ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 19, Number 2, December 2015
Available online at http://acutm.math.ut.ee

Bäcklund transformations according to Bishop frame in E_1^3

Murat Kemal Karacan and Yilmaz Tunçer

Abstract. In this study we have defined Bäcklund transformations of curves according to Bishop frame preserving the natural curvatures under certain assumptions in Minkowski 3-space.

1. Introduction

By the work of Bianchi and Lie it is possible to compute the Gaussian curvature of the focal surfaces of a line congruence in terms of the coefficients of the first fundamental form for the spherical representation and the distance between the corresponding limit points of these surfaces. Bäcklund proved that for pseudospherical congruences satisfying the two additional conditions that the distance r between corresponding limit points is constant and that the normals of the focal surfaces at these points form a constant angle θ, the curvatures must be equal to the same negative constant -$\frac{\sin^2\theta}{r^2}$ (see [11, 14]).

In the classical differential geometry, a Bäcklund transformation takes a given pseudospherical (i.e., constant negative Gauss curvature) surface to a new pseudospherical surface. As explained by Chern and Terng [5], the new surface is connected to the old surface by line segments that are tangent to both surfaces, of a fixed length, and such that the angle between the surface normals at corresponding points is also constant. Moreover, the Bäcklund transformation takes asymptotic lines to asymptotic lines. Since the asymptotic lines on a pseudospherical surface have constant torsion, it is not surprising that we can restrict the Bäcklund transformation to get a transformation that carries constant torsion curves to constant torsion curves (see [11]).

Received March 15, 2012.

2010 *Mathematics Subject Classification.* 53A04, 53B30, 53A17.

Key words and phrases. Bäcklund transformations, Bishop frame, parallel transport frame, Minkowski 3-space.

http://dx.doi.org/10.12697/ACUTM.2015.19.07

In 1998, Calini and Ivey [3] proposed a geometric realization of the Bäcklund transformation for the sine-Gordon equation in the context of curves of constant torsion. Since the asymptotic lines on a pseudospherical surface have constant torsion, the Bäcklund transformation can be restricted to get a transformation that carries constant torsion curves to constant torsion curves. Later the converse of the idea was proved and generalized for the n-dimensional case by Nemeth [11]. In [12], Nemeth studied a similar concept for constant torsion curves in the 3-dimensional constant curvature spaces (see [4]).

In recent years, Gürbüz [7] studied Bäcklund transformations in R_1^n. Using the same method, Özdemir and Cöken [13] have studied Bäcklund transformations of non-lightlike constant torsion curves in Minkowski 3-space. Karacan and Tunçer [10] study Bäcklund transformations according to Bishop frame in Euclidean 3-space.

In this paper, we show that a restriction of Bäcklund theorem on space curves satisfying the given three conditions preserves the first and second curvatures (natural curvatures) of the curves according to the Bishop frame in Minkowski 3-space.

2. Preliminaries

The Minkowski 3-space E_1^3 is the Euclidean 3-space E^3 provided with the standard flat metric given by

$$\langle\, , \rangle_L = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of E_1^3. Since $\langle\, , \rangle_L$ is an indefinite metric, recall that a vector $v \in E_1^3$ can have one of three Lorentzian causal characters: it can be spacelike if $\langle v, v\rangle_L > 0$ or $v = 0$, timelike if $\langle v, v\rangle_L < 0$, and null (lightlike) if $\langle v, v\rangle_L = 0$ and $v \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in E_1^3 can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null (lightlike).The norm of the vector v is defined by $\|v\|_L = \sqrt{|\langle v, v\rangle_L|}$. Associated to that inner product, for any $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$ in E_1^3, the Lorentzian vector product $u \wedge_L v$ of u and v is defined as follows:

$$u \wedge_L v = (-u_2v_3 + u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)\,.$$

Minkowski space is originally from the relativity in physics. In fact, a timelike curve corresponds to the path of an observer moving at less than the speed of light. Denote by $\{T, N, B\}$ the moving Frenet–Serret frame along the curve $\alpha(s)$ in the space E_1^3. For an arbitrary curve $\alpha(s)$ in the space E_1^3, the following Frenet–Serret formulae are given. If α is timelike curve, then the Frenet–Serret formulae read

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{2.1}$$

where

$$\langle T,T\rangle_L = -1,\ \langle N,N\rangle_L = \langle B,B\rangle_L = 1,\ \langle T,N\rangle_L = \langle N,B\rangle_L = \langle T,B\rangle_L = 0.$$

If α is a spacelike curve with a spacelike principal normal, then the Frenet–Serret formulae read

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix},$$

where

$$\langle T,T\rangle_L = \langle N,N\rangle_L = 1,\ \langle B,B\rangle_L = -1,\ \langle T,N\rangle_L = \langle N,B\rangle_L = \langle T,B\rangle_L = 0.$$

If α is a spacelike curve with a spacelike binormal, then the Frenet–Serret formulae read (see [14])

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix},$$

where

$$\langle T,T\rangle_L = \langle B,B\rangle_L = 1,\ \langle N,N\rangle_L = -1,\ \langle T,N\rangle_L = \langle N,B\rangle_L = \langle T,B\rangle_L = 0.$$

The ability to "ride" along a three-dimensional space curve and illustrate the properties of the curve, such as curvature and torsion, would be a great asset to mathematicians. The classic Frenet–Serret frame provides such ability, however the Frenet–Serret frame is not defined for all points along every curve. A new frame is needed for the kind of mathematical analysis that is typically done with computer graphics.

The relatively parallel adapted frame or Bishop frame could provide the desired means to ride along any given space curve. The Bishop frame has many properties that make it ideal for mathematical research. Another area of interest about the Bishop frame is so-called normal developement, or the graph of the twisting motion of Bishop frame. This information along with the initial position and orientation of the Bishop frame provide all of the information necessary to define the curve.

The Bishop frame may have applications in the area of biology and computer graphics. For example, it may be possible to compute information about the shape of sequences of DNA using a curve defined by the Bishop frame. The Bishop frame may also provide a new way to control virtual cameras in computer animations.

4

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. We can parallelly transport an orthonormal frame along a curve simply by parallel transporting each component of the frame. The parallel transport frame is based on the observation that, while $T(s)$ for a given curve model is unique, we may choose any convenient arbitrary basis $(U(s), V(s))$ for the remainder of the frame, so long as it is in the normal plane perpendicular to $T(s)$ at each point. If the derivatives of $(U(s), V(s))$ depend only on $T(s)$ and not each other we can make $U(s)$ and $V(s)$ vary smoothly throughout the path regardless of the curvature.

In addition, suppose the curve α is an arclength-parametrized C^2 curve. Suppose we have C^1 unit vector fields U and $V = T\wedge U$ along the curve α so that

$$\langle T, U\rangle_L = \langle T, V\rangle_L = \langle U, V\rangle_L = 0,$$

i.e., T, U, V will be a smoothly varying right-handed orthonormal frame as we move along the curve. (To this point, the Frenet frame would work just fine if the curve were C^3 with $\kappa \neq 0$.) But now we want to impose the extra condition that $\langle U', V\rangle_L = 0$. We say the unit first normal vector field U is parallel along the curve α. This means that the only change of U is in the direction of T. A Bishop frame can be defined even when a Frenet frame cannot (e.g., when there are points with $\kappa = 0$) (see [8]). Therefore, we have the alternative frame equations

$$\begin{bmatrix} T' \\ U' \\ V' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ U \\ V \end{bmatrix}.$$

One can show that

$$\kappa(s) = \sqrt{k_1^2 + k_2^2}, \theta(s) = \arctan\left(\frac{k_2}{k_1}\right),\ \ k_1 \neq 0,\ \ \tau(s) = -\frac{d\theta(s)}{ds},$$

so that k_1 and k_2 effectively correspond to a Cartesian coordinate system for the polar coordinates κ, θ with $\theta = -\int \tau(s)ds$. The orientation of the parallel transport frame includes the arbitrary choice of integration constant θ_0, which disappears from τ (and hence from the Frenet–Serret–Bartels frame) due to the differentiation (see [5]). Thus the relation matrix may be expressed as

$$\begin{aligned} T &= T, \\ N &= U\cos\theta(s) - V\sin\theta(s), \\ B &= U\sin\theta(s) + V\cos\theta(s). \end{aligned}$$

Bishop curvatures are defined by

$$k_1 = \kappa(s)\cos\theta(s),\ \ k_2 = \kappa(s)\sin\theta(s).$$

If α is a timelike curve, then the Bishop frame is given by (see [9])

$$\begin{bmatrix} T' \\ U' \\ V' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ k_1 & 0 & 0 \\ k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ U \\ V \end{bmatrix}, \tag{2.2}$$

where

$$\kappa(s) = \sqrt{k_1^2 + k_2^2},\ \ \theta(s) = \arctan\left(\frac{k_2}{k_1}\right), \tau(s) = \frac{d\theta(s)}{ds},$$

$\langle T, T\rangle_L = -1,\ \langle U, U\rangle_L = 1,\ \langle V, V\rangle_L = 1$, and the metric is $(-, +, +)$.

If α is a spacelike curve with timelike principal normal, then the Bishop frame is given by (see [2])

$$\begin{bmatrix} T' \\ U' \\ V' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & -k_2 \\ k_1 & 0 & 0 \\ k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ U \\ V \end{bmatrix},$$

where

$$\kappa(s) = \sqrt{\left|k_2^2 - k_1^2\right|},\ \ \theta(s) = \arg\tanh\left(\frac{k_2}{k_1}\right),\ \ \tau(s) = \frac{d\theta(s)}{ds}$$

$\langle T, T\rangle_L = 1,\ \langle U, U\rangle_L = -1,\ \langle V, V\rangle_L = 1$, and the metric is $(+, -, +)$.

If α is a spacelike curve with timelike binormal, then the Bishop frame is given by (see [1])

$$\begin{bmatrix} T' \\ U' \\ V' \end{bmatrix} \begin{bmatrix} 0 & k_1 & -k_2 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ U \\ V \end{bmatrix},$$

where

$$\kappa(s) = \sqrt{\left|k_1^2 - k_2^2\right|},\ \ \theta(s) = \arg\tanh(\frac{k_2}{k_1}),\ \ \tau(s) = -\frac{d\theta(s)}{ds},$$

$\langle T, T\rangle_L = 1,\ \langle U, U\rangle_L = 1,\ \langle V, V\rangle_L = -1$, and the metric is $(+, +, -)$.

3. Bäcklund transformation according to Bishop frame in Minkowski 3-space

In this chapter, we prove the Bäcklund theorem for timelike and spacelike curves in terms of Bishop frame. In [13, Theorems 2 and 3], the Bäcklund theorem is given via the torsion of timelike curve. In this study, it is given in terms of natural curvatures of Bishop frame. They are not the same. Because the geometrical meaning of the curvatures of curve and the geometrical meaning of natural curvatures of Bishop frame are different. There is no geometrical meaning of natural curvatures alone, because natural curvatures of Bishop frame depend on the curvatures of Frenet–Serret frame.

Theorem 1. *Let α be a timelike curve with spacelike first and second normals U_α and V_α. Suppose that ψ is a transformation between two curves α and β in Minkowski 3-space with $\beta = \psi(\alpha)$ such that in the corresponding points:*

1. *the line segment $[\beta(s)\alpha(s)]$ at the intersection of the osculating planes of the curves has constant length r;*
2. *the distance vector $\beta(s) - \alpha(s)$ has the same angle $\gamma \neq \frac{\pi}{2}$ with the tangent vectors of the curves;*
3. *the second normals V_α and V_β of the curves have the same constant angle $\phi \neq 0$.*

Then these curves are congruent with natural curvatures

$$\begin{aligned} k_1^\beta &= k_1^\alpha = -\frac{d\gamma}{ds}, \\ k_2^\beta &= k_2^\alpha = \frac{\tanh\gamma \sin\phi}{r}, \end{aligned}$$

and the transformation of the curves is given by

$$\beta = \alpha + \frac{2C \tanh\gamma}{(k_2^\alpha)^2 + C^2} \left(T_\alpha \cosh\gamma + U_\alpha \sinh\gamma\right), \tag{3.1}$$

where $C = k_2^\alpha \tan\frac{\phi}{2}$ and γ is a solution of the differential equation

$$\frac{d\gamma}{ds} = k_2^\beta \cosh\gamma \tan\frac{\phi}{2} - k_1^\alpha.$$

Proof. Denote by $(T_\alpha, U_\alpha, V_\alpha)$ and $(T_\beta, U_\beta, V_\beta)$ the Bishop frames of the curves α and β in the Minkowski 3-space E_1^3 . Let V_β be a unit second principal normal of β. If we denote by W_1^α the unit vector of $\beta - \alpha$, then we can complete W_1^α,V_α and W_1^α,V_β to the positively oriented orthonormal frames $(W_1^\alpha, W_2^\alpha, W_3^\alpha)$ and $(W_1^\beta, W_2^\beta, W_3^\beta)$, where $W_3^\alpha = V_\alpha$, $W_3^\beta = V_\beta$ and γ is the angle between W_1^α and T_α. The frames $(W_1^\alpha, W_2^\alpha, W_3^\alpha)$ and $(W_1^\beta, W_2^\beta, W_3^\beta)$ can be obtained by rotating the frames $(T_\alpha, U_\alpha, V_\alpha)$ and $(T_\beta, U_\beta, V_\beta)$ around V_α and V_β with an angle γ respectively. So we can write

$$\begin{bmatrix} W_1^\alpha \\ W_2^\alpha \\ W_3^\alpha \end{bmatrix} = \begin{bmatrix} \cosh\gamma & \sinh\gamma & 0 \\ \sinh\gamma & \cosh\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} T_\alpha \\ U_\alpha \\ V_\alpha \end{bmatrix}$$

and

$$\begin{bmatrix} W_1^\alpha \\ W_2^\beta \\ W_3^\beta \end{bmatrix} = \begin{bmatrix} \cosh\gamma & \sinh\gamma & 0 \\ \sinh\gamma & \cosh\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} T_\beta \\ U_\beta \\ V_\beta \end{bmatrix}.$$

Similarly, for a rotation around W_1^α by the angle ϕ,

$$\begin{aligned} W_2^\beta &= W_2^\alpha \cos\phi - W_3^\alpha \sin\phi, \\ W_3^\beta &= W_2^\alpha \sin\phi + W_3^\alpha \cos\phi. \end{aligned}$$

From the above equations we write

$$\begin{aligned} T_\beta = {} & \left(\cosh^2\gamma - \sin^2\gamma\cos\phi\right) T_\alpha \\ & + \left(\cosh\gamma\sinh\gamma\right)\left(1-\cos\phi\right) U_\alpha + \left(\sinh\gamma\sin\phi\right) V_\alpha, \end{aligned} \tag{3.2}$$

$$\begin{aligned} U_\beta = {} & \left(\cosh\gamma\sinh\gamma\right)\left(\cos\phi - 1\right) T_\alpha \\ & + \left(\cosh^2\gamma\cos\phi - \sinh^2\gamma\right) U_\alpha - \left(\cosh\gamma\sin\phi\right) V_\alpha, \end{aligned} \tag{3.3}$$

$$V_\beta = \left(\sinh\gamma\sin\phi\right) T_\alpha + \left(\cosh\gamma\sin\phi\right) U_\alpha + \left(\cos\phi\right) V_\alpha. \tag{3.4}$$

Using (2.2) and (3.2), (3.3), (3.4) for T_β, U_β and V_β, we get

$$\begin{aligned} \frac{dT_\beta}{ds} &= k_1^\beta U_\beta + k_2^\beta V_\beta \\ &= \left[k_1^\beta \cosh\gamma\sinh\gamma(\cos\phi - 1) + k_2^\beta \sinh\gamma\sin\phi\right] T_\alpha \\ &\quad + \left[\left(\cosh^2\gamma\cos\phi - \sinh^2\gamma\right) k_1^\beta + k_2^\beta \cosh\gamma\sin\phi\right] U_\alpha \\ &\quad + \left[\left(k_2^\beta \cos\phi - k_1^\beta \cosh\gamma\sin\phi\right)\right] V_\alpha, \end{aligned}$$

$$\begin{aligned} \frac{dU_\beta}{ds} &= k_1^\beta T_\beta \\ &= k_1^\beta \left(\cosh^2\gamma - \sinh^2\gamma\cos\phi\right) T_\alpha \\ &\quad + k_1^\beta \left(1-\cos\phi\right)\left(\cosh\gamma\sinh\gamma\right) U_\alpha + k_1^\beta \left(\sinh\gamma\sin\phi\right) V_\alpha, \end{aligned}$$

$$\begin{aligned} \frac{dV_\beta}{ds} &= k_2^\beta T_\beta \\ &= k_2^\beta \left(\cosh^2\gamma - \sinh^2\gamma\cos\phi\right) T_\alpha \\ &\quad + k_2^\beta \left(1-\cos\phi\right)\left(\cosh\gamma\sinh\gamma\right) U_\alpha + k_2^\beta \left(\sinh\gamma\sin\phi\right) V_\alpha. \end{aligned}$$

Now, taking derivative of T_β, U_β and V_β in (3.2), (3.3), (3.4) with respect to s, we have that

$$\begin{aligned} \frac{dT_\beta}{ds} = {} & \left[(1-\cos\phi)\left(2\frac{d\gamma}{ds} + k_1^\alpha\right)\cosh\gamma\sinh\gamma + k_2^\alpha \sinh\gamma\sin\phi\right] T_\alpha \\ & + \left[\sinh^2\gamma\left((1-\cos\phi)\frac{d\gamma}{ds} - k_1^\alpha\cos\phi\right)\right. \\ & \left. + \cosh^2\gamma\left(k_1^\alpha + (1-\cos\phi)\frac{d\gamma}{ds}\right)\right] U_\alpha \\ & + \left[\cosh\gamma\left(k_2^\alpha\cosh\gamma + \frac{d\gamma}{ds}\sin\phi\right) - \left(k_2^\alpha \sinh^2\gamma\cos\phi\right)\right] V_\alpha, \end{aligned} \tag{3.5}$$

$$
\begin{aligned}
\frac{dU_\beta}{ds} = & \Big[(\cos\phi - 1)\left(\sinh^2\gamma + \cosh^2\gamma\right)\frac{d\gamma}{ds} \\
& +k_1^\alpha\left(\cosh^2\gamma\cos\phi - \sinh^2\gamma\right) - \left(k_2^\alpha\cosh\gamma\sin\phi\right)\Big] T_\alpha \\
& + \Big[2(\cos\phi - 1)\left(\cosh\gamma\sinh\gamma\right)\frac{d\gamma}{ds} \\
& +k_1^\alpha(\cos\phi - 1)\left(\cosh\gamma\sinh\gamma\right)\Big] U_\alpha \\
& + \left[k_1^\alpha(\cos\phi - 1)\left(\cosh\gamma\sinh\gamma\right) - \left(\sinh\gamma\sin\phi\right)\frac{d\gamma}{ds}\right] V_\alpha,
\end{aligned} \tag{3.6}
$$

and

$$
\begin{aligned}
\frac{dV_\beta}{ds} = & \left[\left(\cosh\gamma\sin\phi\right)\left(\frac{d\gamma}{ds} + k_1^\alpha\right) + \left(k_2^\alpha\cos\phi\right)\right] T_\alpha \\
& + \left[k_1^\alpha\left(\sinh\gamma\sin\phi\right) + \left(\sinh\gamma\sin\phi\right)\frac{d\gamma}{ds}\right] U_\alpha \\
& + \left[k_2^\alpha\left(\sinh\gamma\sin\phi\right)\right] V_\alpha.
\end{aligned} \tag{3.7}
$$

Then, equating the two statements above, we obtain

$$
\begin{aligned}
k_2^\beta &= k_2^\alpha, \\
\frac{d\gamma}{ds} &= k_2^\beta\cosh\gamma\tan\frac{\phi}{2} - k_1^\alpha.
\end{aligned}
$$

Similarly, using (2.2) and (3.5), (3.6), (3.7), we have

$$
k_1^\alpha + k_1^\beta = -2\frac{d\gamma}{ds}
$$

and

$$
k_1^\alpha = k_1^\beta = -\frac{d\gamma}{ds}.
$$

Now α is a unit speed curve. Differentiating

$$
(\beta - \alpha)^2 = r^2
$$

and substituting the distance vector

$$
\beta - \alpha = r\left(T_\alpha\cosh\gamma + U_\alpha\sinh\gamma\right), \tag{3.8}
$$

we find that β is also a unit speed curve. Next, taking the derivative of (3.8), we obtain

$$
\begin{aligned}
T_\beta = & \left[(r\sinh\gamma)\left(k_1^\alpha + \frac{d\gamma}{ds}\right)\right] T_\alpha + \left[(r\cosh\gamma)\left(k_1^\alpha + \frac{d\gamma}{ds}\right)\right] U_\alpha \\
& + \left[\left(rk_2^\alpha\cosh\gamma\right)\right] V_\alpha.
\end{aligned}
$$

From this equation and the Bishop frames (3.2), (3.3) and (3.4) we get

$$
k_2^\beta = k_2^\alpha = \frac{\tanh\gamma\sin\phi}{r}.
$$

Then, rearranging this equality, we get

$$r = \frac{\tanh\gamma \sin\phi}{k_2^\alpha}.$$

Finally, with the aid of (3.8), the Bäcklund transformation of the timelike curves according to Bishop frame is determined by (3.1). □

Theorem 2. *Let α be a spacelike curve with spacelike principal normal. Suppose that ψ is a transformation between two curves α and β in Minkowski 3-space with $\beta = \psi(\alpha)$ such that in the corresponding points:*

(1) *the line segment $[\beta(s)\alpha(s)]$ at the intersection of the osculating planes of the curves has constant length r;*
(2) *the distance vector $\beta(s) - \alpha(s)$ has the same angle $\gamma \neq \frac{\pi}{2}$ with the tangent vectors of the curves;*
(3) *the binormals of the curves have the same constant angle $\phi \neq 0$.*

If the first normal U of the curve α is spacelike, then these curves are congruent with natural curvatures

$$\begin{aligned} k_1^\beta + k_1^\alpha &= -2\frac{d\gamma}{ds}, \\ k_2^\beta = k_2^\alpha &= \frac{\tan\gamma \sinh\phi}{r}, \end{aligned} \tag{3.9}$$

and the transformation of the curves is given by

$$\beta = \alpha + \frac{2C\tanh\gamma}{(k_2^\alpha)^2 - C^2}\left(T_\alpha \cos\gamma + U_\alpha \sin\gamma\right), \tag{3.10}$$

where $C = k_2^\alpha \tan\frac{\phi}{2}$ and γ is a solution of the differential equation

$$\frac{d\gamma}{ds} = -k_2^\alpha \cos\gamma \tanh\frac{\phi}{2} - k_1^\beta.$$

If the second normal V of the curve α is spacelike, then these curves are congruent with natural curvatures

$$\begin{aligned} k_1^\beta + k_1^\alpha &= -2\frac{d\gamma}{ds}, \\ k_2^\beta = k_2^\alpha &= \frac{\tanh\gamma \sinh\phi}{r}, \end{aligned} \tag{3.11}$$

and the transformation of the curves is given by

$$\beta = \alpha + \frac{2C\sinh\gamma}{(k_2^\alpha)^2 - C^2}\left(T_\alpha \cosh\gamma + U_\alpha \sinh\gamma\right), \tag{3.12}$$

where $C = k_2^\alpha \tanh\frac{\phi}{2}$ and γ is a solution of the differential equation

$$\frac{d\gamma}{ds} = -k_2^\alpha \cosh\gamma \tanh\frac{\phi}{2} - k_1^\beta.$$

Proof. If the first normal U of the curve α is spacelike, then, using the arguments of Theorem 1 and equations

$$\begin{aligned} W_1^\alpha &= T_\alpha \cos\gamma + U_\alpha \sin\gamma, \\ W_2^\alpha &= -T_\alpha \sin\gamma + U_\alpha \cos\gamma, \\ W_1^\beta &= T_\beta \cos\gamma + U_\alpha \sin\gamma, \\ W_2^\beta &= -T_\beta \sin\gamma + U_\beta \cos\gamma \end{aligned}$$

and

$$\begin{aligned} W_2^\beta &= W_2^\alpha \cosh\phi + W_3^\alpha \sinh\phi, \\ W_3^\beta &= W_2^\alpha \sinh\phi + W_3^\alpha \cosh\phi, \end{aligned}$$

the natural curvatures of the curves and transformation can be found, respectively, as (3.9) and (3.10), where $C = k_2^\alpha \tan\frac{\phi}{2}$.

If the second normal V of the curve α is spacelike, then, using the equations

$$\begin{aligned} W_1^\alpha &= T_\alpha \cosh\gamma + U_\alpha \sinh\gamma, \\ W_2^\alpha &= T_\alpha \sinh\gamma + U_\alpha \cosh\gamma, \\ W_1^\beta &= T_\beta \cosh\gamma + U_\beta \sinh\gamma, \\ W_2^\beta &= T_\beta \sinh\gamma + U_\beta \cosh\gamma \end{aligned}$$

and

$$\begin{aligned} W_2^\beta &= W_2^\alpha \cosh\phi + W_3^\alpha \sinh\phi, \\ W_3^\beta &= W_2^\alpha \sinh\phi + W_3^\alpha \cosh\phi, \end{aligned}$$

the natural curvatures of the curves and transformation can be found, respectively, as (3.11) and (3.12), where $C = k_2^\alpha \tanh\frac{\phi}{2}$. □

Acknowledgements

The authors would like to thank the referees for their valuable suggestions.

References

[1] B. Bukcu and M. K. Karacan, *Bishop frame of the spacelike curve with a spacelike principal normal in Minkowski 3-space*, Commun. Fac. Sci. Univ. Ank. Sér. A1 Math. Stat. **57**(1) (2008), 13–22.

[2] B. Bukcu and M. K. Karacan, *Bishop frame of the spacelike curve with a spacelike binormal in Minkowski 3-space*, Selçuk J. Appl. Math. **11**(1) (2010), 15–25.

[3] A. Calini and T. Ivey, *Bäcklund transformations and knots of constant torsion*, J. Knot Theor. Ramification **7** (1998), 719–746.

[4] S. Cengiz and N. Gürbüz, *Bäcklund transformations of curves in the Galilean and pseudo-Galilean spaces*, http://arxiv.org/PS_cache/arxiv/pdf/1105/1105.1867v1.pdf, 2011.

[5] S. S. Chern and C. L. Terng, *An analogue of Bäcklund's theorem in affine geometry*, Rocky Mountain J. Math. **10** (1980), 105–124.

[6] L. P. Eisenhart, *A Treatise on the Differential Geometry of Curves and Surfaces*, Ginn and Company, New York, 1909.
[7] N. Gürbüz, *Backlund transformations of constant torsion curves in* R_1^n , Hadronic J. **29** (2006), 213–220.
[8] A. J. Hanson and H. Ma, *Parallel Transport Approach to Curve Framing*, Indiana University, Techreports-TR425, 1995.
[9] M. K. Karacan and B. Bükcü, *Bishop frame of the timelike curve in Minkowski 3-space*, Fen. Derg. **3**(1) (2008), 80–90.
[10] M. K. Karacan and Y. Tunçer, *Bäcklund transformation according to Bishop frame in Euclidean 3-space*, Šiauliai Math. Semin. **7(15)** (2012), 41–49.
[11] S. Z. Nemeth, *Bäcklund transformations of n-dimensional constant torsion curves*, Publ. Math, Debrecen **53** (1998), 271–279.
[12] S. Z. Nemeth, *Bäcklund transformations of constant torsion curves in 3-dimensional constant curvature spaces*, Ital. J. Pure Appl. Math. **7** (2000), 125–138.
[13] M. Özdemir and A. C. Cöken, *Bäcklund transformation for non-lightlike curves in Minkowski 3-space*, Chaos Solitons Fractals **42** (2009), 2540–2545.
[14] J. Walrave, *Curves and Surfaces in Minkowski Space*, PhD Thesis, 1995.

Uşak University, Faculty of Sciences and Arts, Department of Mathematics.1 Eylul Campus, 64200, Uşak-Turkey
E-mail address: `murat.karacan@usak.edu.tr`
E-mail address: `yilmaz.tuncer@usak.edu.tr`