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Some generalizations of the Eneström–Kakeya theorem

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Abstract. Let $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ be a polynomial of degree n, where the coefficients $a_j,\ j = 0, 1, 2, \ldots, n$, are real numbers. We impose some restriction on the coefficients and then prove some extensions and generalizations of the Eneström–Kakeya theorem.

1. Introduction

A classical result due to Eneström [5] and Kakeya [7] concerning the bounds for the moduli of zeros of polynomials having positive coefficients is often stated as follows.

Theorem A. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial with real coefficients satisfying*

$$0 < a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_n.$$

Then all the zeros of $p(z)$ *lie in* $|z| \leq 1$.

In the literature there exist several extensions and generalizations of this result (see [1], [2], [6] and [8]). Joyal et al. [6] extended Theorem A to the polynomials whose coefficients are monotonic but not necessarily nonnegative. In fact, they proved the following result.

Theorem B. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n, *with real coefficients satisfying*

$$a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_n.$$

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Then all the zeros of $p(z)$ *lie in the disk*

$$|z| \leq \frac{1}{|a_n|}(a_n - a_0 + |a_0|).$$

Aziz and Zargar [3] relaxed the hypothesis in several ways and among other things proved the following result.

Theorem C. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *such that for some* $k \geq 1$,

$$0 < a_0 \leq a_1 \leq a_2 \leq \cdots \leq k a_n.$$

Then all the zeros of $p(z)$ *lie in the disk*

$$|z + k - 1| \leq k.$$

In 2012, they further generalized Theorem C which is an interesting extension of Theorem A. In particular, the following theorems are proved in [4].

Theorem D. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n. *If for some positive numbers* k *and* ρ *with* $k \geq 1$, $0 < \rho \leq 1$,

$$0 \leq \rho a_0 \leq a_1 \leq a_2 \leq \cdots \leq k a_n,$$

then all the zeros of $p(z)$ *lie in the disk*

$$|z + k - 1| \leq k + \frac{2a_0}{a_n}(1 - \rho).$$

Theorem E. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n. *If for some positive number* ρ, $0 < \rho \leq 1$, *and for some nonnegative integer* λ, $0 \leq \lambda < n$,

$$\rho a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_{\lambda-1} \leq a_\lambda \geq a_{\lambda+1} \geq \cdots \geq a_{n-1} \geq a_n,$$

then all the zeros of $p(z)$ *lie in the disk*

$$\left|z + \frac{a_{n-1}}{a_n} - 1\right| \leq \frac{1}{|a_n|}\Big[2a_\lambda - a_{n-1} + (2 - \rho)|a_0| - \rho a_0\Big].$$

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Looking at Theorem D, one might want to know what happens if ρa_0 is NOT nonnegative. In this paper we prove some extensions and generalizations of Theorems D and E which in turn give an answer to our enquiry.

2. Main results

Theorem 1. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n. *If for some real numbers* α *and* β,

$$a_0 - \beta \leq a_1 \leq a_2 \leq \cdots \leq a_n + \alpha,$$

then all the zeros of $p(z)$ *lie in the disk*

$$\left|z + \frac{\alpha}{a_n}\right| \leq \frac{1}{|a_n|}\Big[a_n + \alpha - a_0 + \beta + |\beta| + |a_0|\Big].$$

If $\alpha = (k-1)a_n$ and $\beta = (1-\rho)a_0$ with $k \geq 1$, $0 < \rho \leq 1$, then we get the following corollary.

Corollary 1. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n. *If for some postive numbers* $k \geq 1$ *and* ρ *with* $0 < \rho \leq 1$,

$$\rho a_0 \leq a_1 \leq a_2 \leq \cdots \leq k a_n,$$

then all the zeros of $p(z)$ *lie in the disk*

$$|z + k - 1| \leq \frac{1}{|a_n|}\Big[(k a_n - \rho a_0) + |a_0|(2-\rho)\Big].$$

If $a_0 > 0$, then Corollary 1 amounts to Theorem D.

Theorem 2. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n. *If for some real number* s *and for some integer* λ, $0 < \lambda < n$,

$$a_0 - s \leq a_1 \leq a_2 \leq \cdots \leq a_{\lambda-1} \leq a_\lambda \geq a_{\lambda+1} \geq \cdots \geq a_{n-1} \geq a_n,$$

then all the zeros of $p(z)$ *lie in the disk*

$$\left|z + \frac{a_{n-1}}{a_n} - 1\right| \leq \frac{1}{|a_n|}\Big[2a_\lambda - a_{n-1} + s - a_0 + |s| + |a_0|\Big].$$

If we take $s = (1-\rho)a_0$, with $0 < \rho \leq 1$, then Theorem 2 becomes Theorem E. Instead of proving Theorem 2, we shall prove a more general case. In fact, we prove the following result.

Theorem 3. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n. *If for some real numbers* t, s *and for some integer* λ, $0 < \lambda < n$,

$$a_0 - s \leq a_1 \leq a_2 \leq \cdots \leq a_{\lambda-1} \leq a_\lambda \geq a_{\lambda+1} \geq \cdots \geq a_{n-1} \geq a_n + t,$$

then all the zeros of $p(z)$ lie in the disk

$$\Big|z+\frac{a_{n-1}}{a_n}-\Big(1+\frac{t}{a_n}\Big)\Big|\leq\frac{1}{|a_n|}\Big[2a_\lambda-a_{n-1}+s-a_0+|s|+|a_0|+|t|\Big].$$

3. Proofs of the theorems

Proof of Theorem 1. Consider the polynomial

$$\begin{aligned}
g(z) &= (1-z)p(z)\\
&= -a_nz^{n+1}+(a_n-a_{n-1})z^n+(a_{n-1}-a_{n-2})z^{n-1}+\cdots+(a_1-a_0)z+a_0\\
&= -a_nz^{n+1}-\alpha z^n+(a_n+\alpha-a_{n-1})z^n+(a_{n-1}-a_{n-2})z^{n-1}+\dots\\
&\quad+(a_1-a_0+\beta)z-\beta z+a_0\\
&= -z^n(a_nz+\alpha)+(a_n+\alpha-a_{n-1})z^n+(a_{n-1}-a_{n-2})z^{n-1}+\dots\\
&\quad+(a_1-a_0+\beta)z-\beta z+a_0\\
&= -z^n(a_nz+\alpha)+\phi(z),
\end{aligned}$$

where

$$\phi(z)=(a_n+\alpha-a_{n-1})z^n+(a_{n-1}-a_{n-2})z^{n-1}+\cdots+(a_1-a_0+\beta)z-\beta z+a_0.$$

Now for $|z|=1$, we have

$$\begin{aligned}
|\phi(z)| &\leq |a_n+\alpha-a_{n-1}|+|a_{n-1}-a_{n-2}|+\cdots+|a_1-a_0+\beta|+|\beta|+|a_0|\\
&= a_n+\alpha-a_{n-1}+a_{n-1}-a_{n-2}+\cdots+a_1-a_0+\beta+|\beta|+|a_0|\\
&= a_n+\alpha-a_0+\beta+|\beta|+|a_0|.
\end{aligned}$$

Since this is true for all complex numbers with a unit modulus, it must also be true for $1/z$. With this in mind, we have, for all z with $|z|=1$,

$$|z^n\phi(1/z)|\leq a_n+\alpha-a_0+\beta+|\beta|+|a_0|. \tag{1}$$

Also, the function $\Phi(z)=z^n\phi(1/z)$ is analytic in $|z|\leq 1$, hence, inequality (1) holds inside the unit circle by the maximum modulus theorem. That is, for all z with $|z|\leq 1$,

$$|\phi(1/z)|\leq\frac{a_n+\alpha-a_0+\beta+|\beta|+|a_0|}{|z|^n}.$$

Replacing z by $1/z$, we get

$$|\phi(z)|\leq\Big[a_n+\alpha-a_0+\beta+|\beta|+|a_0|\Big]|z|^n$$

if $|z|\geq 1$. Now, for $|z|\geq 1$, we obtain that

$$\begin{aligned}
|g(z)| &= |-z^n(a_nz+\alpha)+\phi(z)|\\
&\geq |z^n||a_nz+\alpha|-|\phi(z)|\\
&\geq |z^n||a_nz+\alpha|-\Big[a_n+\alpha-a_0+\beta+|\beta|+|a_0|\Big]|z|^n
\end{aligned}$$

$$= |z^n|\Big(|a_n z + \alpha| - \Big[a_n + \alpha - a_0 + \beta + |\beta| + |a_0|\Big]\Big)$$
$$> 0$$

if and only if

$$|a_n z + \alpha| > \Big[a_n + \alpha - a_0 + \beta + |\beta| + |a_0|\Big]$$

or, equivalently, if and only if

$$\Big|z + \frac{\alpha}{a_n}\Big| > \frac{1}{|a_n|}\Big[a_n + \alpha - a_0 + \beta + |\beta| + |a_0|\Big].$$

Thus, all the zeros of $g(z)$ whose modulus is greater than or equal to 1 lie in

$$\Big|z + \frac{\alpha}{a_n}\Big| \leq \frac{1}{|a_n|}\Big[a_n + \alpha - a_0 + \beta + |\beta| + |a_0|\Big]. \tag{2}$$

But those zeros of $p(z)$ whose modulus is less than 1 already satisfy (2), because $|\phi(z)| \leq a_n + \alpha - a_0 + \beta + |\beta| + |a_0|$ for $|z| = 1$ and $\phi(z) = g(z) + z^n(a_n z + \alpha)$. Also, all the zeros of $p(z)$ are zeros of $g(z)$. That completes the proof of Theorem 1. □

Proof of Theorem 3. Consider the polynomial

$$\begin{aligned} g(z) &= (1 - z)p(z) \\ &= -a_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \dots + (a_1 - a_0)z + a_0 \\ &= -a_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \dots \\ &\quad + (a_{\lambda+1} - a_\lambda)z^{\lambda+1} + (a_\lambda - a_{\lambda-1})z^\lambda + \dots + (a_1 - a_0)z + a_0 \\ &= -z^n[a_n z - a_n + a_{n-1} - t] - tz^n + (a_{n-1} - a_{n-2})z^{n-1} + \dots \\ &\quad + (a_{\lambda+1} - a_\lambda)z^{\lambda+1} + (a_\lambda - a_{\lambda-1})z^\lambda + \dots + (a_1 - a_0 + s)z - sz + a_0 \\ &= -z^n[a_n z - a_n + a_{n-1} - t] + \psi(z), \end{aligned}$$

where

$$\begin{aligned} \psi(z) = -tz^n + (a_{n-1} - a_{n-2})z^{n-1} + \dots + (a_{\lambda+1} - a_\lambda)z^{\lambda+1} \\ + (a_\lambda - a_{\lambda-1})z^\lambda + \dots + (a_1 - a_0 + s)z - sz + a_0. \end{aligned}$$

For $|z| = 1$ we get

$$\begin{aligned} |\psi(z)| &\leq |t| + |a_{n-1} - a_{n-2}| + \dots + |a_{\lambda+1} - a_\lambda| + |a_\lambda - a_{\lambda-1}| + \dots \\ &\quad + |a_1 - a_0 + s| + |s| + |a_0| \\ &= |t| + a_{n-2} - a_{n-1} + \dots + a_\lambda - a_{\lambda+1} + a_\lambda - a_{\lambda-1} + \dots \\ &\quad + a_1 - a_0 + s + |s| + |a_0| \\ &= |t| - a_{n-1} + 2a_\lambda - a_0 + s + |s| + |a_0|. \end{aligned}$$

It is clear that

$$|z^n\psi(1/z)| \le |t| - a_{n-1} + 2a_\lambda - a_0 + s + |s| + |a_0| \tag{3}$$

on the unit circle. Since the function $\Psi(z) = z^n\psi(1/z)$ is analytic in $|z| \le 1$, inequality (3) holds inside the unit circle by the maximum modulus theorem. That is,

$$|\psi(1/z)| \le \frac{|t| - a_{n-1} + 2a_\lambda - a_0 + s + |s| + |a_0|}{|z|^n}$$

for $|z| \le 1$. Replacing z by $1/z$ we get

$$|\psi(z)| \le \Big[|t| - a_{n-1} + 2a_\lambda - a_0 + s + |s| + |a_0|\Big]|z|^n$$

for $|z| \ge 1$. Now for $|z| \ge 1$, we have

$$\begin{aligned}|g(z)| &\ge |z^n||a_nz - a_n + a_{n-1} - t| - |\psi(z)| \\ &\ge |z^n||a_nz - a_n + a_{n-1} - t| \\ &\quad - \Big[|t| - a_{n-1} + 2a_\lambda - a_0 + s + |s| + |a_0|\Big]|z|^n \\ &= |z^n|\Big(|a_nz - a_n + a_{n-1} - t| - \Big[|t| - a_{n-1} + 2a_\lambda - a_0 + s + |s| + |a_0|\Big]\Big) \\ &> 0\end{aligned}$$

if and only if

$$|a_nz - a_n + a_{n-1} - t| > \Big[|t| - a_{n-1} + 2a_\lambda - a_0 + s + |s| + |a_0|\Big].$$

But this holds if and only if

$$\left|z + \frac{a_{n-1}}{a_n} - \Big(1 + \frac{t}{a_n}\Big)\right| > \frac{1}{|a_n|}\Big[|t| - a_{n-1} + 2a_\lambda - a_0 + s + |s| + |a_0|\Big].$$

Hence, the zeros of $p(z)$ with modulus greater or equal to 1 are in the closed disk

$$\left|z + \frac{a_{n-1}}{a_n} - \Big(1 + \frac{t}{a_n}\Big)\right| \le \frac{1}{|a_n|}\Big[|t| - a_{n-1} + 2a_\lambda - a_0 + s + |s| + |a_0|\Big].$$

Also, those zeros of $p(z)$ whose modulus is less than 1 already satisfy the above inequality since $\psi(z) = g(z) + z^n[a_nz - a_n + a_{n-1} - t]$ and, for $|z| = 1$, $|\psi(z)| \le |t| - a_{n-1} + 2a_\lambda - a_0 + s + |s| + |a_0|$. That completes the proof. □

4. Demonstrating examples

Example 1. Let us consider the polynomial

$$p(z) = 3z^5 + 4z^4 + 3z^3 + 2z^2 + z - 1.$$

The coefficients here are $a_5 = 3$, $a_4 = 4$, $a_3 = 3$, $a_2 = 2$, $a_1 = 1$ and $a_0 = -1$. We cannot apply Theorems A, B, C and D. But we can apply Theorem 1 to determine where all the zeros of the polynomial lie. Using MATLAB, we obtain the following zeros : $-0.9154 + 0.4962i$, $-0.9154 - 0.4962i$, $0.0530 +$

$0.8845i$, $0.0530 - 0.8845i$, 0.3916. Taking $\alpha = 2$ and $\beta = 0$, Theorem 1 gives that all the zeros of the polynomial lie in the closed disk $|3z + 2| \leq 7$.

Example 2. Next, consider

$$q(z) = -z^6 + 2z^5 + 2z^4 + 3z^3 + z^2 - 2.$$

The coefficients of $q(z)$ are $a_6 = -1$, $a_5 = 2$, $a_4 = 2$, $a_3 = 3$, $a_2 = 1$, $a_1 = 0$ and $a_0 = -2$. Using MATLAB, we obtain the following zeros: 3.0197, $-0.7682+0.5814i$, $-0.7682-0.5814i$, $-0.0803+1.0233i$, $-0.0803-1.0233i$, 0.6773. Taking $\lambda = 3$, $t = 1$ and $s = 0$, Theorem 3 gives that the zeros lie in $|z - 2| \leq 9$.

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