 0$. Of course when $\sigma_b^2 = 0$, the RCA(1) model reduces to the standard AR(1) model. Since the information is larger for the RCA(1) model, this would give rise to estimators with a smaller variance.

3. Conclusion

The objective of this paper was to study the Whittle estimates for the RCA(1) model. It was shown that the Whittle information of the autoregressive parameter in an RCA(1) model is larger than the corresponding information in an AR(1) model. The implication of this study is that since the information is larger for the RCA(1) model, this would give rise to estimators with a smaller variance, and hence more accurate estimates.

Acknowledgement

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