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On uniform convergence for distribution functions generated by stable laws

A. H. Arakelyan

Abstract. Two two-parametric families of densities generated by stable laws are considered. The uniform convergence by both parameters for the introduced families is established.

0. Introduction

The stable laws are a rich class of probability families that allow an incresingly wide range of applications in such fields as finance, physics, engineering and bioinformatics. Historically the class was extracted by Paul Lévy in his study of infinitely divisible distributions in the 1920s. More precisely, the canonical representations for the logarithm of characteristic function of infinitely divisible distributions has been found. It allowed to obtain the logarithm of characteristic function for all stable laws. Such a canonical representation of stable laws became a powerful tool for many new results. For instance, the existence of continuous densities for series expansions for them. Unfortunately, the lack of closed formulas for densities and distribution functions for all but a few stable distributions (normal, Lévy's, Cauchy's) has been a major drawback to the use of stable distributions by practitioners. Anyway, the series expansions and integral representations became a strong source for obtaining many facts on stable laws (see [3], [7]).

A basic topic of any statistical inference of biomolecular systems is the characterization of the distributions of object frequencies for a population, so-called frequency distributions (FD), say $\{p_n\}$ (see [2]). Based on huge datasets of such systems several common statistical facts on the frequency distribution have been discovered. From the mathematical point of view

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these are: skewness to the right; regular variation at infinity; upward/downward convexity; continuity by parameters (stability); unimodality, etc. of frequency distributions (see [1]–[3], [5], [6]). Some well-known distributions are widely used in large-scale biomolecular systems. But the variety of such systems requires to generate new ones that satisfy the empirical facts above.

Stable densities concentrated on $[0,+\infty)$ vary regularly at infinity with exponent $-\rho$, $\rho \in (1,2)$. Then their distribution functions have only (right) tail which varies regularly at infinity with exponent $-\rho+1$. They satisfy all other above statistical facts and at once may be suggested as infinite differentiable analogs of empirical frequency distributions in large-scale biomolecular models. Other right-side stable densities are concentrated on $(-\infty,+\infty)$. But their left tails are extremely small in comparison with the right tails. Therefore, we may give a manner how to construct with the help of them infinite differentiable analogs of $\{p_n\}$. But more perspective is to use two-parametric family of symmetric stable densities in order to construct infinite differentiable analogs of $\{p_n\}$. It is possible if we consider the random variables being the absolute values of symmetrically distributed stable random variables. Usually stable laws are functions of four parameters. Let $S(x;\alpha,\beta,\sigma,\gamma)$ denote a stable distribution function. Let us describe the parameters of stable laws. The first essential parameter $\alpha \in (0,2]$ is the exponent, which defines the exponent $-\rho$ of stable law density's regular variation $\rho = \alpha+1$. Excluding the normal law $\alpha = 2$ any stable law $\alpha \in (0,2)$ has infinite variance. Denoting by S_α the stable law with exponent $\alpha \in (0,2)$ consider its two tails: $S_\alpha(-x)$ (left tail) and $1-S_\alpha(x)$ (right tail) for $x \in R^+$. The second essential parameter for S_α is the asymmetry, i.e., the value of the limit

$$\beta = \lim_{x\to\infty} \frac{1-S_\alpha(x)-S_\alpha(-x)}{1-S_\alpha(x)+S_\alpha(-x)} \in [-1,1],$$

which always exists. In other words, the asymmetry is nothing else, but the measure of skewness for $S_\alpha(x)$. Due to empirical facts, we are interested in stable laws with maximal skewness to the right, i.e., in $S_\alpha(x)$ with $\beta = +1$. The remained two parameters (shifting parameter and scale factor) are non-essential. The next condition, which has to be fulfilled, if we want to use stable laws in bioinformatics, consists in the following. The extracted densities of stable laws, which are assumed to be continuous analogs of FDs, must be concentrated in $[0,+\infty)$. Denote by $s(x;\alpha,\beta)$ a stable density with exponent α and asymmetry β.

For our purposes we may use not only $s(x;\alpha,1)$, $0<\alpha<1$ (only in this case $s(x;\alpha,1)$ is concentrated on $[0,+\infty)$), but also $2\cdot s(x;\alpha,0)$, $0<\alpha<2$, for $x \in [0,+\infty)$. The density $s(x;\alpha,0)$ for $x \in R^1$ is symmetric, so $2\cdot s(x;\alpha,0)$ for $x \in [0,+\infty)$ is concentrated on $[0,+\infty)$ and has skewness to the right.

Now, the following families of two-parametric densities

$$\{\, \hat f_{\alpha,\sigma}(x) = \sigma^{-1/\alpha}\, s(x\,\sigma^{-1/\alpha};\ \alpha, 1) : \quad 0<\alpha<1\,,\ \sigma \in R^+\}, \tag{0.1}$$

$$\{\, \hat f_{\alpha,\sigma}(x) = 2\sigma^{-1/\alpha}\, s(x\,\sigma^{-1/\alpha};\ \alpha, 0) : \quad 1<\alpha<2\,,\ \sigma \in R^+\,\} \tag{0.2}$$

are candidates to be continuous analogs of FDs. From the general case, considered in 2.7 of [7], p.173, we extract that these families satisfy the majority of statistical facts and in the present paper we deal with the uniform convergence of distribution functions of two-parametric families of densities generated by stable laws. Namely, we deal with the following series expansions (see [7], pp. 109–110). For $x \in R^+$, let

$$s(x;\alpha,1) = \frac{1}{\pi}\sum_{n\ge 1}(-1)^{n-1}\frac{\Gamma(n\alpha+1)}{n\,!}\,\frac{1}{x^{n\alpha+1}}\,\sin(\pi n\alpha), \quad 0<\alpha<1,$$

and

$$s(x;\ \alpha,0) = \frac{1}{\pi}\sum_{m\ge 1}(-1)^{m-1}\frac{\Gamma(\frac{2m-1}{\alpha}+1)}{(2m-1)\,!}\,x^{2m-2}, \quad 1<\alpha<2.$$

Here $\Gamma(\cdot)$ is the Euler gamma function. It can be shown (see [7], Chapter 2) that the integral representations for corresponding distribution functions (DF) are given as follows:

$$\hat F_{\alpha,\sigma}(x) = \frac{1}{\pi}\int_0^{\pi} \exp\left(-(\sigma/x^{\alpha})^{1/(1-\alpha)}\overline{U}_{\alpha}(\varphi)\right)d\varphi\,, \text{ for } 0<\alpha<1,$$

with

$$\overline{U}_{\alpha}(\varphi) = \left(\frac{\sin(\alpha\varphi)}{\sin\varphi}\right)^{\frac{\alpha}{1-\alpha}}\frac{\sin((1-\alpha)\,\varphi))}{\sin\,\varphi}\,, \quad \varphi\in[0\,,\,\pi],$$

and

$$\hat F_{\alpha,\sigma}(x) = 1-\frac{2}{\pi}\int_0^{\pi/2} \exp\left(-(x^{\alpha}/\sigma)^{1/(\alpha-1)}\overline{V}_{\alpha}(\varphi)\right)d\varphi, \text{ for } 1<\alpha<2,$$

with

$$\overline{V}_{\alpha}(\varphi) = \left(\frac{\cos\varphi}{\sin(\alpha\varphi)}\right)^{\frac{\alpha}{\alpha-1}}\frac{\cos((\alpha-1)\,\varphi)}{\cos\varphi}\,, \quad \varphi\in[0,\,\pi/2].$$

1. Problem and results

Consider the families of DFs

$$\{\hat F_{\alpha,\sigma}(x) : 0<\alpha<1,\ \sigma\in R^+\} \tag{1.1}$$

and

$$\{\hat F_{\alpha,\sigma}(x) : 1<\alpha<2,\ \sigma\in R^+\}. \tag{1.2}$$

Denote

$$(\alpha,\sigma;\alpha',\sigma') = E\left(\alpha',\sigma';\alpha,\sigma\right) = \int_{0-}^{+\infty}\left|\hat f_{\alpha,\sigma}(x)-\hat f_{\alpha',\sigma'}(x)\right|dx, \tag{1.3}$$

where $\sigma, \sigma' \in R^+$ and either $\alpha, \alpha' \in (0,1)$ or $\alpha, \alpha' \in (1,2)$.

Let the constants $\underline{\sigma}$ and $\overline{\sigma}$ be fixed and satisfy the inequalities $0 < \underline{\sigma} < \overline{\sigma} < +\infty$. The constants $\underline{\alpha}$ and $\overline{\alpha}$ are fixed too and for the family of DFs (1.1) satisfy the inequalities $0 < \underline{\alpha} \le \overline{\alpha} < 1$. For the family of DFs (1.2) these constants satisfy the inequalities $1 < \underline{\alpha} \le \overline{\alpha} < 2$.

Theorem 1. *The limit*

$$\lim_{|\alpha-\alpha'|+|\sigma-\sigma'|\to 0} E\left(\alpha, \sigma; \alpha', \sigma'\right) = 0.$$

exists uniformly in $\alpha \in [\underline{\alpha}, \overline{\alpha}]$, $\alpha' \in [\underline{\alpha}, \overline{\alpha}]$, $\sigma \in [\underline{\sigma}, \overline{\sigma}]$, *and* $\sigma' \in [\underline{\sigma}, \overline{\sigma}]$.

Denote

$$\begin{aligned} E_1\left(\alpha; \sigma, \sigma'\right) &= E_1\left(\alpha; \sigma', \sigma\right) \\ &= \int_{0-}^{+\infty} \left|\hat{f}_{\alpha,\sigma}(x) - \hat{f}_{\alpha,\sigma'}(x)\right| dx \left(= E\left(\alpha, \sigma; \alpha, \sigma'\right)\right), \\ E_2\left(\sigma; \alpha, \alpha'\right) &= E_2\left(\sigma; \alpha', \alpha\right) \\ &= \int_{0-}^{+\infty} \left|\hat{f}_{\alpha,\sigma}(x) - \hat{f}_{\alpha',\sigma}(x)\right| dx \left(= E\left(\alpha, \sigma; \alpha', \sigma\right)\right). \end{aligned}$$

According to the inequalities

$$0 \le E\left(\alpha, \sigma; \alpha', \sigma'\right) \le E_1\left(\alpha; \sigma, \sigma'\right) + E_2\left(\sigma'; \alpha, \alpha'\right)$$

we may formulate the following remark.

Remark 1. In order to prove the statement of Theorem 1 it is enough to do it in the two particular cases. Namely, it suffices to show that the limits

$$\lim_{|\sigma-\sigma'|\to 0} E_1\left(\alpha; \sigma, \sigma'\right) = 0 \tag{1.4}$$

and

$$\lim_{|\alpha-\alpha'|\to 0} E_2\left(\sigma; \alpha, \alpha'\right) = 0 \tag{1.5}$$

simultaneously exist uniformly in $\alpha \in [\underline{\alpha}, \overline{\alpha}]$, $\alpha' \in [\underline{\alpha}, \overline{\alpha}]$, $\sigma \in [\underline{\sigma}, \overline{\sigma}]$, and $\sigma' \in [\underline{\sigma}, \overline{\sigma}]$.

2. Preliminary estimations I

Lemma 1. *Given* $\varepsilon \in (0,1)$ *in conditions of Theorem 1 for DFs there is a number* $x_0 \in R^+$ *(*x_0 *does not depend on* α *and* σ*) such that for all* $x \in [x_0, +\infty)$ *we have*

$$1 - \hat{F}_{\alpha,\sigma}(x) < \frac{\varepsilon}{16}. \tag{2.1}$$

Proof. The case $0 < \alpha < 1$. We deal with the following series expansion ([7], pp. 108–109):

$$1 - \hat{F}_{\alpha,1}(x) = \frac{1}{\pi\alpha} \sum_{n\geq 1} (-1)^{n-1} \frac{\Gamma(n\alpha+1)}{n \cdot n\,!} \sin(\pi n\alpha) \frac{1}{x^{n\alpha}}. \tag{2.2}$$

By the asymptotic formula (see [4], formula 8.327 on p. 886)

$$\Gamma(x) \approx x^{x-(1/2)} \cdot e^{-x} \cdot \sqrt{2\pi}, \quad x \to \infty, \tag{2.3}$$

we get

$$\frac{\Gamma(n\overline{\alpha}+1)}{n \cdot n!} \approx \frac{\overline{\alpha}^{1/2}}{n} \cdot \left(\frac{e}{n}\right)^{n(1-\overline{\alpha})} \cdot \overline{\alpha}^{\overline{\alpha}\cdot n}, \quad n \to \infty,$$

which proves the convergence of the series $\frac{1}{\pi\underline{\alpha}} \sum_{n\geq 1} \frac{\Gamma(n\overline{\alpha}+1)}{n \cdot n!}$. That is why for a given $\varepsilon \in (0,1)$ there is an integer $n_0 > 1$ such that

$$\frac{1}{\pi\underline{\alpha}} \sum\nolimits_{n\geq n_0} \frac{\Gamma(n\overline{\alpha}+1)}{n \cdot n!} < \frac{\varepsilon}{16}. \tag{2.4}$$

Note that $1 - \hat{F}_{\alpha,\sigma}(x) = 1 - \hat{F}_{\alpha,1}(\sigma^{-1/\alpha}x)$. Thus for any $x \in (\max(1, (\overline{\sigma})^{1/\alpha}), +\infty)$ from (2.2) we have

$$\begin{aligned} 0 \leq 1 - \hat{F}_{\alpha,\sigma}(x) &\leq \frac{1}{\pi\underline{\alpha}} \sum_{n\geq 1} \frac{\Gamma(n\overline{\alpha}+1)}{n \cdot n\,!} \frac{\overline{\sigma}^n}{x^{n\underline{\alpha}}} \\ &< \frac{1}{\pi\underline{\alpha}} \sum_{n\geq 1} \frac{\Gamma(n\overline{\alpha}+1)}{n \cdot n\,!}, \end{aligned}$$

which, due to (2.4), proves Lemma 1 in this case.

The case $1 < \alpha < 2$. Denote by $S(x; \alpha, 0)$ the DF corresponding to the density $s(x;\ \alpha, 0)$. According to (0.2), we have

$$\begin{aligned} 1 - \hat{F}_{\alpha,\sigma}(x) &= 2(1 - S(\sigma^{-1/\alpha}x; \alpha, 0)) \leq 2(1 - S(\overline{\sigma}^{-1/\alpha}x; \alpha, 0)) \\ &\leq 2(1 - S(\overline{\sigma}^{-1/\underline{\alpha}}x; \underline{\alpha}, 0)) = 1 - \hat{F}_{\underline{\alpha},\overline{\sigma}}(x). \end{aligned} \tag{2.5}$$

Since

$$2(1 - S(\overline{\sigma}^{-1/\underline{\alpha}}x; \underline{\alpha}, 0)) \approx \frac{2\overline{\sigma}\Gamma(\underline{\alpha})\sin(\pi(2-\underline{\alpha})/2)}{\pi x^{\underline{\alpha}}}(1 + O(x^{-2\underline{\alpha}})),$$

where constants in $O(\cdot)$ depends only on $\underline{\alpha}$ and $\overline{\sigma}$ (see [7], p. 116, with $\alpha = \underline{\alpha}$ and $\beta = 0$), $1 - \hat{F}_{\underline{\alpha},\overline{\sigma}}(x)$ varies regularly at infinity with exponent $-\underline{\alpha}$. Hence, for a given $\varepsilon \in (0,1)$ there is a number $x_0 \in R^+$ such that $1 - \hat{F}_{\underline{\alpha},\overline{\sigma}}(x) < \varepsilon/16$ for all $x \in (x_0, +\infty)$, which together with (2.5) and $x \in (1, +\infty)$ prove (2.1) in this case. □

3. Preliminary estimations II

Denote

$$\gamma(\alpha,\sigma,\alpha',\sigma') = \left|\hat{f}_{\alpha,\sigma}(0) - \hat{f}_{\alpha',\sigma'}(0)\right|. \tag{3.1}$$

According to (0.1)–(0.2), in the case $1<\alpha<2$,

$$\hat{f}_{\alpha,\sigma}(0) = \sigma^{-1/\alpha}\frac{2}{\pi}\Gamma\left(1+\frac{1}{\alpha}\right), \tag{3.2}$$

and in the case $0<\alpha<1$, obviously, $\hat{f}_{\alpha,\sigma}(0) = 0$.

Lemma 2. *In conditions of Theorem 1, for a given $\varepsilon \in (0,1)$, the inequalities*

$$0 \le \overline{\lim_{|\alpha-\alpha'|+|\sigma-\sigma'|\to 0}} \gamma(\alpha,\sigma,\alpha',\sigma') < \frac{\varepsilon}{16}$$

hold uniformly in $\alpha \in [\underline{\alpha},\overline{\alpha}], \alpha' \in [\underline{\alpha},\overline{\alpha}], \sigma \in [\underline{\sigma},\overline{\sigma}]$, and $\sigma' \in [\underline{\sigma},\overline{\sigma}]$. This limit relationship implies that, for $|\alpha-\alpha'| + |\sigma-\sigma'|$ small enough,

$$\gamma(\alpha,\sigma,\alpha',\sigma') < \frac{\varepsilon}{8}.$$

Proof. The case $0<\alpha<1$ is obvious. Consider the case $1<\alpha<2$. Due to (3.2), we have

$$\begin{aligned}
\frac{\pi}{2}\gamma(\alpha,\sigma,\alpha',\sigma') &= \left|\sigma^{-1/\alpha}\Gamma\left(1+\frac{1}{\alpha}\right) - (\sigma')^{-1/\alpha'}\Gamma\left(1+\frac{1}{\alpha'}\right)\right| \\
&= \frac{1}{\sigma^{1/\alpha}(\sigma')^{1/\alpha'}}\left|(\sigma')^{1/\alpha'}\Gamma\left(1+\frac{1}{\alpha}\right) - \sigma^{1/\alpha}\Gamma\left(1+\frac{1}{\alpha'}\right)\right| \\
&\le A_1\left|(\sigma')^{1/\alpha'} - (\sigma')^{1/\alpha}\right| + A_1\left|(\sigma')^{1/\alpha} - \sigma^{1/\alpha}\right| \\
&\quad + A_2\left|\Gamma\left(1+\frac{1}{\alpha}\right) - \Gamma\left(1+\frac{1}{\alpha'}\right)\right|,
\end{aligned} \tag{3.3}$$

where

$$A_1 = (\underline{\sigma})^{-2/\overline{\alpha}}\Gamma\left(1+\tfrac{1}{\underline{\alpha}}\right), \quad A_2 = (\overline{\sigma})^{1/\underline{\alpha}}.$$

Here the monotonicity of gamma function was used. Next, we have

$$\lim_{|\alpha-\alpha'|\to 0}\left|\Gamma\left(1+\frac{1}{\alpha}\right) - \Gamma\left(1+\frac{1}{\alpha'}\right)\right| = 0 \tag{3.4}$$

uniformly in $\alpha \in [\underline{\alpha},\overline{\alpha}]$ and $\alpha' \in [\underline{\alpha},\overline{\alpha}]$. Similarly,

$$\lim_{|\alpha'-\alpha|\to 0}\left|(\sigma')^{1/\alpha'} - (\sigma')^{1/\alpha}\right| = 0 \ \text{ and } \lim_{|\sigma'-\sigma|\to 0}\left|(\sigma')^{1/\alpha} - \sigma^{1/\alpha}\right| = 0 \tag{3.5}$$

uniformly in $\alpha \in [\underline{\alpha},\overline{\alpha}], \sigma \in [\underline{\sigma},\overline{\sigma}]$, and $\sigma' \in [\underline{\sigma},\overline{\sigma}]$. The relations (3.3)–(3.5) imply the inequalities in Lemma 2. □

For $\tau \in (0,1)$ denote

$$I_\tau(\alpha,\sigma) = \int_{0-}^{\tau} \left| \hat{f}_{\alpha,\sigma}(0) - \hat{f}_{\alpha,\sigma}(u) \right| du. \tag{3.6}$$

Lemma 3. *1. There is a constant $B \in R^+$ such that uniformly in $\alpha \in [\underline{\alpha}, \overline{\alpha}]$ and $\sigma \in [\underline{\sigma}, \overline{\sigma}]$, for all $x \in R^+$,*

$$\left| \frac{d}{dx} \hat{f}_{\alpha,\sigma}(x) \right| \le B. \tag{3.7}$$

2. For a given $\varepsilon \in (0,1)$ and any $\tau \in (0, \varepsilon/(8B))$, uniformly in $\alpha \in [\underline{\alpha}, \overline{\alpha}]$ and $\sigma \in [\underline{\sigma}, \overline{\sigma}]$,

$$I_\tau(\alpha,\sigma) < \frac{\varepsilon}{8}. \tag{3.8}$$

Proof. We need in the following general fact on standard stable densities derivatives of order n ([4], p. 106): for $x \in R^+$,

$$\left| \frac{d^n}{dx^n} s(x,\alpha,\beta) \right| \le \frac{1}{\pi\alpha} \Gamma\left(\frac{n+1}{\alpha} \right) \left(\cos\left(\frac{\pi}{2} K(\alpha)\, \beta \right) \right)^{-\frac{n+1}{\alpha}}, \tag{3.9}$$

where

$$K(\alpha) = \alpha - 1 + \operatorname{sign}(1-\alpha). \tag{3.10}$$

Here α and β are the exponent and asymmetry of the standard stable density. Let us apply (3.9) and (3.10) to the cases: 1) $0 < \alpha < 1$, $\beta = 1$, $n = 1$, and 2) $1 < \alpha < 2$, $\beta = 0$, $n = 1$.

In the case 1) we have

$$\left| \frac{d}{dx} \hat{f}_{\alpha,1}(x) \right| \le \frac{1}{\pi\alpha} \Gamma\left(\frac{2}{\alpha} \right) \frac{1}{(\cos(\pi\alpha/2))^{2/\alpha}}. \tag{3.11}$$

Since $\Gamma(x)$ increases for $x > 1$ and $\cos(\pi\alpha/2)$ decreases (remind that $\alpha \in (0,1)$), we may write

$$\Gamma\left(\frac{2}{\alpha} \right) \le \Gamma\left(\frac{2}{\underline{\alpha}} \right), \quad \left(\cos \frac{\pi\alpha}{2} \right)^{-2/\alpha} \le \left(\cos \frac{\pi\overline{\alpha}}{2} \right)^{-2/\overline{\alpha}}. \tag{3.12}$$

Inequalities (3.11) and (3.12)) imply

$$\left| \frac{d}{dx} \hat{f}_{\alpha,1}(x) \right| \le \frac{1}{\pi\underline{\alpha}} \Gamma\left(\frac{2}{\underline{\alpha}} \right) \frac{1}{(\cos(\pi\overline{\alpha}/2))^{2/\overline{\alpha}}}.$$

So,

$$\begin{aligned} \left| \frac{d}{dx} \hat{f}_{\alpha,\sigma}(x) \right| &= \sigma^{-2/\alpha} \left| \frac{d}{dx} \hat{f}_{\alpha,1}(y) \right| \\ &\le (\underline{\sigma})^{-2/\overline{\alpha}} \frac{1}{\pi\underline{\alpha}} \Gamma\left(\frac{2}{\underline{\alpha}} \right) \frac{1}{(\cos(\pi\overline{\alpha}/2))^{2/\overline{\alpha}}} \end{aligned}$$

with $y = \sigma^{-1/\alpha} \cdot x$, which implies (3.7).

In the case 2) we have

$$\left|\frac{d}{dx}\hat{f}_{\alpha,1}(x)\right| \le \frac{2}{\pi\alpha}\Gamma\left(\frac{2}{\alpha}\right) \le \frac{2}{\pi\underline{\alpha}}\Gamma\left(\frac{2}{\underline{\alpha}}\right).$$

Thus, arguing in the same way we conclude that (3.7) holds. Now, by the mean value theorem, from (3.6) with the help of (3.7), we come for both cases to the same type inequality

$$I_\tau(\alpha,\sigma) = \left|\frac{d}{dx}\hat{f}_{\alpha,\sigma}(x)\right|_{x=\theta\,\tau} \cdot \tau \le \tau\, B,$$

where $\theta = \theta_\tau \in (0,1)$. Thus, from the last inequality, for $\tau \in (0, \varepsilon/(8B))$, we obtain (3.8). □

4. Preliminary estimations III

Let $\tau \in (0,1)$, $x \in [\tau, 1/\tau]$, $\alpha \in [\underline{\alpha}, \overline{\alpha}]$, $\sigma \in [\underline{\sigma}, \overline{\sigma}]$, and $N > 1$. We consider the series with the partial sums

$$\hat{f}_{\alpha,\sigma,N}(x) = \frac{1}{\pi}\sum_{n=1}^{N}(-1)^{n-1}\frac{\Gamma(n\alpha+1)}{n!}\,\frac{\sigma^n}{x^{n\alpha+1}}\sin(\pi n\alpha) \tag{4.1}$$

in the case $0 < \alpha < 1$ and

$$\hat{f}_{\alpha,\sigma,N}(x) = \frac{2}{\pi}\sum_{n=1}^{N}(-1)^{n-1}\frac{\Gamma\left(\frac{2n-1}{\alpha}+1\right)}{(2n-1)!}\,\frac{x^{2n-2}}{\sigma^{(2n-1)/\alpha}} \tag{4.2}$$

in the case $1 < \alpha < 2$. The functions (4.1) and (4.2) represent the partial sums of series expansions for the density $\hat{f}_{\alpha,\sigma}(x)$, $0 < \alpha < 1$. Denoting

$$J_{\alpha,\sigma,N}(\tau) = \int_\tau^{1/\tau}\left|\hat{f}_{\alpha,\sigma}(x) - \hat{f}_{\alpha,\sigma,N}(x)\right| dx,$$

we have that

$$J_{\alpha,\sigma,N}(\tau) = \frac{1}{\pi}\int_\tau^{1/\tau}\left(\sum_{n>N}(-1)^{n-1}\frac{\Gamma(n\alpha+1)}{n!}\,\frac{\sigma^n}{x^{n\alpha+1}}\sin(\pi n\alpha)\right)dx \tag{4.3}$$

for $0 < \alpha < 1$ and

$$J_{\alpha,\sigma,N}(\tau) = \frac{2}{\pi}\int_\tau^{1/\tau}\left(\sum_{n>N}(-1)^{n-1}\frac{\Gamma\left(\frac{2n-1}{\alpha}+1\right)}{(2n-1)!}\,\frac{x^{2n-2}}{\sigma^{(2n-1)/\alpha}}\right)dx \tag{4.4}$$

for $1 < \alpha < 2$.

Lemma 4. *1. The integrals at the right-hand-side of* (4.3) *and* (4.4) *exist.*

2. For given $\varepsilon \in (0,1)$ and $\tau \in (0,1)$ there is an integer $N > 1$ (N does not depend on α and σ) such that

$$J_{\alpha,\sigma,N}(\tau) < \frac{\varepsilon}{8}. \tag{4.5}$$

Proof. The case $0 < \alpha < 1$. From (4.3) we have

$$\begin{aligned} J_{\alpha,\sigma,N}(\tau) &\leq \frac{1}{\pi} \int_{\tau}^{1/\tau} \left(\sum_{n>N} \frac{\Gamma(n\alpha+1)}{n!} \frac{\sigma^n}{x^{n\alpha+1}} \right) dx \\ &\leq \frac{1}{\pi} \left(-\tau + \frac{1}{\tau} \right) \sum_{n>N} \frac{\Gamma(n\overline{\alpha}+1)}{n!} \overline{\sigma}^n \, \tau^{-(n\overline{\alpha}+1)}. \end{aligned} \tag{4.6}$$

According to (4.6), with the help of (2.3), for N large enough, we get that

$$J_{\alpha,\sigma,N}(\tau) < \frac{2}{\pi\tau} \left(-\tau + \frac{1}{\tau} \right) \overline{\alpha}^{1/2} \sum_{n>N} \left(\frac{c_\tau(\overline{\alpha}, \overline{\sigma})}{n^{1-\overline{\alpha}}} \right)^n, \tag{4.7}$$

where $c_\tau(\alpha, \sigma) = \exp(1-\alpha)\, \sigma\, \tau^{-\alpha}$. At the right-hand-side of (4.7) we have a convergent series, which proves the statement 1 of Lemma 4. The statement 2 is proved too because the last series does not depend on α and σ.

The case $1 < \alpha < 2$. From (4.4) we have

$$\begin{aligned} J_{\alpha,\sigma,N}(\tau) &\leq \frac{2}{\pi} \int_{\tau}^{1/\tau} \left(\sum_{n>N} \frac{\Gamma\left(\frac{2n-1}{\alpha}+1\right)}{(2n-1)!} \frac{x^{2n-2}}{\sigma^{(2n-1)/\alpha}} \right) dx \\ &\leq \begin{cases} \frac{2}{\pi} \left(-\tau + \frac{1}{\tau}\right) \sum_{n>N} \frac{\Gamma\left(\frac{2n-1}{\underline{\alpha}}+1\right)}{(2n-1)!} \frac{1}{\underline{\sigma}^{(2^{n}-1)/\overline{\alpha}}\, \tau^{2n-2}} & \text{if } \underline{\sigma} \in (1, +\infty), \\ \frac{2}{\pi} \left(-\tau + \frac{1}{\tau}\right) \sum_{n>N} \frac{\Gamma\left(\frac{2n-1}{\underline{\alpha}}+1\right)}{(2n-1)!} \frac{1}{\underline{\sigma}^{(2^{n}-1)/\underline{\alpha}}\, \tau^{2n-2}} & \text{if } \underline{\sigma} \in (0, 1). \end{cases} \end{aligned} \tag{4.8}$$

Similarly to the case $0 < \alpha < 1$ estimations of the series at the right-hand-side of (4.8), with the help of (2.3), imply the statements 1 and 2 of Lemma 4 in this case. □

5. Solution to problem

Proof of Theorem 1. In conditions of Theorem 1 choose, for a given $\varepsilon \in (0, 1)$, a number τ satisfying the restrictions $\tau \in (0, \varepsilon/(8B))$, $1/\tau > x_0$. Then, for $|\alpha - \alpha'| + |\sigma - \sigma'|$ small enough, from Lemmas 1 and 3 we have the inequalities

$$\begin{aligned} \int_{0-}^{\tau} \left| \hat{f}_{\alpha,\sigma}(x) - \hat{f}_{\alpha',\sigma'}(x) \right| dx &\leq \int_{0-}^{\tau} \left| \hat{f}_{\alpha,\sigma}(0) - \hat{f}_{\alpha',\sigma'}(0) \right| dx \\ &+ \int_{0-}^{\tau} \left| \hat{f}_{\alpha,\sigma}(x) - \hat{f}_{\alpha,\sigma}(0) \right| dx + \int_{0-}^{\tau} \left| \hat{f}_{\alpha',\sigma'}(x) - \hat{f}_{\alpha',\sigma'}(0) \right| dx = \\ &= I_\tau(\alpha, \sigma) + I_\tau(\alpha', \sigma') + \tau \cdot \gamma(\alpha, \sigma; \alpha', \sigma') \\ &< \frac{\varepsilon}{8} + \frac{\varepsilon}{8} + \frac{\varepsilon}{8} = \frac{3\varepsilon}{8}, \end{aligned} \tag{5.1}$$

where the monotonicity of $\hat{f}_{\alpha,\sigma}$ around the origin (point zero) and (3.1), (3.5) were used, and

$$\begin{aligned}
&\int_{1/\tau}^{+\infty} \left|\hat{f}_{\alpha,\sigma}\left(x\right) - \hat{f}_{\alpha',\sigma'}\left(x\right)\right| dx \\
&\qquad \leq \int_{1/\tau}^{+\infty} \hat{f}_{\alpha,\sigma}\left(x\right) dx + \int_{1/\tau}^{+\infty} \hat{f}_{\alpha',\sigma'}\left(x\right) dx \\
&\qquad = (1 - \hat{F}_{\alpha,\sigma}(\frac{1}{\tau})) + (1 - \hat{F}_{\alpha',\sigma'}(\frac{1}{\tau})) < \frac{\varepsilon}{16} + \frac{\varepsilon}{16} = \frac{\varepsilon}{8}.
\end{aligned} \tag{5.2}$$

In accordance with (5.1) and (5.2), for given ε and τ, from (1.3) we obtain that

$$\begin{aligned}
E\left(\alpha, \sigma; \alpha', \sigma'\right) &= \int\limits_{0-}^{+\infty} \left|\hat{f}_{\alpha,\sigma}(x) - \hat{f}_{\alpha',\sigma'}(x)\right| dx \\
&= \int\limits_{0-}^{\tau} \left|\hat{f}_{\alpha,\sigma}(x) - \hat{f}_{\alpha',\sigma'}(x)\right| dx + \int_{\tau}^{1/\tau} \left|\hat{f}_{\alpha,\sigma}(x) - \hat{f}_{\alpha',\sigma'}(x)\right| dx \\
&\quad + \int_{1/\tau}^{+\infty} \left|\hat{f}_{\alpha,\sigma}\left(x\right) - \hat{f}_{\alpha',\sigma'}\left(x\right)\right| dx \\
&\leq \frac{\varepsilon}{2} + \int_{\tau}^{1/\tau} \left|\hat{f}_{\alpha,\sigma}(x) - \hat{f}_{\alpha',\sigma'}(x)\right| dx.
\end{aligned} \tag{5.3}$$

Now, let us choose an integer $N > 1$ such that, for given ε and τ, (4.5) takes place, and fix N. Then, by (5.3), for $|\alpha - \alpha'| + |\sigma - \sigma'|$ small enough, we come to the inequalities

$$\begin{aligned}
E\left(\alpha, \sigma; \alpha', \sigma'\right) &\leq \int_{\tau}^{1/\tau} \left|\hat{f}_{\alpha,\sigma,N}(x) - \hat{f}_{\alpha',\sigma',N}(x)\right| dx + \frac{\varepsilon}{2} \\
&\quad + \int_{\tau}^{1/\tau} \left|(\hat{f}_{\alpha,\sigma}(x) - \hat{f}_{\alpha,\sigma,N}(x))\right| dx \\
&\quad + \int_{\tau}^{1/\tau} \left|(\hat{f}_{\alpha',\sigma'}(x) - \hat{f}_{\alpha',\sigma',N}(x))\right| dx \\
&\leq \frac{3\varepsilon}{4} + \int_{\tau}^{1/\tau} \left|\hat{f}_{\alpha,\sigma,N}(x) - \hat{f}_{\alpha',\sigma',N}(x)\right| dx.
\end{aligned} \tag{5.4}$$

If we proceed as above in the cases $E_1(\alpha;\sigma,\sigma')$ and $E_2(\sigma';\alpha,\alpha')$, then we obtain the following analogs of (5.4):

$$\begin{aligned} E_1\left(\alpha;\sigma,\sigma'\right) &\le \frac{3\varepsilon}{4} + \int_\tau^{1/\tau} \left|\hat{f}_{\alpha,\sigma,N}(x) - \hat{f}_{\alpha,\sigma',N}(x)\right| dx, \\ E_2\left(\sigma;\alpha,\alpha'\right) &\le \frac{3\varepsilon}{4} + \int_\tau^{1/\tau} \left|\hat{f}_{\alpha,\sigma,N}(x) - \hat{f}_{\alpha',\sigma,N}(x)\right| dx. \end{aligned} \tag{5.5}$$

In the last inequality we take σ instead of σ', which changes nothing. If we prove that for given ε, τ, N and $|\alpha-\alpha'|+|\sigma-\sigma'|$ small enough, in conditions of Theorem 1,

$$\begin{aligned} T_N^{(1)}(\tau) &= \int_\tau^{1/\tau} \left|\hat{f}_{\alpha,\sigma,N}(x) - \hat{f}_{\alpha,\sigma',N}(x)\right| dx < \frac{\varepsilon}{4}, \\ T_N^{(2)}(\tau) &= \int_\tau^{1/\tau} \left|\hat{f}_{\alpha,\sigma,N}(x) - \hat{f}_{\alpha',\sigma,N}(x)\right| dx < \frac{\varepsilon}{4}, \end{aligned} \tag{5.6}$$

then (5.5) and (5.6) imply (1.4) and (1.5), which, due to Remark 1, prove Theorem 1. Indeed, assuming that (5.6) is true, we may rewrite inequalities (5.5) in the form

$$0 \le E_1\left(\alpha;\sigma,\sigma'\right) < \varepsilon \text{ and } 0 \le E_2\left(\sigma';\alpha,\alpha'\right) < \varepsilon, \tag{5.7}$$

for $|\alpha-\alpha'|+|\sigma-\sigma'|$ small enough. Tending $|\alpha-\alpha'|+|\sigma-\sigma'| \to 0$, from (5.7) we obtain that

$$0 \le \overline{\lim}_{|\sigma-\sigma'|\to 0}\delta_i(\alpha;\sigma,\sigma') < \varepsilon.$$

Now, tending $\varepsilon \to 0$, we prove Theorem 1. □

Remark 2. Since in (4.1) and (4.2) under the signs of sums continuous functions on α and σ are written, and (α,σ) belongs to the compact sets

$$B_1 = \{(\alpha,\sigma) :\, 0 < \underline{\alpha} \le \alpha \le \overline{\alpha} < 1, \quad 0 < \underline{\sigma} \le \sigma \le \overline{\sigma} < +\infty\}$$

and

$$B_2 = \{(\alpha,\sigma) :\, 1 < \underline{\alpha} \le \alpha \le \overline{\alpha} < 2, \quad 0 < \underline{\sigma} \le \sigma \le \overline{\sigma} < +\infty\},$$

respectively, according to Cantor's theorem, they are uniformly continuous on these sets. Therefore, $\hat{f}_{\alpha,\sigma,N}(x)$, for $0<\alpha<1$ and $1<\alpha<2$, as finite sums of uniformly continuous functions on B_1 and B_2, respectively, are also uniformly continuous on these compact sets. Hence, for fixed τ, the relations (5.6) take place.

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References

[1] J. Astola and E. A. Danielyan, *Regularly varying skewed distributions generated by birth-death process*, Tampere, TICSP Series **27** (2004), 1–94.
[2] J. Astola, E. A. Danielyan, and A. H. Arakelyan, *Frequency distributions in growing biomolecular networks based on stable densities*, Dokl. Nats. Acad. Nauk Armen. **107**(1) (2007), 26–36.
[3] T. Gisinger, *Scale invariance in biology: coincidence of footprint of universal mechanism?*, Biol. Rev. Camb. Philos. Soc. **76**(2) (2001), 161–209.
[4] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press, San Diego, 2000.
[5] V. A. Kuznetsov, *Distribution associated with stochastic processes of gene expression in a single Eukaryotic cell*, EURASIP J. Appl. Signal Process. **2001**(4) (2001), 285–296.
[6] V. A. Kuznetsov, *Family of skewed distributions associated with the gene expression and proteome evolution*, Signal Process. **83**(4) (2003), 889–910.
[7] V. M. Zolotarev, *One-Dimensional Stable Distributions*, Nauka, Moscow, 1983. (Russian)

Faculty of Applied Mathematics, State Engeniering University of Armenia, Yerevan, Teryan st.105, 375049, Armenia
E-mail address: arman.arakelyan@hotmail.com