ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 19, Number 2, December 2015
Available online at http://acutm.math.ut.ee

Spaces of entire functions represented by vector valued Dirichlet series of two complex variables

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Abstract. Let Y be the space of all entire functions $f:\mathbb{C}^2\to E$ defined by the vector valued Dirichlet series, where E is a complex Banach algebra with the unit element. We study various topologies defined on the space Y and characterize continuous linear transformations on Y.

1. Introduction

Let

$$f(s)=\sum_{n=1}^{\infty} a_n e^{s\lambda_n},\quad s=\sigma+it \quad (\sigma, t \text{ are real variables}), \tag{1.1}$$

where a_n $(n\in\mathbb{N})$ are complex numbers and the real sequence $\{\lambda_n\}$ satisfies the conditions: $0<\lambda_1<\lambda_2<\cdots<\lambda_n<\cdots$, $\lambda_n\to\infty$ as $n\to\infty$, and

$$\limsup_{n\to\infty}\frac{\log|a_n|}{\lambda_n}=-\infty\,,$$

$$\limsup_{n\to\infty}\frac{\log\lambda_n}{n}<\infty\,.$$

Then the Dirichlet series (1.1) represents an entire function $f(s)$. Kamthan and Gautam ([3], [4]) defined various norms on this space. They obtained the properties of bases of the space using the growth parameters of entire Dirichlet series. In [1] and [2], S. Daoud studied properties of the space X of entire functions defined by Dirichlet series of two complex variables.

In [5], B. L. Srivastava considered the vector valued Dirichlet series where the coefficients $\{a_n\}$ belong to a complex Banach space. He also defined the growth parameters such as order, type, lower order and lower type of the

Received March 13, 2012.
2010 *Mathematics Subject Classification.* 32A15, 30H05.
Key words and phrases. Dirichlet series, entire functions, linear transformations.
http://dx.doi.org/10.12697/ACUTM.2015.19.06

vector valued entire Dirichlet series. He also obtained coefficient characterizations of order and type.

Let the coefficients $a_{m,n}$ $(m, n = 0, 1, \dots)$ belong to a complex commutative Banach algebra $(E, \|\cdot\|)$ with the unit element ω, and the real sequences $\{\lambda_m\}$ and $\{\mu_n\}$ satisfy the following conditions: $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_m < \cdots$, $\lambda_m \to \infty$ as $m \to \infty$, $0 = \mu_0 < \mu_1 < \cdots < \mu_n < \cdots$, $\mu_n \to \infty$ as $n \to \infty$, and

$$\limsup_{m,n\to\infty} \frac{\ln(m+n)}{\lambda_m + \mu_n} = D < +\infty\,, \tag{1.2}$$

$$\limsup_{m,n\to\infty} \frac{\ln\|a_{m,n}\|}{\lambda_m + \mu_n} = -\infty\,. \tag{1.3}$$

In the following we may assume, without loss of generality, that $\lambda_1, \mu_1 \geqslant 1$.

Let us consider the mapping $f : \mathbb{C}^2 \to E$ defined as

$$f(s_1, s_2) = \sum_{m,n=0}^{\infty} a_{m,n} \exp(\lambda_m s_1 + \mu_n s_2) \ \ (s_j = \sigma_j + it_j,\ \ j = 1, 2)\,. \tag{1.4}$$

Then $f(s_1, s_2)$ is an entire function (see [5]). In [6], the authors introduced two equivalent topologies on the space Y of entire functions (1.4) and obtained some properties of bases in Y. In this paper we prove some additional properties of the space Y. We also give a characterization of certain continuous linear transformations on the space Y.

2. Topologies on the space Y

Let us assume that $\{\sigma_1^{(k)}\}$ and $\{\sigma_2^{(k)}\}$ are two non-decreasing sequences of positive numbers such that $\sigma_1^{(k)} \to \infty$ and $\sigma_2^{(k)} \to \infty$ with $k \to \infty$. For each $f \in Y$ we put (see [6], p. 84)

$$\|f; \sigma_1^{(k)}, \sigma_2^{(k)}\| = \sum_{m,n=0}^{\infty} \|a_{m,n}\| \exp(\lambda_m \sigma_1^{(k)} + \mu_n \sigma_2^{(k)})\,,$$

where $f(s_1, s_2)$ is a vector valued entire function defined by (1.4), and define a metric topology on Y with the metric

$$\rho(f, g) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\|f - g; \sigma_1^{(k)}, \sigma_2^{(k)}\|}{1 + \|f - g; \sigma_1^{(k)}, \sigma_2^{(k)}\|}, \quad f, g \in Y.$$

Another metric topology on Y is determined by the metric (see [6], p. 84)

$$T(f, g) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{M(f - g, \sigma_1^{(j)}, \sigma_2^{(j)})}{1 + M(f - g, \sigma_1^{(j)}, \sigma_2^{(j)})}, \quad f, g \in Y\,,$$

where, for $0 < \sigma_1, \sigma_2 < \infty$,

$$M(f;\sigma_1,\sigma_2) = \sup_{-\infty<t_1,t_2<\infty} \|f(\sigma_1 + it_1, \sigma_2 + it_2)\|.$$

For each $f \in Y$, let us define a function

$$p(f) = \sup\left\{\|a_{0,0}\|, \|a_{m,n}\|^{1/(\lambda_m+\mu_n)}:\ m,n \geqslant 0,\ m+n \neq 0\right\}$$

which is well defined in view of (1.3). The function p satisfies the following properties:

(i) $p(f) = 0 \iff f = 0$,
(ii) $p(-f) = p(f)$,
(iii) $p(f+g) \le p(f) + p(g), \quad f, g \in Y$.

Indeed, (i) and (ii) are obvious. To prove (iii) let $f, g \in Y$, where f is defined by (1.4) and

$$g(s_1,s_2) = \sum_{m,n=0}^{\infty} b_{m,n} \exp(\lambda_m s_1 + \mu_n s_2).$$

Then

$$(f+g)(s_1,s_2) = \sum_{m,n=0}^{\infty} (a_{m,n} + b_{m,n}) \exp(\lambda_m s_1 + \mu_n s_2)$$

and so,

$$p(f+g) = \sup\left\{\|a_{0,0}+b_{0,0}\|,\ \|a_{m,n} + b_{m,n}\|^{1/(\lambda_m+\mu_n)} : m,n \ge 0,\ m+n \neq 0\right\}.$$

Therefore, using the inequality

$$\|a_{m,n} + b_{m,n}\|^{1/(\lambda_m+\mu_n)} \le \|a_{m,n}\|^{1/(\lambda_m+\mu_n)} + \|b_{m,n}\|^{1(\lambda_m+\mu_n)},$$

we get (iii).

Let us put $d(f,g) = p(f-g)$. Then from (i), $d(f,g) = 0$ if and only if $f = g$. The symmetry of d is evident from (ii) and the triangle inequality follows from (iii). Hence $d(f,g)$ defines a metric on Y.

Now we prove our first two results.

Proposition 1. *The three topologies on Y defined, respectively, by ρ, d and T are equivalent.*

Proof. First we consider the topologies defined by the metrics ρ and d. For this we take a sequence $\{f_\beta\} \subset Y$,

$$f_\beta(s_1,s_2) = \sum_{m,n=0}^{\infty} a_{m,n}^{(\beta)} \exp(\lambda_m s_1 + \mu_n s_2), \quad \beta = 1, 2, \ldots,$$

and assume that $f_\beta \to f$ in the metric d, where $f \in Y$ is definad by (1.4). For an arbitrary large number r we have

$$\|a_{0,0}^{(\beta)} - a_{0,0}\| < 1/r, \quad \beta \geq \beta_0(r),$$

and, for $m,n \geq 0,\ m+n \neq 0$,

$$\|a_{m,n}^{(\beta)} - a_{m,n}\|^{1/(\lambda_m+\mu_n)} < 1/r, \quad \beta \geq \beta_0(r)\,.$$

Hence, as in [1, p. 413], for each k we get

$$\begin{aligned}\|f_\beta - f; \sigma_1^{(k)}, \sigma_2^{(k)}\| &< 1/r + \sum_{m,n>0} \exp\left\{(\sigma_1^{(k)} - \log r)\lambda_m + (\sigma_1^{(k)} - \log r)\mu_n\right\} \\ &< 1/r + O(1)\ \exp\{-(\lambda_1 + \mu_1)\log\sqrt{r}\}.\end{aligned}$$

Therefore $f_\beta \to f$ with respect to each norm $||f; \sigma_1^{(k)}, \sigma_2^{(k)}||$. Hence $f_\beta \to f$ in the metric ρ.

Now, suppose that $f_\beta \to f$ in the metric ρ. Then $f_\beta \to f$ with respect to each norm $||f; \sigma_1^{(k)}, \sigma_2^{(k)}||$. Therefore, for a given $\varepsilon > 0$ we can choose k large enough such that $\exp(-\sigma_k) < \varepsilon$, where $\sigma_k = \min(\sigma_1^{(k)}, \sigma_2^{(k)})$, and there exists $\beta_0 = \beta_0(\varepsilon)$ such that

$$\|a_{0,0}^{(\beta)} - a_{0,0}\| \ < \varepsilon, \quad \beta \geq \beta_0,$$

and, for $m,n \geq 0,\ m+n \neq 0$,

$$\|a_{m,n}^{(\beta)} - a_{m,n}\| < \varepsilon \exp[-(\lambda_m \sigma_1^{(k)} + \mu_n \sigma_2^{(k)})], \quad \beta \geq \beta_0,\ k \geq 1.$$

Thus

$$\begin{aligned}\sup_{m,n} \|a_{m,n}^{(\beta)} - a_{m,n}\|^{1/(\lambda_m+\mu_n)} &< \exp(-\sigma_k)\sup_{m,n}\varepsilon^{1/(\lambda_m+\mu_n)} \\ &< \varepsilon, \quad \beta \geq \beta_0.\end{aligned}$$

Hence $d(f_\beta, f) < \varepsilon$ for all $\beta \geq \beta_0$ and, consequently, $f_\beta \to f$ in the metric defined by p. Combining the two sides, we obtain the equivalence of ρ and d.

It has been shown earlier by the authors (see [6], p. 85) that the topologies defined by ρ and T are equivalent. This gives the result. $\square$

Proposition 2. *The space Y endowed with any one of the topologies given by ρ, d or T is complete.*

Proof. In view of Proposition 1, it is sufficient to prove the completeness of Y under one of the above topologies. We consider the topology generated by the metric d. Let $\{f_\beta\}$ be a Cauchy sequence in Y. Then for a given $\varepsilon,\ 0 < \varepsilon < 1$, there exists an integer $N = N(\varepsilon)$ such that

$$\|a_{0,0}^{(\beta)} - a_{0,0}^{(\gamma)}\| < \varepsilon, \quad \beta, \gamma \geq N,$$

and

$$\|a^{(\beta)}_{m,n} - a^{(\gamma)}_{m,n}\|^{1/(\lambda_m+\mu_n)} < \varepsilon, \quad \beta,\gamma \geq N,\ \ m,n \geq 0, m+n \neq 0.$$

Hence, the sequence $\left\{a^{(\beta)}_{m,n}\right\}$is a Cauchy sequence in the Banach algebra E for each fixed m and n . Therefore, $\left\{a^{(\beta)}_{m,n}\right\} \to a_{m,n}$ as $\beta \to \infty$ for each $m,n \geq 0$.

Now

$$\begin{aligned}\|a_{m,n}\| &\leq |\|a^{(\beta)}_{m,n} - a_{m,n}\| + \|a^{(\beta)}_{m,n}\| \\ &\leq \varepsilon + \exp\{-(\lambda_m+\mu_n)k\}, \quad \beta \geq N,\ \ m+n \geq n_0(k),\end{aligned}$$

in view of (1.3). This shows that

$$\|a_{m,n}\|^{1/(\lambda_m+\mu_n)} \leq \exp(-k)\,, \quad m+n \geq n_0(k).$$

Thus $f(s_1,s_2) = \sum_{m,n=0}^{\infty} a_{m,n}\exp(\lambda_m s_1 + \mu_n s_2)$ is an entire function. For all $\beta \geq N$ we have

$$\begin{aligned}d(f_\beta, f) &= p(f_\beta - f) \\ &= \sup\left\{\|a^{(\beta)}_{0,0} - a_{0,0}\|,\ \|a^{(\beta)}_{m,n} - a_{m,n}\|^{1/(\lambda_m+\mu_n)} : m,n \geq 0,\ m+n \neq 0\right\} \\ &\leq \varepsilon,\end{aligned}$$

where $f \in Y$. The result is proved. □

3. Linear transformations on the space Y

In this section, we characterize certain linear transformations on Y. In what follows, for each $f \in Y$, let $p(f)$ be defined as above and for $f,g \in Y$ we have $d(f,g) = p(f-g)$.

Theorem 1. *Let* $\{c_{m,n}\}_{m,n=0}^{\infty}$ *be a sequence of complex numbers and* $a_{m,n} \in E$ $(m,n \geq 0)$. *If sequence* $\{a_{m,n}\}$ *satisfies condition* (1.3), *then the series* $\sum_{m,n=0}^{\infty} c_{m,n}a_{m,n}$ *converges absolutely in* E *if and only if the sequence*

$$\left\{\ln|c_{0,0}|,\ \ln|c_{m,n}|^{1/(\lambda_m+\mu_n)}\right\}_{m,n\geq 0,\, m+n\neq 0} \tag{3.1}$$

is bounded.

Proof. Let us assume that the series $\sum\limits_{m,n=0}^{\infty} c_{m,n}a_{m,n}$ is absolutely convergent but the sequence (3.1) is not bounded. Then we have

$$|c_{m_k,n_k}| \geq \exp\{k(\lambda_{m_k}+\mu_{n_k})\}, \quad k \geq 1.$$

Now we define a sequence $\{a_{m,n}\} \subseteq E$ such that

$$a_{m,n} = \begin{cases}\omega\ \exp\{-k(\lambda_{m_k}+\mu_{n_k})\} & \text{if } m = m_k,\ n = n_k, \\ 0 & \text{otherwise,}\end{cases}$$

where ω is the unit element of E. Then (1.4) holds but

$$\|a_{m_k,n_k}c_{m_k,n_k}\| \geq 1, \quad k \geq 1.$$

Therefore, the sequence $\|c_{m,n}\, a_{m,n}\|$ does not tend to zero as $m+n \to \infty$ and so, the series $\sum_{m,n=0}^{\infty} \|c_{m,n}\, a_{m,n}\|$ does not converge. This is a contradiction and hence, the necessary part is proved.

To prove the sufficiency part let us suppose that the sequence (3.1) is bounded. Then there is a positive constant M such that

$$|c_{0,0}| < e^{M} \text{ and } |c_{m,n}|^{1/(\lambda_m+\mu_n)} \leq e^{M}, \quad m,n \geq 0,\, m+n \neq 0,$$

and (see [1], Lemma 1) the series $\sum_{m+n>0} \exp\{-M(\lambda_m+\mu_n)\}$ is convergent. Now, using (1.3), we find a positive integer n_0 such that

$$\|a_{m,n}\| < \exp\{-2M(\lambda_m + \mu_n)\}, \quad m+n \geq n_0,$$

or

$$\|c_{m,n}\, a_{m,n}\| < \exp\{-M(\lambda_m + \mu_n)\}, \quad m+n \geq n_0.$$

Hence

$$\begin{aligned}\sum_{m,n=0}^{\infty} \|c_{m,n}\, a_{m,n}\| &\leq O(1) + \sum_{m+n\geq n_0} \|c_{m,n}\, a_{m,n}\| \\ &\leq O(1) + \sum_{m+n\geq n_0} \exp\{-M(\lambda_m + \mu_n)\}\end{aligned}$$

which shows that the series $\sum_{m,n=0}^{\infty} \|c_{m,n}\, a_{m,n}\|$ converges. The proof of Theorem 1 is complete. □

Theorem 2. *Let $\{c_{m,n}\}$ be a sequence of complex numbers. The transformation $\phi : Y \to E$ of the form*

$$\phi(f) = \sum_{m,n=0}^{\infty} c_{m,n}\, a_{m,n} \tag{3.2}$$

is linear and continuous if the sequence (3.1) *is bounded and Y is endowed with any one of the topologies given by metrics ρ, d or T.*

Proof. The transformation ϕ is correctly defined in view of Theorem 1. The linearity of ϕ is immediately clear.

To prove that ϕ is continuous it is sufficient to show that if $\{f_q\}_{q=1}^{\infty}$ is a sequence in Y and $f_q \to 0$ as $q \to \infty$, then $\phi(f_q) \to 0$. Since all three topologies on Y are equivalent by Proposition 1, we take the metric d on Y for our proof. Let

$$f_q(s_1, s_2) = \sum_{m,n=0}^{\infty} a_{m,n}^{(q)} \exp(\lambda_m s_1 + \mu_n s_2)\,, \quad q \geq 1,$$

and suppose that

$$M = \sup\{\ln |c_{0,0}|,\ \ln |c_{m,n}|^{1/(\lambda_m+\mu_n)}:\ \ m, n \geq 0,\ m+n \neq 0\}.$$

Since (1.2) is satisfied, by [1, Lemma 1] there exists a number α, $0 < \alpha < \infty$, such that the series $\sum_{m+n>0} \exp\{-\alpha(\lambda_m + \mu_n)\}$ is convergent. Now we choose $\varepsilon > 0$ so that $M - 1/\varepsilon \leq -\alpha$. Since $f_q \to 0$ as $q \to \infty$, there exists a positive integer $Q = Q(\varepsilon)$ such that

$$\|a_{0,0}^{(q)}\| < \exp(-1/\varepsilon),\quad \|a_{m,n}^{(q)}\| < \exp\{-(\lambda_m + \mu_n)/\varepsilon\},\quad q \geq Q.$$

Then, for $q \geq Q$,

$$\begin{aligned}\|\phi(f_q)\| &= \left\| \sum_{m,n=0}^{\infty} a_{m,n}^{(q)}\, c_{m,n} \right\| \leq \sum_{m,n=0}^{\infty} \left\|a_{m,n}^{(q)}\right\|\ |c_{m,n}| \\ &< \exp(M - 1/\varepsilon) + \sum_{m+n>0} \exp\{(M - 1/\varepsilon)\,(\lambda_m + \mu_n)\} \\ &\to 0 \ \text{ as } \ \varepsilon \to 0.\end{aligned}$$

Hence ϕ is continuous and the proof of Theorem 2 is complete. □

Acknowledgements

The authors are very much thankful to the referee and professor Enno Kolk for their valuable comments which helped very much in improving the paper.

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3