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New fractional integral inequalities associated with pathway operator

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Abstract. This paper deals with the derivation of certain integral inequalities involving fractional integral operators of Chebyshev type by an application of the pathway fractional integral operator. The results obtained are of general character and provide extension of results given by Belarbi and Dahmani.

1. Introduction and preliminaries

Let $f, g : [a, b] \to \mathbb{R}$ be two integrable functions. The Chebyshev functional is defined by (see [2])

$$T(f,g) = \frac{1}{b-a} \int_a^b f(x)\, g(x)\, dx - \left(\frac{1}{b-a} \int_a^b f(x)\, dx \right) \left(\frac{1}{b-a} \int_a^b g(x)\, dx \right),$$

where f and g are synchronous on $[a, b]$, i.e.,

$$[f(x) - f(y)][g(x) - g(y)] \geq 0, \quad x, y \in [a, b].$$

The notation and definitions related to fractional calculus are given below. (For more details see Gorenflo and Mainardi [3], Podlubny [9], and Kilbas et al. [4]).

Recall that the left-sided Riemann–Liouville fractional integral operator is defined by (see, for example, Samko et al. [10])

$$\left(I_{0+}^{\eta} f \right)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - \mathrm{t})^{\alpha - 1} f(t)\, dt,$$

where $f \in L(a, b)$ and $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$. A generalization of this notion is contained in the following definition.

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Definition 1 (see [8])**.** Let $f \in L(a,b)$, $\eta \in \mathbb{C}$ with $\Re(\eta) > 0$, and $0 < a < 1$. The *pathway fractional integration operator* is defined in the form

$$\left(P_{0+}^{(\eta,\,\alpha)} f\right)(x) \;=\; x^{\eta} \int_0^{\left[\frac{x}{a\,(1-\alpha)}\right]} \left[1 - \frac{a\,(1-\alpha)\,t}{x}\right]^{\frac{\eta}{1-\alpha}} f\,(t)\;dt. \tag{1.1}$$

The pathway model has been introduced by Mathai [5] and studied further by Mathai and Haubold [6, 7]. For real scalar case the pathway model is represented by the probability density function

$$f\,(x) \;=\; c\,|\,x\,|^{\gamma-1}\,\left[\,1 - a\,(1-\alpha)\;|x|^{\;\delta}\,\right]^{\,\frac{\beta}{1-\alpha}}, \tag{1.2}$$

where $\delta\, >\, 0,\ \beta\, \geq\, 0,\ 1 - a\,(1-\alpha)\;|x|^{\delta}\, >\, 0,\ \gamma\, >\, 0$, c is the normalizing constant, and α is the pathway parameter. For real α, the normalizing constant c is calculated in [8]. It is interesting to observe that for $\alpha <\, 1$, (1.2) is a finite range density with $1\, -\, a\,(1-\alpha)\,|x|^{\delta} > 0$ and it is an extended generalized type-1 beta model.

If $\alpha >\, 1$, then, writing $1-\alpha = -(\alpha - 1)$, we have

$$f\,(x) \;=\; c\;|x|^{\gamma-1}\,\left[\,1 + a\,(\alpha-1)\;|x|^{\;\delta}\,\right]^{\,-\,\frac{\beta}{\alpha-1}}\,,$$

which is the extended generalized type-2 beta model for real x.

Belarbi and Dahmani [1] investigated new integral inequalities for Chebyshev functional by making use of the Riemann–Liouville fractional integral operator. Saxena et al. [11] investigated some integral inequalities for Chebyshev functional by using of the Saigo fractional integral operator. This has motivated the authors to derive certain integral inequalities associated with pathway fractional integral operator (see Section 2).

For the proof of main theorems we need the following lemma.

Lemma 1 (see [8])**.** *Let* $\eta, \beta \in \mathbb{C},\ \ \Re\,(\eta) > 0,$ *and* $\alpha < 1$*. If* $\Re(\beta) > 0,\ \ \Re\left(\frac{\eta}{1-\alpha}\right) > -1$*, then for* $f(t) = 1$ *in equation* (1.1) *we have*

$$P_{0+}^{(\eta,\,\alpha)}\,(1) \;=\; x^{\eta} \int_0^{\left[\frac{x}{a\,(1-\alpha)}\right]} \left[1 - \frac{a\,(1-\alpha)\,t}{x}\right]^{\frac{\eta}{1-\alpha}} dt. \tag{1.3}$$

When $f\,(t) = x^{\beta-1}$*, then it gives*

$$P_{0+}^{(\eta,\,\alpha)}\;\left(x^{\beta-1}\right) = \frac{x^{\eta\,+\,\beta}}{[a\,(1-\alpha)]^{\,\beta}}\,\frac{\Gamma\,(\beta)\,\Gamma\,\left[1 + \frac{\eta}{1-\alpha}\right]}{\Gamma\,\left[\,\frac{\eta}{1-\alpha} + \beta + 1\right]}. \tag{1.4}$$

When $\beta\, =\, 1$*, then equation* (1.4) *yields*

$$P_{0\,+}^{(\eta\,,\,\alpha)}\,(1) = \frac{x^{\eta\,+\,1}\,\Gamma\,\left[1 + \frac{\eta}{1-\alpha}\,\right]}{[a\,(1-\alpha)]^{\,\beta}\,\Gamma\,\left[\,\frac{\eta}{1-\alpha}\, + 2\right]}. \tag{1.5}$$

2. Main results

Theorem 1. *Let f and g be two synchronous functions on $[0,\infty]$. Then the following inequality holds:*

$$\left(P_{0+}^{\eta,\alpha} fg\right)(x) \geq \frac{[a\,(1-\alpha)]\;\Gamma\left(2+\frac{\eta}{1-\alpha}\right)}{x^{\eta+1}\,\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}\left(P_{0+}^{\eta,\alpha} f\right)(x)\,\left(P_{0+}^{\eta,\alpha} g\right)(x). \tag{2.1}$$

Proof. Let f and g be two synchronous functions on $[0,\infty]$. Then for all $\tau,\rho\geq 0$ we clearly have

$$f(\tau)\,g(\tau) + f(\rho)\,g(\rho) \geq f(\tau)\,g(\rho) + f(\rho)\,g(\tau).$$

Multiplying both sides of this inequality by

$$x^{\eta+1}\left(1-\frac{a\,(1-\alpha)\,\tau}{x}\right)^{\frac{\eta}{1-\alpha}}$$

and integrating over the interval $\left(0,\,\frac{x}{a\,(1-\alpha)}\right)$, we find that

$$\begin{aligned}
&x^{\eta+1}\int_0^{\frac{x}{a\,(1-\alpha)}}\left(1-\frac{a\,(1-\alpha)\,\tau}{x}\right)^{\frac{\eta}{1-\alpha}} f(\tau)\,g(\tau)\,\mathrm{d}\tau\\
&\quad + x^{\eta+1}\int_0^{\frac{x}{a\,(1-\alpha)}}\left(1-\frac{a\,(1-\alpha)\,\tau}{x}\right)^{\frac{\eta}{1-\alpha}} f(\rho)\,g(\rho)\,\mathrm{d}\tau\\
&\geq x^{\eta+1}\int_0^{\frac{x}{a\,(1-\alpha)}}\left(1-\frac{a\,(1-\alpha)\,\tau}{x}\right)^{\frac{\eta}{1-\alpha}} f(\tau)\,g(\rho)\,\mathrm{d}\tau\\
&\quad + x^{\eta+1}\int_0^{\frac{x}{a\,(1-\alpha)}}\left(1-\frac{a\,(1-\alpha)\,\tau}{x}\right)^{\frac{\eta}{1-\alpha}} f(\rho)\,g(\tau)\,\mathrm{d}\tau.
\end{aligned}$$

Using equations (1.1) and (1.3), this simplifies to

$$\left(P_{0+}^{\eta,\alpha} fg\right)(x)+f(\rho)\,g(\rho)\,P_{0+}^{\eta,\alpha}(1) \geq g(\rho)\left(P_{0+}^{\eta,\alpha} f\right)(x)+f(\rho)\left(P_{0+}^{\eta,\alpha} g\right)(x). \tag{2.2}$$

Now, multiplying both sides of (2.2) by

$$x^{\eta+1}\left(1-\frac{a\,(1-\alpha)\,\rho}{x}\right)^{\frac{\eta}{1-\alpha}}$$

and integrating again over $\left(0,\,\frac{x}{a\,(1-\alpha)}\right)$, we find that

$$\begin{aligned}
&\left(P_{0+}^{\eta,\alpha} fg\right)(x)\int_0^{\frac{x}{a\,(1-\alpha)}} x^{\eta+1}\left(1-\frac{a\,(1-\alpha)\,\rho}{x}\right)^{\frac{\eta}{1-\alpha}} d\rho\\
&\quad + P_{0+}^{\eta,\alpha}(1)\int_0^{\frac{x}{a\,(1-\alpha)}} x^{\eta+1}\left(1-\frac{a\,(1-\alpha)\,\rho}{x}\right)^{\frac{\eta}{1-\alpha}} f(\rho)\,g(\rho)\,d\rho\\
&\geq \left(P_{0+}^{\eta,\alpha} f\right)(x)\int_0^{\frac{x}{a\,(1-\alpha)}} x^{\eta+1}\left(1-\frac{a\,(1-\alpha)\,\rho}{x}\right)^{\frac{\eta}{1-\alpha}} g(\rho)\,d\rho\\
&\quad + \left(P_{0+}^{\eta,\alpha} g\right)(x)\int_0^{\frac{x}{a\,(1-\alpha)}} x^{\eta+1}\left(1-\frac{a\,(1-\alpha)\,\rho}{x}\right)^{\frac{\eta}{1-\alpha}} f(\rho)\,d\rho.
\end{aligned}$$

By virtue of (1.1) and (1.3) it is seen that

$$\left(P_{0+}^{\eta,\alpha} fg\right)(x) \;\geq\; \frac{1}{P_{0+}^{\eta,\alpha}\,(1)}\;\left(P_{0+}^{\eta,\alpha} f\right)(x)\;\left(P_{0+}^{\eta,\alpha} g\right)(x),$$

and (2.1) follows in view of (1.5). □

If we set $\alpha = 0$, $a = 1$, and replace η with $\eta - 1$ in equation (2.1), then it reduces to the result for the Riemann–Liouville fractional integral operator, given by Belarbi and Dahmani [1].

Next we establish the following theorem.

Theorem 2. *Let f and g be two synchronous functions on $[0,\infty]$. Then for all $\alpha > 0$ and $\gamma, \eta \in \mathbb{R}$ we have*

$$\begin{aligned}&\left(P_{0+}^{\eta,\alpha} fg\right)(x)\; P_{0+}^{\eta,\gamma}\,(1) + P_{0+}^{\eta,\alpha}\,(1)\;\left(P_{0+}^{\eta,\gamma} fg\right)(x)\\ &\geq \left(P_{0+}^{\eta,\alpha} f\right)(x)\;\left(P_{0+}^{\eta,\gamma} g\right)(x) + \left(P_{0+}^{\eta,\gamma} f\right)(x)\;\left(P_{0+}^{\eta,\alpha} g\right)(x),\end{aligned} \tag{2.3}$$

where $P_{0+}^{\eta,\alpha}\,(1)$ and $P_{0+}^{\eta,\gamma}\,(1)$ are determined by (1.5).

Proof. Using similar arguments as in the proof of Theorem 1, we can write

$$\begin{aligned}&x^{\eta+1}\left(1-\frac{a\,(1-\gamma)\,\rho}{x}\right)^{\frac{\eta}{1-\gamma}}\;\left(P_{0+}^{\eta,\alpha} fg\right)(x)\\ &\quad + P_{0+}^{\eta,\alpha}\,(1)\;x^{\eta+1}\left(1-\frac{a\,(1-\gamma)\,\rho}{x}\right)^{\frac{\eta}{1-\gamma}}\;f(\rho)\,g(\rho)\\ &\geq x^{\eta+1}\left(1-\frac{a\,(1-\gamma)\,\rho}{x}\right)^{\frac{\eta}{1-\gamma}}\;g(\rho)\;\left(P_{0+}^{\eta,\alpha} f\right)(x)\\ &\quad + x^{\eta+1}\left(1-\frac{a\,(1-\gamma)\,\rho}{x}\right)^{\frac{\eta}{1-\gamma}}\;f(\rho)\;\left(P_{0+}^{\eta,\alpha} g\right)(x)\,).\end{aligned}$$

If we integrate this equation over $\left(0, \frac{x}{a\,(1-\gamma)}\right)$, we find that

$$\begin{aligned}&\left(P_{0+}^{\eta,\alpha} fg\right)(x)\int_0^{\frac{x}{a\,(1-\gamma)}} x^{\eta+1}\left(1-\frac{a\,(1-\gamma)\,\rho}{x}\right)^{\frac{\eta}{1-\gamma}}\,d\rho\\ &\quad + P_{0+}^{\eta,\alpha}\,(1)\int_0^{\frac{x}{a\,(1-\gamma)}} x^{\eta+1}\left(1-\frac{a\,(1-\gamma)\,\rho}{x}\right)^{\frac{\eta}{1-\gamma}}\,f(\rho)\,g(\rho)\,d\rho\\ &\geq \left(P_{0+}^{\eta,\alpha} f\right)(x)\int_0^{\frac{x}{a\,(1-\gamma)}} x^{\eta+1}\left(1-\frac{a\,(1-\gamma)\,\rho}{x}\right)^{\frac{\eta}{1-\gamma}}\,g\,(\rho)\,d\rho\\ &\quad + \left(P_{0+}^{\eta,\alpha} g\right)(x)\int_0^{\frac{x}{a\,(1-\gamma)}} x^{\eta+1}\left(1-\frac{a\,(1-\gamma)\,\rho}{x}\right)^{\frac{\eta}{1-\gamma}}\,f(\rho)\,d\rho.\end{aligned}$$

Using equalities (1.3) and (1.5), this yields (2.3). Theorem is proved. □

If we set $\alpha = 0$, $a = 1$, and use $\eta - 1$ instead of η in equation (2.3), then it reduces to the result for the Riemann–Liouville fractional integral operator, given by Belarbi and Dahmani [1].

Finally we prove the following result.

Theorem 3. *Let f_i, $i = 1, 2, \dots n$, be positive increasing functions on $[0, \infty]$. Then for any $t > 0$, $\alpha > 0$, and $\eta \in \mathbb{R}$ we have*

$$\left(P_{0+}^{\eta,\alpha} \prod_{i=1}^{n} f_i \right)(x) \geq \left[P_{0+}^{\eta,\alpha}\, (1) \right]^{1-n} \prod_{i=1}^{n} \left(P_{0+}^{\eta,\alpha} f_i \right)(x) . \tag{2.4}$$

Proof. We prove this theorem by the method of mathematical induction. It is clear that (2.4) holds in the case when $n = 1$. Suppose that (2.4) is satisfied for $k - 1$, $2 \leq k \leq n$, i.e.,

$$\left(P_{0+}^{\eta,\alpha} \prod_{i=1}^{k-1} f_i \right)(x) \geq \left[P_{0+}^{\eta,\alpha}\, (1) \right]^{2-k} \prod_{i=1}^{k-1} \left(P_{0+}^{\eta,\alpha} f_i \right)(x) . \tag{2.5}$$

Since f_i, $i = 1, 2, \dots k$, are positive increasing functions, $\prod_{i=1}^{k-1} f_i$ is also an increasing function. Hence we can apply Theorem 1 to the functions $f = \prod_{i=1}^{k-1} f_i$ and $g = f_k$ to obtain

$$\left(P_{0+}^{\eta,\alpha} \prod_{i=1}^{k} f_i \right)(x) \geq \left[P_{0+}^{\eta,\alpha}\, (1) \right]^{-1} \left(P_{0+}^{\eta,\alpha} \prod_{i=1}^{k-1} f_i \right)(x) \ \left(P_{0+}^{\eta,\alpha} f_k \right)(x) .$$

Now, taking into account (2.5), we get (2.4) with $n = k$. □

If we set $\alpha = 0$, $a = 1$, and write $\eta - 1$ instead of η in equation (2.4), then it reduces to the result for the Riemann–Liouville fractional integral operator, given by Belarbi and Dahmani [1].

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