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Automorphisms with annihilator condition in prime rings

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Abstract. Let R be a prime ring, I a nonzero ideal of R, and $a \in R$. Suppose that σ is a nontrivial automorphism of R such that $a\{(\sigma(x \circ y))^n - (x \circ y)^m\} = 0$ or $a\{(\sigma([x, y]))^n - ([x, y])^m\} = 0$ for all $x, y \in I$, where n and m are fixed positive integers. We prove that if $\mathrm{char}(R) > n + 1$ or $\mathrm{char}(R) = 0$, then either $a = 0$ or R is commutative.

1. Introduction

In all that follows, unless stated otherwise, R will be an associative ring, Z the center of R, and Q its two sided Martindale quotient ring. The center of Q, denoted by C, is called the extended centroid of R. We refer the reader to [3] for the definitions and related properties of these objects. For $x, y \in R$, we denote $[x, y] = xy - yx$, the commutator of x and y, and $x \circ y = xy + yx$, the anti-commutator (skew-commutator) of x and y. Recall that a ring R is prime if for any $a, b \in R$, the equality $aRb = 0$ implies that $a = 0$ or $b = 0$. In addition, s_4 denotes the standard identity in 4 variables. For a subset S of R, a mapping $f\colon S \to R$ is called commuting (centralizing) if $[f(x), x] = 0$ (resp. $[f(x), x] \in Z$) for all $x \in S$. A mapping $f\colon S \to R$ is called skew-commuting (skew-centralizing) on R if $f(x) \circ x = 0$ (resp. $f(x) \circ x \in Z$) holds for all $x \in S$. The study of commuting and centralizing mappings goes back to 1955 when Divinsky [17] proved that a simple Artinian ring is commutative if it has a commuting automorphism different from the identity mapping. Two year later, Posner [27] showed that a prime ring must be commutative if it possesses a nonzero centralizing derivation. In 1970, Luh [24] generalized Divinsky's result to prime rings. Later, Mayne [26] obtained the analogous result to Posner's for nonidentity centralizing automorphisms. Similar results were extended to the case of left ideals by Bell and Martindale [4] and Lanski [22].

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In [5], Bresar obtained a characterization of commuting additive mappings on prime rings. Based on this result, Bresar initiated the study of functional identities. We refer the reader to Bresar [7] for an introductory account on functional identities and their applications. In [6], Bresar proved that there are no nonzero skew-commuting additive mappings on a 2-torsion free semiprime rings. In other words, if R is a 2-torsion free semiprime ring and $f\colon R \to R$ an additive mapping such that $f(x) \circ x = 0$ for all $x \in R$, then $f = 0$.

A well known theorem of Posner [27] states that if R is prime and the commutator $[d(x), x] \in Z$ for all $x \in R$, then either $d = 0$ or R is commutative. This result of Posner was generalized in many directions by several authors and they studied the relationship between the structure of prime or semiprime ring and the behaviour of additive maps satisfying various conditions. Some authors have studied derivations with annihilator conditions in prime and semiprime rings. For example, we refer the reader to [14], [15], and [16].

In 2002, Ashraf and Rehman [2] proved that if R is a prime ring, I is a nonzero ideal of R, and d is a derivation of R such that $d(x \circ y) - (x \circ y) = 0$ for all $x, y \in I$, then R is commutative. Further, Argac and Inceboz [1] generalized this result by obtaining the following: if R is a prime ring, I is a nonzero ideal of R and d is a derivation of R, n is a fixed positive integer and $(d(x \circ y))^n - (x \circ y) = 0$ or is central for all $x, y \in I$, then R is commutative. We continued this line of investigation in [28] by examining what happens in the case when the derivation is replaced by an automorphism, and proved that if R is a prime ring with center Z, I is a nonzero ideal of R and σ a nontrivial automorphism of R such that $\{\sigma(x \circ y) - (x \circ y)\}^n \in Z$ for all $x, y \in I$, and either $\operatorname{char}(R) > n$ or $\operatorname{char}(R) = 0$, then R satisfies s_4, the standard identity in 4 variables. Also, Daif and Bell [12] proved that if R is a semiprime ring, I is a nonzero ideal of R, and d is a derivation of R such that $d([x, y]) - [x, y] = 0$ for all $x, y \in I$, then I is central. In particular, in the prime case, R is commutative. De Filippis [13] obtained a result which has the same flavour replacing the derivation d by a nontrivial automorphism σ. To be more precise, it is proved that if R is a non-commutative prime ring, I is a nonzero ideal of R, and σ is nontrivial automorphism of R such that $\sigma([x, y]) - [x, y]$ is zero or invertible for all $x, y \in I$, then either R is a division ring or R is the ring of all 2×2 matrices over a division ring. Motivated by these cited results, in the present paper we study the above situation under an annihilator condition in a more general way.

2. Main results

An automorphism σ of R is called Q-inner if there exists an invertible element $b \in Q$ such that $\sigma(x) = bxb^{-1}$ for all $x \in R$. Otherwise, σ is called outer. We denote by A the group of all automorphisms of R, and by G_i the group consisting of all Q-inner automorphisms of R. Recall that a subset S of A is said to be independent (modulo G_i) if, for any $g_1, g_2 \in S$, $g_1 g_2^{-1} \in G_i$ implies $g_1 = g_2$. For instance, if g is an outer automorphism of R, then 1 and g are independent (modulo G_i). The following useful result is due to Chuang [9]. We only state a special case needed for our proofs below, and refer for its proof to that of Chuang [9].

Lemma 1 ([9], Theorem 3)**.** *Suppose that R is a prime ring and S an independent subset of A modulo G_i. Let $\phi = \psi(x_i{}^{g_j}) = 0$ be a generalized identity with automorphisms of R reduced with respect to S. If, for all $x_i \in X$ and $g_j \in S$, the $x_i{}^{g_j}$-word degree of $\phi = \psi(x_i{}^{g_j})$ is strictly less than* $\mathrm{char}(R)$ *when* $\mathrm{char}(R) \neq 0$*, then $\psi(z_{ij}) = 0$ is also a generalized polynomial identity of R.*

Now, we are well equipped to prove our main result.

Theorem 2. *Let R be a prime ring, I a nonzero ideal of R, σ a nontrivial automorphism of R, $a \in R$, and let $n, m \geq 1$ be fixed integers. Suppose that $a\{(\sigma(x \circ y))^n - (x \circ y)^m\} = 0$ for all $x, y \in I$. If either* $\mathrm{char}(R) = 0$ *or* $\mathrm{char}(R) > n + 1$*, then either $a = 0$ or R is commutative.*

Proof. Assume that $a \neq 0$. The aim is to prove that R is commutative. We have

$$a\{(\sigma(x \circ y))^n - (x \circ y)^m\} = 0 \text{ for all } x, y \in I. \tag{1}$$

Case 1. Let σ be not Q-inner. Since either $\mathrm{char}(R) > n+1$ or $\mathrm{char}(R) = 0$, by Lemma 1, we arrive at

$$a\{(s \circ t)^n - (x \circ y)^m\} = 0 \text{ for all } x, y, s, t \in I. \tag{2}$$

In particular, if $s = 0$, we have

$$a(x \circ y)^m = 0 \text{ for all } x, y \in I.$$

Since $a \neq 0$, by Chuang and Lee [11], it follows that $(x \circ y)^m = 0$ for all $x, y \in I$. This implies that $(2x^2)^m = 0$ (for $x = y$), which is a contradiction for prime rings (note that $\mathrm{char}(R) > n + 1$ means that $\mathrm{char}(R) \neq 2$).

Case 2. Suppose that σ is Q-inner. Then there exists an invertible element $b \in Q - C$ such that $\sigma(x) = b^{-1}xb$ for all $x \in I$. By Chuang [8, Theorem 2], I and Q satisfy the same generalized polynomial identities (GPIs). From (1), we obtain that

$$a\{(b^{-1}(x \circ y)b)^n - (x \circ y)^m\} = 0 \text{ for all } x, y \in Q. \tag{3}$$

Denote by $\bar{C}$ either the algebraic closure of C or C according as C is either infinite or finite, respectively. Then, by a standard argument (see for instance [23, Proposition]), (3) is also a GPI for $Q \otimes_C \bar{C}$. Since $Q \otimes_C \bar{C}$ is a centrally closed prime $\bar{C}$- algebra [18, Theorem 2.5 and Theorem 3.5], by replacing R

and C with $Q \otimes_C \bar{C}$ and $\bar{C}$, respectively, we may assume that R is centrally closed and C is either finite or algebraic closed. By Martindale [25, Theorem 3], R is a primitive ring having nonzero socle H with C as its associated division ring. Hence, by Jacobson's theorem [20, p.75], R is isomorphic to a dense ring of linear transformations of some vector space V over C, and H consists of finite rank linear transformations in R.

Firstly, remark that, since $b \notin C$, then R cannot be commutative and so $\dim_C V \geq 2$. We want to show that v and bv are linearly C-dependent for all $v \in V$. If $bv = 0$ then $\{v, bv\}$ are C-dependent. Suppose that $bv \neq 0$. Assume that $\{v, bv\}$ is C-independent for some $v \in C$.

If $b^{-1}v \notin \text{Span}_C\{v, bv\}$, then $\{v, bv, b^{-1}v\}$ are linearly C-independent. By the density of R, there exist $x, y \in R$ such that

$$\begin{array}{lll} xv = 0, & xbv = b^{-1}v, & xb^{-1}v = 0, \\ yv = 0, & ybv = 0, & yb^{-1}v = bv, \end{array}$$

and it follows from (3) that $0 = a\{(b^{-1}(xy + yx)b)^n - (xy + yx)^m\}v = av$. This implies that if $av \neq 0$, then $b^{-1}v \in \text{Span}_C\{v, bv\}$, a contradiction.

Now suppose that $av = 0$. Since $a \neq 0$, there exists $b^{-1}v \in V$ such that $ab^{-1}v \neq 0$, and then $a(v + b^{-1}v) = ab^{-1}v \neq 0$. By the previous argument, we find that $b^{-1}v$ and v are linearly C-dependent, and also so are $v + b^{-1}v$ and $b(v + b^{-1}v)$. Thus there exist $\alpha, \beta \in C$ such that $bb^{-1}v = \alpha b^{-1}v$ and $b(v + b^{-1}v) = \beta(v + b^{-1}v)$. Moreover, v and $b^{-1}v$ are linearly C-independent and so by density there exist $x, y \in R$ such that

$$\begin{array}{ll} xb^{-1}v = -b^{-1}v, & xv = 0, \\ yb^{-1}v = (\alpha - \beta)b^{-1}v, & yv = v. \end{array}$$

Then we obtain that $0 = a\{(b^{-1}(xy+yx)b)^n - (xy+yx)^m\}b^{-1}v = 2^m ab^{-1}v(\alpha - \beta)^m$. Since $\text{char}(R) \neq 2$, this implies that $ab^{-1}v(\alpha - \beta)^m = 0$. Since $ab^{-1}v \neq 0$, one has $\alpha = \beta$, and so $bv = \alpha v$ are linearly C-dependent, that is $bv = \alpha_v v$ for some $\alpha_v \in C$. It is very easy to prove that α_v is independent of the choice of $v \in V$. Thus we can write $bv = \alpha v$ for all $v \in V$ and $\alpha \in C$ fixed, a contradiction.

If $b^{-1}v \in \text{Span}_C\{v, bv\}$, then $b^{-1}v = \gamma v + \delta bv$ for some $\gamma, \delta \in C$. Since v and bv are linearly C- independent, by the density of R, there exist $x, y \in R$ such that

$$\begin{array}{ll} xv = v, & xbv = 0, \\ yv = -v, & ybv = bv, \end{array}$$

and it follows from (3) that $0 = a\{(b^{-1}(xy + yx)b)^n - (xy + yx)^m\}v = 2^m av$. Since $\text{char}(R) \neq 2$, this implies that $av = 0$. Now, by the previous proof, we have that v and bv are linearly C-dependent, and we arrive at a contradiction. This completes the proof. $\square$

Replace the anti-commutator by commutator and use the same techniques with necessary variations to get the following result.

Theorem 3. *Let R be a prime ring, I a nonzero ideal of R, σ a nontrivial automorphism of R, $a \in R$, and let $n, m \geq 1$ be fixed integers. Suppose that $a\{(\sigma([x,y]))^n - ([x,y])^m\} = 0$ for all $x, y \in I$. If either* $\mathrm{char}(R) = 0$ *or* $\mathrm{char}(R) > n+1$, *then either $a = 0$ or R is commutative.*

The following example demonstrates that the hypothesis R to be prime is essential in the above theorems.

Example 4. Let $M_2(F)$ and $N_2(F)$ denote the 2×2 upper triangular and srtictly upper triangular matrix ring over a field F, respectively. Clearly $N_2(F)$ is an ideal of $M_2(F)$. Let $R = M_2(F) \times M_2(F)$ and $I = N_2(F) \times 0$; then R is a semiprime ring and I is a nonzero ideal of R. Let $a(\neq 0)$ be an element in R. Define an automorphism σ of R by $\sigma(x_1, x_2) = (x_2, x_1)$ for all $x_1, x_2 \in M_2(F)$. Clearly, $a\{(\sigma(x \circ y))^n - (x \circ y)^m\} = 0$ and $a\{(\sigma([x,y]))^n - ([x,y])^m\} = 0$ for all $x, y \in I$, but R is not commutative.

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