ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 19, Number 1, June 2015
Available online at http://acutm.math.ut.ee

Testing equality of scale parameters of two Weibull distributions in the presence of unequal shape parameters

Khurshid Alam and Sudhir Paul

Abstract. Data in the form of survival times arise in many fields of studies such as engineering, manufacturing, aeronautics and bio-medical sciences. A popular model for survival data is the two parameter Weibull distribution. Often lifetime or survival time data that are collected in the form of two independent samples are assumed to have come from two independent Weibull populations with different shape and scale parameters. In such a situation it may be of interest to test the equality of the scale parameters with the shape parameters being unspecified. This is equivalent to testing the equality of the location parameters with the shape parameters being unspecified in two extreme value distributions. Also, this is analogous to the traditional Behrens–Fisher problem of testing the equality of the means μ_1 and μ_2 of two normal populations where the variances σ_1^2 and σ_2^2 are unknown. We develop four test procedures, namely, a likelihood ratio test, a $C(\alpha)$ test based on the maximum likelihood estimates of the nuisance parameters, a $C(\alpha)$ test based on the method of moments estimates of the nuisance parameters by Cran (1988), and a $C(\alpha)$ test based on the method of moments estimates of the nuisance parameters by Teimouri and Gupta (2013). These test statistics are then compared, in terms of empirical size and power, using a simulation study.

1. Introduction

The Weibull distribution has a long history in describing data in the form of survival times since its initiation by the Swedish physicist Waloddi Weibull and is one of the most popular distributions in survival analysis. This distribution has been considered as an appropriate model in reliability studies and

Received October 14, 2014.
2010 *Mathematics Subject Classification.* 62F03.
Key words and phrases. $C(\alpha)$ test, maximum likelihood estimate, method of moments estimate, scale parameter, score test, Weibull model.
http://dx.doi.org/10.12697/ACUTM.2015.19.02

life-testing experiments and thus has versatile use in the fields such as engineering, manufacturing, aeronautics and bio-medical sciences. For recent review see Murthy et al. [9].

Let Y be a random variable that follows a two parameter Weibull distribution with shape parameter β and scale parameter α. Then the probability density function of Y can be written as

$$f(y) = \frac{\beta}{\alpha}\left(\frac{y}{\alpha}\right)^{(\beta-1)} \exp\left[-\left(\frac{y}{\alpha}\right)^{\beta}\right], \quad y \geq 0, \;\; \beta, \alpha > 0. \tag{1.1}$$

Often lifetime or survival time data that are collected in the form of two samples are assumed to have come from two independent Weibull populations with different shape and scale parameters. In such a situation it may be of interest to test the equality of scale parameters with the shape parameters being unspecified. For example, Pike [14] gives the times from insult with the carcinogen DMBA to mortality from vaginal cancer in two groups of rats. Two groups were distinguished by a pretreatment regime and the data are reproduced in Kalbfleisch and Prentice [5]. For details of analysis see Pike [14].

However, the problem of testing the equality of scale parameters with the shape parameters being unspecified is equivalent to testing the equality of location parameters with the shape parameters being unspecified in two extreme value distributions. Also, this is analogous to the traditional Behrens–Fisher problem of testing the equality of the means μ_1 and μ_2 of two normal populations where the variances σ_1^2 and σ_2^2 are unknown.

Let $y_{11}, y_{12}, \cdots, y_{1n_1}$ and $y_{21}, y_{22}, \cdots, y_{2n_2}$ be samples from two independent Weibull populations with parameters (α_1, β_1) and (α_2, β_2) respectively. As a motivational example we refer to the data from Lawless [6] on times to fatigue failure in units of millions of cycles of 10 high-speed turbine engine bearings made out of five different compounds. The data are reproduced in Table 7 and the maximum likelihood estimates of the parameters (α, β) for the five compounds are $(12.0607, 2.5881)$, $(6.8596, 2.3202)$, $(9.6847, 3.1324)$, $(7.5107, 4.0912)$, and $(16.3507, 3.6518)$. Obviously, the α as well as the β parameters differ from compound to compound. So, it will be of interest to do a pairwise comparison of the α parameters where the β may be different.

Thus, our objective is to test the null hypothesis $H_0 : \alpha_1 = \alpha_2$, where β_1 and β_2 are unspecified. We develop four test procedures, namely, a likelihood ratio test, a $C(\alpha)$ test based on the maximum likelihood estimates of the nuisance parameters, a $C(\alpha)$ test based on the method of moments estimates of the nuisance parameters by Cran [4], and a $C(\alpha)$ test based on the method of moments estimates of the nuisance parameters by Teimouri and Gupta [16]. These test statistics are then compared, in terms of empirical size and power, using a simulation study.

Paul and Islam [13] give a brief description on the construction mechanism as well as the advantages of the $C(\alpha)$ or score tests. The $C(\alpha)$ test is constructed by regressing the residuals of the score function for the parameter(s) of interest on the score function for the nuisance parameters. Afterwards, the nuisance parameters are replaced by $\sqrt{n}$ (where n = number of observations used in estimating the parameters) consistent estimates. The maximum likelihood estimates (MLEs) are $\sqrt{n}$ consistent and once the nuisance parameters are replaced by their MLEs then the $C(\alpha)$ statistic reduces to the score statistic (see Rao [15]). Many authors have shown (see, for example, Moran [8] and Cox and Hinkley [3]) that the $C(\alpha)$ or score test is asymptotically equivalent to the likelihood ratio test and to tests using the MLEs (i. e., Wald tests). Some of the worth mentioning advantages of $C(\alpha)$ or score tests are: (i) it often maintains, at least approximately, a preassigned level of significance, say (α) (see Bartoo and Puri [1]), (ii) it requires estimates of the parameters only under the null hypothesis, and (iii) it often produces a statistic that is simple to calculate. For more details on the choice of score tests see Breslow [2].

The estimates of the parameters as needed are given in Section 2 and the tests are developed in Section 3. A simulation study is conducted in Section 4. A data set is analyzed in Section 5 and a discussion follows in Section 6.

2. Estimates of the parameters

2.1. The maximum likelihood estimates. The log-likelihood under the alternative hypothesis, apart from a constant, can be written as

$$l_1 = \sum_{i=1}^{2} \left[n_i \log\left(\frac{\beta_i}{\alpha_i}\right) + (\beta_i - 1)\left\{ \sum_{j=1}^{n_i} \log(y_{ij}) - n_i \log(\alpha_i) \right\} - \frac{\sum_{j=1}^{n_i} y_{ij}^{\beta_i}}{\alpha_i^{\beta_i}} \right].$$

The maximum likelihood estimates of the parameters α_i and β_i are obtained by solving the following estimating equations obtained from l_1

$$-\frac{n_i \beta_i}{\alpha_i} + \frac{\beta_i}{\alpha_i^{\beta_i + 1}} \sum_{j=1}^{n_i} y_{ij}^{\beta_i} = 0$$

and

$$\frac{n_i}{\beta_i} + \sum_{j=1}^{n_i} \log(y_{ij}) - n_i \log(\alpha_i) + \frac{\log(\alpha_i)}{\alpha_i^{\beta_i}} \sum_{j=1}^{n_i} y_{ij}^{\beta_i} - \frac{1}{\alpha_i^{\beta_i}} \sum_{j=1}^{n_i} y_{ij}^{\beta_i} \log(y_{ij}) = 0$$

4

simultaneously. The log-likelihood under the null hypothesis is

$$l_0 = \sum_{i=1}^{2} \left[n_i \log\left(\frac{\beta_i}{\alpha}\right) + (\beta_i - 1)\left\{ \sum_{j=1}^{n_i} \log(y_{ij}) - n_i \log(\alpha) \right\} - \frac{\sum_{j=1}^{n_i} y_{ij}^{\beta_i}}{\alpha^{\beta_i}} \right]$$

and the maximum likelihood estimates of the parameters α, β_1 and β_2 are obtained by solving the following estimating equations

$$\sum_{i=1}^{2} \left[-\frac{n_i \beta_i}{\alpha} + \frac{\beta_i}{\alpha^{\beta_i+1}} \sum_{j=1}^{n_i} y_{ij}^{\beta_i} \right] = 0,$$

$$\frac{n_1}{\beta_1} + \sum_{j=1}^{n_1} \log(y_{1j}) - n_1 \log(\alpha) + \frac{\log(\alpha)}{\alpha^{\beta_1}} \sum_{j=1}^{n_1} y_{1j}^{\beta_1} - \frac{1}{\alpha^{\beta_1}} \sum_{j=1}^{n_1} y_{1j}^{\beta_1} \log(y_{1j}) = 0$$

and

$$\frac{n_2}{\beta_2} + \sum_{j=1}^{n_2} \log(y_{2j}) - n_2 \log(\alpha) + \frac{\log(\alpha)}{\alpha^{\beta_2}} \sum_{j=1}^{n_2} y_{2j}^{\beta_2} - \frac{1}{\alpha^{\beta_2}} \sum_{j=1}^{n_2} y_{2j}^{\beta_2} \log(y_{2j}) = 0$$

simultaneously. Denote the maximum likelihood estimates of $\delta = (\alpha,\ \beta_1,\ \beta_2)$ by $\hat{\delta}_{ml} = \left(\hat{\alpha}_{ml},\ \hat{\beta}_{1ml},\ \hat{\beta}_{2ml}\right)$.

2.2. The method of moments estimates by Cran. Cran [4] proposes moments estimates of the parameters for the three-parameter Weibull distribution and applies this procedure for the two-parameter model considering the location parameter as zero. Following Cran [4] the estimates of the parameters α_i and β_i, under the alternative hypothesis, are

$$\hat{\alpha}_{ic} = \frac{\bar{m}_1}{\Gamma\left(1 + \frac{1}{\hat{\beta}_{ic}}\right)} \text{ and } \hat{\beta}_{ic} = \frac{ln\,(2)}{ln\,(\bar{m}_1) - ln\,(\bar{m}_2)},$$

where

$$\bar{m}_k = \sum_{r=0}^{n-1} \left(1 - \frac{r}{n}\right)^k \left\{ y_{(r+1)} - y_{(r)} \right\} \text{ with } y_{(0)} = 0$$

and $y_{(r)}$ is the r^{th} ordered observation.

Note that the estimate of β_i is independent of α_i, so, it should be the same under the null and the alternative hypotheses. As a moment estimate of the common value α of α_1 and α_2 under the null hypothesis we use a weighted average of $\hat{\alpha}_{ic}$ as

$$\hat{\alpha}_c = \sum_{i=1}^{2} w_i \hat{\alpha}_{ic} / \sum_{i=1}^{2} w_i,$$

where

$$w_i = \frac{n_i}{\hat{V}_{ic}}, \quad i = 1,\, 2, \quad \hat{V}_{ic} = \hat{\alpha}_{ic}^2 \left[\Gamma\left(1 + \frac{2}{\hat{\beta}_{ic}}\right) - \left\{ \Gamma\left(1 + \frac{1}{\hat{\beta}_{ic}}\right) \right\}^2 \right],$$

and V_{ic} is the variance of a random variable from the Weibull $(\alpha_i,\, \beta_i)$ population (see Lawless [6]). Denote these method of moments estimates by $\hat{\delta}_{cr} = \left(\hat{\alpha}_c,\, \hat{\beta}_{1c},\, \hat{\beta}_{2c}\right)$.

2.3. The method of moments estimates by Teimouri and Gupta. In a recent article Teimouri and Gupta [16] propose a method of moment estimate of the shape parameter of a three-parameter Weibull distribution and apply this method to a two-parameter Weibull distribution for estimating the shape parameter of a two-parameter Weibull distribution. As the estimate by Cran [4] this estimate is also independent of the estimate of the scale parameter α (see Cran [4] for details). Following Teimouri and Gupta [16] the moment estimate of β_i is

$$\hat{\beta}_{itg} = \frac{-ln2}{ln\left[1 - \frac{r_i}{\sqrt{3}} CV_i \sqrt{\frac{n_i + 1}{n_i - 1}}\right]},$$

where r_i and CV_i denote the i^{th} sample correlation coefficient between the observations and their ranks and the coefficient of variation respectively. Using this estimate of $\hat{\beta}_{itg}$, the estimate of $\hat{\alpha}_{itg}$ is

$$\hat{\alpha}_{itg} = \frac{\bar{m}_1}{\Gamma\left(1 + \frac{1}{\hat{\beta}_{itg}}\right)}.$$

As in Section 2.2 we estimate the common value α of α_1 and α_2 under the null hypothesis as a weighted average of $\hat{\alpha}_{itg}$ but using $\hat{\beta}_{itg}$ instead of $\hat{\beta}_{ic}$ as

$$\hat{\alpha}_{tg} = \sum_{i=1}^{2} w_i \hat{\alpha}_{itg} / \sum_{i=1}^{2} w_i, \text{ where } w_i = \frac{n_i}{\hat{V}_{itg}}, \quad i = 1,\, 2,$$

$$\hat{V}_{itg} = \hat{\alpha}_{itg}^2 \left[\Gamma\left(1 + \frac{2}{\hat{\beta}_{itg}}\right) - \left\{ \Gamma\left(1 + \frac{1}{\hat{\beta}_{itg}}\right) \right\}^2 \right].$$

Denote these method of moments estimates by $\hat{\delta}_{tg} = \left(\hat{\alpha}_{tg},\, \hat{\beta}_{1tg},\, \hat{\beta}_{2tg}\right)$.

3. The tests

3.1. The likelihood ratio test. Let $\hat{l}_1$ and $\hat{l}_0$ be the maximized log-likelihood under the alternative and the null hypothesis respectively. Then the likelihood ratio test statistic is $LR = 2\left(\hat{l}_1 - \hat{l}_0\right)$; which, under the null hypothesis, follows a χ^2 distribution with 1 degree of freedom.

3.2. The $C\,(\alpha)$ tests. Suppose the alternative hypothesis is represented by $\alpha_i = \alpha + \phi_i$, $i = 1, 2$, with $\phi_2 = 0$. Then the null hypothesis $H_0 : \alpha_1 = \alpha_2$ can equivalently be written as $H_0 : \phi_1 = 0$ with α, β_1 and β_2 treated as nuisance parameters. With this reparameterization, the log-likelihood can then be written as

$$l = \sum_{i=1}^{2} \left[n_i \log\left(\frac{\beta_i}{\alpha + \phi_i}\right) + (\beta_i - 1)\left\{\sum_{j=1}^{n_i} \log\left(y_{ij}\right) - n_i \log\left(\alpha + \phi_i\right)\right\} - \frac{1}{(\alpha + \phi_i)^{\beta_i}} \sum_{j=1}^{n_i} y_{ij}^{\beta_i} \right]. \tag{3.1}$$

Now, let $\phi = \phi_1$ and $\delta = (\alpha, \beta_1, \beta_2)'$ and define

$$\psi = \frac{\partial l}{\partial \phi}\Big|_{\phi=0}, \ \gamma_1 = \frac{\partial l}{\partial \alpha}\Big|_{\phi=0}, \ \gamma_2 = \frac{\partial l}{\partial \beta_1}\Big|_{\phi=0}, \ \text{and } \gamma_3 = \frac{\partial l}{\partial \beta_2}\Big|_{\phi=0}.$$

Then the $C(\alpha)$ statistic is based on the adjusted score $S(\delta) = \psi - a_1\gamma_1 - a_2\gamma_2 - a_3\gamma_3$, where a_1, a_2, and a_3 are partial regression coefficient of ψ on γ_1, ψ on γ_2, and ψ on γ_3 respectively. The variance-covariance of $S\,(\delta)$ is $D - AB^{-1}A'$ and the regression coefficients $a = (a_1, a_2, a_3) = AB^{-1}$, where D is 1×1, A is 1×3 and B is 3×3 with elements $D = E\left[-\frac{\partial^2 l}{\partial \phi^2}\Big|_{\phi=0}\right]$, $A_1 = E\left[-\frac{\partial^2 l}{\partial \phi \partial \alpha}\Big|_{\phi=0}\right]$, $A_2 = E\left[-\frac{\partial^2 l}{\partial \phi \partial \beta_1}\Big|_{\phi=0}\right]$, $A_3 = E\left[-\frac{\partial^2 l}{\partial \phi \partial \beta_2}\Big|_{\phi=0}\right]$, $B_{11} = E\left[-\frac{\partial^2 l}{\partial \alpha^2}\Big|_{\phi=0}\right]$, $B_{12} = B_{21} = E\left[-\frac{\partial^2 l}{\partial \alpha \partial \beta_1}\Big|_{\phi=0}\right]$, $B_{13} = B_{31} = E\left[-\frac{\partial^2 l}{\partial \alpha \partial \beta_2}\Big|_{\phi=0}\right]$, $B_{22} = E\left[-\frac{\partial^2 l}{\partial \beta_1^2}\Big|_{\phi=0}\right]$, $B_{23} = B_{32} = E\left[-\frac{\partial^2 l}{\partial \beta_1 \partial \beta_2}\Big|_{\phi=0}\right]$ and $B_{33} = E\left[-\frac{\partial^2 l}{\partial \beta_2^2}\Big|_{\phi=0}\right]$.

Derivation of the above elements based on the Weibull log-likelihood (3.1) are given in the Appendix.

Substituting $\sqrt{n}$ (where $n = n_1 + n_2$) consistent estimate of δ in S, D, A and B, the $C(\alpha)$ statistic can be obtained as

$$C = S^2 \,/\, \left(D - AB^{-1}A'\right), \tag{3.2}$$

which is approximately distributed as a chi-squared with 1 degree of freedom (see Neyman [10], Neyman and Scott [12], Moran [8], and Neyman [11]).

If the maximum likelihood estimate $\hat{\delta}_{ml}$ of δ is used then the maximized scores γ_1, γ_2, and γ_3 are all zero and hence $S = \psi$, and the $C(\alpha)$ statistic reduces to a score statistic (see Rao [15])

$$C_{ml} = \psi^2 \,/\, (D - AB^{-1}A'). \tag{3.3}$$

Further, two $C(\alpha)$ statistics are obtained from equation (3.2) by using $\hat{\delta}_{cr}$ and $\hat{\delta}_{tg}$ in all the expressions of S, D, A and B. Denote these $C(\alpha)$ statistics by C_{cr} and C_{tg} respectively. Each of the statistics C_{ml}, C_{cr} and C_{tg} is approximately distributed as a chi-squared with 1 degree of freedom.

4. Simulation study

We conduct a simulation study to compare the performance of the test procedures, namely, LR, C_{ml}, C_{cr} and C_{tg} that were developed in Section 3. The performance of the test procedures are compared on the basis of empirical level and power. To compare the statistics in terms of empirical level we considered sample sizes $n_1 = n_2 = 5$, 10, 20, 50, values of scale parameter $\alpha_1 = \alpha_2 = 5$, 10, 15, the values of first shape parameter $\beta_1 =$ 3, 6, 10, and the values of second shape parameter $\beta_2 = \beta + \beta_1$ with $\beta = 0.00, 0.50, 1.00, 1.50, 2.00, 2.50, 3.00$. The results are given in Table 1 and Table 2 which are summarized in what follows.

Performance of score statistic is the worst in the sense that it shows most conservative behaviour, ever for sample size as large as $n_1 = n_2 = 50$. The best overall performance is of the $C(\alpha)$ statistic C_{cr}. Even for sample size as small as $n_1 = n_2 = 5$ the level never drops below 4.1%.

To compare power performance of the four statistics we considered the same sample sizes as for the study of performance of the statistics in terms of empirical level. The combinations of $(\beta_1,\ \beta_2)$ considered were $(\beta_1,\ \beta_2)$ = (3, 4), (5, 8). Further, we considered $\alpha_2 = \alpha_1 + \alpha$, with $\alpha_1 = 5, 10, 15$ and $\alpha = 0.00, 0.50, 1.00, 1.50, 2.00, 2.50, 3.00, 4.00, 5.00$. Except for the fact that as sample size increases the power of all the statistics increases, the comparative performance for $n_1 = n_2 = 5$, $n_1 = n_2 = 10$, $n_1 = n_2 = 20$, and $n_1 = n_2 = 50$ are similar. So, we present the power results in Table 3 for $n_1 = n_2 = 5$ and in Table 4 for $n_1 = n_2 = 50$.

From Table 3 and Table 4 we see that power performance of the two $C(\alpha)$ statistics C_{cr} and C_{tg} are similar and best overall, although the former has some edge over the latter. Power performance of the score test statistic C_{ml} is the worst, as expected, as its level is the lowest.

5

5. Examples

Lawless [6] presents a set of data (originally given by McCool [7]) that represent the times to fatigue failure in units of millions of cycles of 10 high-speed turbine engine bearings made out of five different compounds. The data are given in Table 7. We conduct a pairwise comparison of the five different compound types. The maximum likelihood estimates of parameters, under both alternative and null hypotheses, and the methods of moments estimates are presented in Table 5 and the values of the test statistics along with the corresponding p-values are given in Table 6.

Out of the 10 pairwise comparisons, conclusion of whether to reject or not to reject the hypothesis of equality of the scale parameters is the same for six pairs, namely the pairs (I, II), (I, III), (I, IV), (II, V), (III, IV), and (III, IV) by all four methods. For three of the remaining four pairs, namely, the pairs (I, V), (II, III), and (IV, V), the statistic C_{cr} rejects the null hypothesis of equality of the scale parameters at 5% level of significance, whereas, this hypothesis is not rejected by the other three statistics. For the remaining pair (II, IV) both the statistics C_{cr} and C_{tg} reject the null hypothesis, whereas the other two statistics LR and C_{ml} do not reject the null hypothesis. Further, rejection by the the statistic C_{cr} is stronger (p-value is 0.0005) than by the statistic C_{tg} (p-value is 0.0011). The analysis here agree with the finding in the simulation study that the statistic C_{cr} is likely to be most powerful among the four statistics studies.

6. Discussion

In this section we dealt with the survival data that follow Weibull distribution and we developed four test procedures to test the equality of scale parameters of two Weibull distributions where the shape parameters are assumed unknown and unequal. We developed four test procedures, namely, a likelihood ratio statistic LR, a $C(\alpha)$ (score) statistic based on maximum likelihood estimates of the nuisance parameters C_{ml}, a $C(\alpha)$ statistic based on method of moments estimates of the nuisance parameters by Cran [4] C_{cr}, and a $C(\alpha)$ statistic based on method of moments estimates of the nuisance parameters by Teimouri and Gupta [16] C_{tg}.

A simulation study in terms of empirical level show the best overall performance of the $C(\alpha)$ statistic C_{cr}. Even for sample size as small as $n_1 = n_2 = 5$ the level never drops below 4.1%. Performance of score statistic is the worst in the sense that it shows most conservative behaviour, ever for sample size as large as $n_1 = n_2 = 50$. Further simulations show that power performance of the two $C(\alpha)$ statistics C_{cr} and C_{tg} are similar and best overall, although the former has some edge over the latter. Power performance of the score test statistic C_{ml} is the worst which is expected, as its level is the lowest.

The advantage of the $C(\alpha)$ or the score test is discussed in the introduction. Further, based on the fact that a $C(\alpha)$ statistic, such as C_{cr}, performs best overall, it would be of interest to explore whether the $C(\alpha)$ or the score test can be used to analyze two groups of survival data following a three-parameter distribution such as the three-parameter Weibull distribution discussed by Teimouri and Gupta [16]. This problem will be investigated in a future paper.

Appendix

A. Derivation of the elements of S, D, A, and B of the $C(\alpha)$ statistic based on the Weibull likelihood (3.1).

After detailed calculation we obtain

$$\psi = -\frac{n_1\beta_1}{\alpha} + \frac{\beta_1}{\alpha^{\beta_1+1}}\sum_{j=1}^{n_1} y_{1j}^{\beta_1},\ \ \gamma_1 = \sum_{i=1}^{2}\left[-\frac{n_i\beta_i}{\alpha} + \frac{\beta_i}{\alpha^{\beta_i+1}}\sum_{j=1}^{n_i} y_{ij}^{\beta_i}\right],$$

$$\gamma_2 = \frac{n_1}{\beta_1} + \sum_{j=1}^{n_{1j}} \log(y_{1j}) - n_1\log(\alpha) + \frac{\log(\alpha)}{\alpha^{\beta_1}}\sum_{j=1}^{n_1} y_{1j}^{\beta_1} - \frac{1}{\alpha^{\beta_1}}\sum_{j=1}^{n_1} y_{1j}^{\beta_1}\log(y_{1j}),$$

$$\gamma_3 = \frac{n_2}{\beta_2} + \sum_{j=1}^{n_{2j}} \log(y_{2j}) - n_2\log(\alpha) + \frac{\log(\alpha)}{\alpha^{\beta_2}}\sum_{j=1}^{n_2} y_{2j}^{\beta_2} - \frac{1}{\alpha^{\beta_2}}\sum_{j=1}^{n_2} y_{2j}^{\beta_2}\log(y_{2j}),$$

$$D = -\frac{n_1\beta_1}{\alpha^2} + \frac{n_1\beta_1(\beta_1+1)E\left(y_{1j}^{\beta_1}\right)}{\alpha^{\beta_1+2}},\ \ A_1 = -\frac{n_1\beta_1}{\alpha^2} + \frac{n_1\beta_1(\beta_1+1)E\left(y_{1j}^{\beta_1}\right)}{\alpha^{\beta_1+2}},$$

$$A_2 = \frac{n_1}{\alpha} - \frac{n_1\{1-\beta_1\log(\alpha)\}E\left(y_{1j}^{\beta_1}\right)}{\alpha^{\beta_1+1}} - \frac{n_1\beta_1 E\left\{y_{1j}^{\beta_1}\log(y_{1j})\right\}}{\alpha^{\beta_1+1}},\ \ A_3 = 0,$$

$$B_{11} = \sum_{j=1}^{n_i}\left[-\frac{n_i\beta_i}{\alpha^2} + \frac{n_i\beta_i(\beta_i+1)E\left(y_{ij}^{\beta_i}\right)}{\alpha^{\beta_i+2}}\right],$$

$$B_{12} = B_{21} = \frac{n_1}{\alpha} - \frac{n_1\{1-\beta_1\log(\alpha)\}E\left(y_{1j}^{\beta_1}\right)}{\alpha^{\beta_1+1}} - \frac{n_1\beta_1 E\left\{y_{1j}^{\beta_1}\log(y_{1j})\right\}}{\alpha^{\beta_1+1}},$$

$$B_{13} = B_{31} = \frac{n_2}{\alpha} - \frac{n_2\{1-\beta_2\log(\alpha)\}E\left(y_{2j}^{\beta_2}\right)}{\alpha^{\beta_2+1}} - \frac{n_2\beta_2 E\left\{y_{2j}^{\beta_2}\log(y_{2j})\right\}}{\alpha^{\beta_2+1}},$$

$$B_{22} = \frac{n_1}{\beta_1^2} + \frac{n_1\{\log(\alpha)\}^2 E\left(y_{1j}^{\beta_1}\right) + n_1 E\left\{y_{1j}^{\beta_1}(\log(y_{1j}))^2\right\}}{\alpha^{\beta_1}},\ \ B_{23} = B_{32} = 0,$$

$$B_{33} = \frac{n_2}{\beta_2^2} + \frac{n_2\{\log(\alpha)\}^2 E\left(y_{2j}^{\beta_2}\right) + n_2 E\left\{y_{2j}^{\beta_2}(\log(y_{2j}))^2\right\}}{\alpha^{\beta_2}}.$$

These expressions are then evaluated at $\alpha = \hat{\alpha}$, $\beta_1 = \hat{\beta}_1$ and $\beta_2 = \hat{\beta}_2$, where, for example, $\hat{\alpha}$ is either $\hat{\alpha}_{ml}$, $\hat{\alpha}_{cr}$ and $\hat{\alpha}_{tg}$ in C_{ml}, C_{cr} and C_{tg} respectively. Expectation of a function $f(y, \hat{\alpha}, \hat{\beta}_1, \hat{\beta}_2)$ of a Weibull random variable y is calculated as $\int_0^\infty f(y, \hat{\alpha}, \hat{\beta_1}, \hat{\beta_2}) f(y) dy$, where $f(y)$ is as given in equation (1.1).

B. The tables.

TABLE 1. Empirical level (%) of the test statistics LR, C_{ml}, C_{cr}, and C_{tg} for $\alpha_1 = \alpha_2 = \alpha$ and $\beta_2 = \beta + \beta_1$; based on 5000 iterations, and nominal level = 0.05.

n_1, n_2	Statistics	(α, β_1)	β						
		(5, 3)	0.00	0.50	1.00	1.50	2.00	2.50	3.00
14*5, 5	LR		3.2	3.3	3.3	3.8	3.9	3.6	3.5
	C_{ml}		2.9	3.1	3.2	3.4	3.6	3.3	3.1
	C_{cr}		4.1	4.4	4.3	4.2	4.5	4.8	4.4
	C_{tg}		3.8	3.7	3.9	4.1	4.5	4.2	4.0
		(10, 6)							
	LR		3.2	3.5	3.7	4.1	4.2	4.1	3.9
	C_{ml}		3.0	3.1	3.4	3.4	3.8	3.6	3.3
	C_{cr}		4.3	4.6	4.9	5.1	4.9	5.2	4.8
	C_{tg}		3.7	4.1	4.4	4.7	4.5	4.8	4.4
		(15, 10)							
	LR		3.4	3.8	4.1	4.2	4.3	4.5	4.4
	C_{ml}		3.2	3.5	3.5	3.8	4.1	3.8	3.6
	C_{cr}		4.5	4.7	4.8	5.0	5.4	5.2	5.4
	C_{tg}		3.8	4.1	4.4	4.5	4.8	4.6	4.8
		(5, 3)	0.00	0.50	1.00	1.50	2.00	2.50	3.00
14*10, 10	LR		3.6	3.8	4.2	4.4	4.4	4.4	4.5
	C_{ml}		3.2	3.3	3.6	3.8	4.0	3.9	3.8
	C_{cr}		4.1	4.6	5.1	5.1	5.2	5.1	5.4
	C_{tg}		4.0	4.4	4.7	4.8	5.0	5.1	5.2
		(10, 6)							
	LR		3.8	3.8	4.1	4.4	4.5	4.4	4.3
	C_{ml}		3.4	3.5	4.0	4.0	4.3	4.4	4.1
	C_{cr}		4.5	4.7	5.1	5.0	5.3	5.4	5.0
	C_{tg}		4.1	4.3	4.5	4.7	4.7	5.1	4.8
		(15, 10)							
	LR		3.4	3.9	4.1	4.6	4.6	4.7	4.4
	C_{ml}		3.4	3.7	4.1	4.4	4.5	4.5	4.0
	C_{cr}		4.5	5.2	5.3	5.2	5.3	5.4	5.6
	C_{tg}		4.2	4.7	4.8	5.0	4.9	5.0	5.1

TABLE 2. Empirical level (%) of the test statistics LR, C_{ml}, C_{cr}, and C_{tg} for $\alpha_1 = \alpha_2 = \alpha$ and $\beta_2 = \beta + \beta_1$; based on 5000 iterations, and nominal level = 0.05.

n_1, n_2	Statistics	(α, β_1)	β						
		(5, 3)	0.00	0.50	1.00	1.50	2.00	2.50	3.00
14*20, 20	LR		3.7	3.7	4.3	4.5	4.7	4.7	4.7
	C_{ml}		3.4	3.7	3.7	3.9	4.4	4.0	3.9
	C_{cr}		4.5	5.0	5.2	5.2	5.1	5.4	5.2
	C_{tg}		4.3	4.6	4.9	5.0	5.1	5.3	5.2
		(10, 6)							
	LR		3.8	4.0	4.5	4.4	4.8	4.6	4.6
	C_{ml}		3.6	3.6	3.9	4.1	4.6	4.5	4.4
	C_{cr}		4.7	5.1	5.1	5.0	5.3	5.6	5.4
	C_{tg}		4.2	4.6	5.1	4.8	5.0	5.2	5.0
		(15, 10)							
	LR		3.9	4.2	4.4	4.7	5.1	4.7	4.8
	C_{ml}		3.7	3.9	4.2	4.4	4.7	4.6	4.5
	C_{cr}		4.5	4.8	5.4	5.5	5.5	5.4	5.7
	C_{tg}		4.4	4.6	5.1	5.2	5.4	5.1	5.4
		(5, 3)	0.00	0.50	1.00	1.50	2.00	2.50	3.00
14*50, 50	LR		4.3	4.5	4.8	5.0	4.9	4.8	4.8
	C_{ml}		3.9	4.2	4.2	4.2	4.6	4.4	4.2
	C_{cr}		4.9	5.1	5.5	5.5	5.4	5.3	5.5
	C_{tg}		4.6	4.8	5.0	5.2	5.0	5.3	5.1
		(10, 6)							
	LR		4.3	4.6	4.7	4.7	5.1	4.9	4.9
	C_{ml}		4.1	4.2	4.5	4.6	4.7	4.7	4.6
	C_{cr}		5.1	5.1	5.5	5.4	5.3	5.6	5.6
	C_{tg}		4.8	5.0	5.2	5.1	5.3	5.4	5.2
		(15, 10)							
	LR		4.4	4.7	4.8	5.1	5.2	5.1	5.0
	C_{ml}		4.4	4.5	4.5	4.7	4.7	4.8	4.8
	C_{cr}		5.5	5.7	5.7	5.4	5.5	5.6	5.4
	C_{tg}		5.2	5.2	5.5	5.4	5.5	5.4	5.0

6

TABLE 3. Empirical power (%) of test statistics LR, C_{ml}, C_{cr}, and C_{tg} for $\alpha_2 = \alpha_1 + \alpha$; based on 5000 iterations, $n_1 = n_2 = 5$, and nominal level = 0.05.

2*Statistics	2*$(\alpha_1, \beta_1, \beta_2)$	α								
	(5, 3, 4)	0.00	0.50	1.00	1.50	2.00	2.50	3.00	4.00	5.00
LR		3.1	3.8	6.0	10.6	16.7	24.8	36.8	57.6	81.9
C_{ml}		2.7	3.3	5.4	9.7	15.3	22.7	34.4	54.8	78.5
C_{cr}		4.0	4.8	7.0	11.9	17.5	26.1	38.5	59.4	83.7
C_{tg}		3.6	4.2	6.4	10.8	17.0	24.9	37.6	58.3	82.6
	(5, 5, 8)									
LR		3.2	3.8	6.0	11.2	17.1	25.3	37.1	58.2	82.8
C_{ml}		2.8	3.4	5.5	10.3	15.9	23.1	35.0	55.3	78.8
C_{cr}		4.0	4.7	7.1	12.4	18.3	27.5	39.7	61.0	84.1
C_{tg}		3.7	4.5	6.7	11.7	17.7	26.0	37.9	58.8	83.1
	(10, 3, 4)									
LR		3.1	3.7	5.6	10.3	16.5	24.7	36.3	57.4	81.4
C_{ml}		2.8	3.3	5.3	9.8	15.3	22.3	34.5	54.1	76.2
C_{cr}		4.1	4.7	6.7	11.5	18.0	27.1	38.9	60.6	83.6
C_{tg}		3.5	4.0	6.0	10.7	17.2	25.6	37.2	58.3	82.3
	(10, 5, 8)									
LR		3.1	3.6	5.9	10.8	17.1	25.1	37.1	57.7	83.1
C_{ml}		2.9	3.4	5.5	10.2	15.8	23.0	35.3	55.2	77.5
C_{cr}		4.3	4.9	7.1	12.0	18.6	27.9	39.8	60.9	84.7
C_{tg}		3.6	4.2	6.3	11.1	17.6	26.3	38.7	59.4	83.9
	(15, 3, 4)									
LR		3.2	3.9	5.7	10.5	16.8	24.3	35.9	56.8	82.5
C_{ml}		3.1	3.6	5.4	10.0	15.3	22.4	35.0	54.6	76.4
C_{cr}		4.5	5.1	6.9	11.8	17.8	27.1	38.4	59.3	83.5
C_{tg}		3.6	4.2	6.1	11.0	17.0	25.8	37.8	58.0	82.6
	(15, 5, 8)									
LR		3.5	4.2	6.1	11.2	17.1	24.9	36.7	57.3	82.9
C_{ml}		3.3	3.9	5.7	10.5	15.8	23.7	35.4	55.3	77.1
C_{cr}		4.7	5.2	7.1	12.7	18.3	27.6	39.0	60.1	83.9
C_{tg}		3.6	4.4	6.2	11.9	17.6	26.4	38.3	58.8	83.1

TABLE 4. Empirical power (%) of test statistics LR, C_{ml}, C_{cr}, and C_{tg} for $\alpha_2 = \alpha_1 + \alpha$; based on 5000 iterations, $n_1 = n_2 = 50$, and nominal level = 0.05.

2*Statistics	2*$(\alpha_1, \beta_1, \beta_2)$	α								
	(5, 3, 4)	0.00	0.50	1.00	1.50	2.00	2.50	3.00	4.00	5.00
LR		4.6	5.9	8.0	13.4	23.2	35.0	48.9	75.4	100
C_{ml}		4.1	5.5	7.5	12.7	22.0	33.7	47.1	73.2	100
C_{cr}		5.5	6.7	8.7	14.2	24.7	36.9	52.0	78.4	100
C_{tg}		4.9	6.2	8.1	13.8	23.9	35.8	50.3	77.0	100
	(5, 5, 8)									
LR		4.7	6.1	8.1	13.7	24.5	36.4	50.3	77.2	100
C_{ml}		4.1	5.6	7.5	13.1	23.4	34.3	48.4	75.1	100
C_{cr}		5.4	7.0	9.0	14.6	26.3	38.1	53.4	79.6	100
C_{tg}		5.1	6.5	8.6	14.5	25.8	37.3	52.1	78.6	100
	(10, 3, 4)									
LR		4.6	5.9	7.9	14.7	25.9	38.0	52.6	79.8	100
C_{ml}		4.5	5.8	7.6	13.9	24.1	34.9	49.1	76.4	100
C_{cr}		5.5	6.9	8.7	15.8	29.4	41.1	57.9	84.2	100
C_{tg}		5.1	6.4	8.4	15.3	27.9	39.8	55.5	81.6	100
	(10, 5, 8)									
LR		4.7	6.2	8.1	15.6	26.3	38.6	53.2	80.3	100
C_{ml}		4.5	5.9	7.9	14.3	24.7	35.5	49.7	77.3	100
C_{cr}		5.4	6.9	8.9	17.1	32.7	44.0	61.4	86.3	100
C_{tg}		5.1	6.5	8.4	16.4	29.1	41.4	57.5	83.1	100
	(15, 3, 4)									
LR		4.6	6.1	8.0	14.1	23.6	35.3	48.7	74.8	99.7
C_{ml}		4.4	5.7	7.5	13.6	22.9	33.2	46.4	71.7	98.8
C_{cr}		5.6	7.0	8.8	15.7	26.0	37.7	53.1	78.9	100
C_{tg}		5.2	6.5	8.3	15.0	24.9	36.1	51.7	78.2	100
	(15, 5, 8)									
LR		4.9	6.2	8.0	15.3	25.8	37.9	52.6	79.4	100
C_{ml}		4.8	6.0	7.8	14.0	24.5	35.2	49.0	76.8	100
C_{cr}		5.3	6.6	8.8	17.0	32.4	43.4	60.7	85.6	100
C_{tg}		5.1	6.5	8.5	16.3	28.6	41.0	56.8	82.4	100

Table 5. Estimates of parameters obtained by different methods for compound combinations of bearing specimens data in Table 7

2*Estimates	Compound type combinations				
	(I, II)	(I, III)	(I, IV)	(I, V)	(II, III)
$\hat{\alpha}_0$	9.0056	10.4848	11.7213	14.7887	8.5093
$\hat{\beta}_{10}$	1.8385	2.2491	2.5351	2.4628	2.3276
$\hat{\beta}_{20}$	2.2376	3.2077	1.9758	3.1844	2.6780
$\hat{\alpha}_{1a}$	12.0607	12.0607	12.0607	12.0607	6.8596
$\hat{\alpha}_{2a}$	6.8596	9.6847	7.5107	16.3507	9.6847
$\hat{\beta}_{1a}$	2.5881	2.5881	2.5881	2.5881	2.3202
$\hat{\beta}_{2a}$	2.3202	3.1324	4.0912	3.6518	3.1324
$\hat{\alpha}_{cr}$	7.9472	10.4246	11.4968	13.1814	7.8480
$\hat{\beta}_{1cr}$	2.4941	2.4941	2.4941	2.4941	2.6956
$\hat{\beta}_{2cr}$	2.6956	3.0152	2.5244	3.5348	3.0152
$\hat{\alpha}_{tg}$	7.9686	10.4688	11.5176	14.2777	7.9218
$\hat{\beta}_{1tg}$	2.0733	2.0733	2.0733	2.0733	2.2457
$\hat{\beta}_{2tg}$	2.2457	2.5192	2.0992	2.9636	2.5192
	(II, IV)	(II, V)	(III, IV)	(III, V)	(IV, V)
$\hat{\alpha}_0$	7.1645	9.5075	8.6567	13.8549	15.0676
$\hat{\beta}_{10}$	2.3718	2.1549	2.3160	2.2909	1.9228
$\hat{\beta}_{20}$	1.5713	1.4804	1.3236	2.8340	3.2844
$\hat{\alpha}_{1a}$	6.8596	6.8596	9.6847	9.6847	7.5107
$\hat{\alpha}_{2a}$	7.5107	16.3507	7.5107	16.3507	16.3507
$\hat{\beta}_{1a}$	2.3202	2.3202	3.1324	3.1324	4.0912
$\hat{\beta}_{2a}$	4.0912	3.6518	4.0912	3.6518	3.6518
$\hat{\alpha}_{cr}$	7.8763	8.8612	10.1640	11.7659	13.4134
$\hat{\beta}_{1cr}$	2.6956	2.6956	3.0152	3.0152	2.5244
$\hat{\beta}_{2cr}$	2.5244	3.5348	2.5244	3.5348	3.5348
$\hat{\alpha}_{tg}$	7.8992	8.9299	10.2093	11.8609	13.5077
$\hat{\beta}_{1tg}$	2.2457	2.2457	2.5192	2.5192	2.0992
$\hat{\beta}_{2tg}$	2.0992	2.9636	2.0992	2.9636	2.9636

Table 6. Test statistics along with p-values for compound combinations of bearing specimens data in Table 7

2*Statistics	Compound type combinations				
	(I, II)	(I, III)	(I, IV)	(I, V)	(II, III)
2*LR	7.0443	1.6233	0.5067	3.4073	3.4310
	(0.0080)	(0.2026)	(0.4766)	(0.0649)	(0.0640)
2*C_{ml}	5.4028	1.4351	0.1191	2.6298	2.7254
	(0.0201)	(0.2309)	(0.7300)	(0.1049)	(0.0988)
2*C_{cr}	13.1311	1.9438	0.8346	4.0383	4.5123
	(0.0003)	(0.1633)	(0.3609)	(0.0445)	(0.0337)
2*C_{tg}	11.8042	1.1957	0.0147	2.5652	3.3701
	(0.0006)	(0.2742)	(0.9034)	(0.1092)	(0.0664)
	(II, IV)	(II, V)	(III, IV)	(III, V)	(IV, V)
2*LR	7.4859	18.8333	1.0934	10.1554	2.9559
	(0.0062)	(0.0000)	(0.2957)	(0.0014)	(0.0856)
2*C_{ml}	3.6041	8.5357	0.8114	5.8944	2.6429
	(0.0576)	(0.0035)	(0.3677)	(0.0152)	(0.1040)
2*C_{cr}	12.0014	21.4434	2.3691	16.8222	4.3819
	(0.0005)	(0.0000)	(0.1238)	(0.0000)	(0.0363)
2*C_{tg}	10.6301	98.0519	0.9900	15.7253	3.8369
	(0.0011)	(0.0000)	(0.3198)	(0.0001)	(0.0501)

Table 7. Failure times of different bearing specimens

Type of compound				
I	II	III	IV	V
3.03	3.19	3.46	5.88	6.43
5.53	4.26	5.22	6.74	9.97
5.60	4.47	5.69	6.90	10.39
9.30	4.53	6.54	6.98	13.55
9.92	4.67	9.16	7.21	14.45
12.51	4.69	9.40	8.14	14.72
12.95	5.78	10.19	8.59	16.81
15.21	6.79	10.71	9.80	18.39
16.04	9.37	12.58	12.28	20.84
16.84	12.75	13.41	25.46	21.51

7

Acknowledgements

This research was partially supported by the Natural Sciences and Engineering Research Council of Canada. We thank a referee for comments and suggestions which have improved the paper.

References

[1] J. B. Bartoo and P. S. Puri, *On optimal asymptotic tests of composite statistical hypothesis*, Ann. Math. Statist. **38** (1967), 1845–1852.

[2] N. Breslow, *Tests of hypotheses in over-dispersed Poisson regression and other quasi-likelihood models*, J. Amer. Statist. Assoc. **85** (1990), 565–571.

[3] D. R. Cox and D. V. Hinkley, *Theoretical Statistics*, Chapman and Hall, London, 1974.

[4] G. W. Cran, Moment estimators for the 3-parameter Weibull distribution. IEEE Trans. Reliab. **37**(4) (1988), 360–363.

[5] J. D. Kalbfleisch and R. L. Prentice, *The Statistical Analysis of Failure Time Data*, John Wiley & Sons, Inc., New York, 1980.

[6] J. F. Lawless, *Statistical Models and Methods for Lifetime Data*, John Wiley & Sons, Inc., New York, 1982.

[7] J. I. McCool, *Analysis of single classification experiments based on censored samples from the two-parameter Weibull distribution*, J. Statist. Plann. Inference **3** (1979), 39–68.

[8] P. A. P. Moran, *On asymptotically optimal tests of composite hypotheses*, Biometrika **57** (1970), 45–75.

[9] D. N. P. Murthy, M. Xie, and R. Jiang, *Weibull Models*, John Wiley & Sons, Inc., New Jersey, 2004.

[10] J. Neyman, *Optimal asymptotic tests for composite hypotheses*, in: Probability and Statistics, John Wiley & Sons, Inc., New York, 1959, pp. 213–234.

[11] J. Neyman, $C(\alpha)$ *tests and their use*, Sankhyā Ser. A **41** (1979), 1–21.

[12] J. Neyman and E. L. Scott, *On the use of* $C(\alpha)$ *optimal tests of composite hypotheses*, Bull. Internat. Statist. Inst. **41** (1966), 477–497.

[13] S. R. Paul and A. S. Islam, *Analysis of proportions in the presence of over-/under-dispersion*, Biometrics **51** (1995), 1400–1410.

[14] M. C. Pike, *A method of analysis of a certain class of experiments in carcinogenesis*, Biometrics **22** (1966), 142–161.

[15] C. R. Rao, *Large sample tests of statistical hypotheses concerning several parameters with application to problems of estimation*, Math. Proc. Cambridge Philos. Soc. **44** (1948), 50–57.

[16] M. Teimouri and A. K. Gupta, *On the three-parameter Weibull distribution shape parameter estimation*, J. Data Sci. **11** (2013), 403–414.

Department of Mathematics and Statistics, University of Windsor, 401 Sunset, Windsor, Ont., N9B 3P4, Canada

E-mail address: `alaml@uwindsor.ca`

E-mail address: `smjp@uwindsor.ca`