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Approximation of periodic integrable functions in terms of modulus of continuity

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Abstract. We estimate the pointwise approximation of periodic functions belonging to $L^p(\omega)_\beta$-class, where ω is an integral modulus of continuity type function associated with f, using product means of the Fourier series of f generated by the product of two general linear operators. We also discuss the case $p = 1$ separately. This case has not been mentioned in the earlier results given by various authors. The deviations obtained in our theorems are free from p and more sharper than the earlier results.

1. Introduction

Let f be a 2π periodic function belonging to the space $L^p := L^p[0, 2\pi]$ $(p \geq 1)$. The trigonometric Fourier series of f is defined as

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx). \tag{1}$$

The $(k+1)^{\text{th}}$ partial sum of the Fourier series (1),

$$s_0(f;x) := \frac{a_0}{2}, \quad s_k(f;x) := \frac{a_0}{2} + \sum_{\nu=1}^{k}(a_\nu \cos \nu x + b_\nu \sin \nu x), \quad k \in \mathbb{N},$$

is called the trigonometric polynomial of degree or order k (see [7]).

Let $T \equiv (a_{n,k})$ be a lower triangular matrix. Then the sequence to sequence transformation

$$t_n(f;x) = \sum_{k=0}^{n} a_{n,k} s_k(f;x), \quad n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\},$$

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defines the matrix means of $\{s_n(f;x)\}$. The Fourier series (1) is said to be summable to s by T-means, if $\lim_{n\to\infty} t_n(f;x) = s$, where s is a finite number.

Let $A \equiv (a_{n,m})$ and $B \equiv (b_{n,m})$ be infinite lower triangular matrices of real numbers such that

$$A(\text{or } B) = \begin{cases} a_{n,m}(\text{or } b_{n,m}) \geq 0, & m = 0,1,2,\ldots,n, \\ a_{n,m}(\text{or } b_{n,m}) = 0, & m > n, \end{cases}$$

$$\sum_{m=0}^{n} a_{n,m} = 1 \text{ and } \sum_{m=0}^{n} b_{n,m} = 1, \text{ where } n = 0,1,2,\ldots,$$

and let

$$A_{n,r} = \sum_{m=0}^{r} a_{n,m} \text{ and } \bar{A}_{n,r} = \sum_{m=r}^{n} a_{n,m},$$

$$B_{n,r} = \sum_{m=0}^{r} b_{n,m} \text{ and } \bar{B}_{n,r} = \sum_{m=r}^{n} b_{n,m},$$

so that $A_{n,n} = B_{n,n} = 1 = \bar{A}_{n,0} = \bar{B}_{n,0}$.

When we superimpose the B-summability on A-summability, we get $B \cdot A$ means of $\{s_k(f;x)\}$ defined by (see [1, 4])

$$\begin{aligned} t_n^{B\cdot A}(f;x) &= \sum_{m=0}^{n} b_{n,m} \left(\sum_{k=0}^{m} a_{m,k} s_k(f;x) \right) \\ &= \sum_{m=0}^{n} \sum_{k=0}^{m} b_{n,m} a_{m,k} s_k(f;x), \quad n = 0,1,2,\ldots. \end{aligned} \tag{2}$$

We write $(B \cdot A)_n(t)$ as

$$(B \cdot A)_n(t) = \frac{1}{2\pi} \sum_{m=0}^{n} \sum_{k=0}^{m} b_{n,m} a_{m,k} \frac{\sin{(k+1/2)t}}{\sin{(t/2)}},$$

and we also write

$$\phi(t) \equiv \phi(x,t) := f(x+t) + f(x-t) - 2f(x), \quad x \in [0,2\pi], \ \ t \in [0,\pi].$$

The L^p norm of $f \in L^p[0,2\pi]$ is defined by

$$\|f\|_p = \begin{cases} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(x)|^p dx \right)^{1/p}, & 1 \leq p < \infty, \\ \text{ess sup}_{x\in[0,2\pi]} |f(x)|, & p = \infty. \end{cases}$$

The degree of approximation $E_n(f)$ of a function $f \in L^p$ by a trigonometric polynomial $T_n(x)$ of degree n is given by

$$E_n(f) = \min_{T_n} \| f(x) - T_n(x) \|_p .$$

This method of approximation is called the trigonometric Fourier approximation.

Lenski and Szal [3] defined the generalized modulus of continuity of f in L^p by

$$\omega_\beta f(\delta)_{L^p} = \sup_{0\le|t|\le\delta}\left\{\left|\sin\frac{t}{2}\right|^{\beta p}\int_0^{2\pi}|\phi(t)|^p dx\right\}^{1/p}, \quad \beta\ge 0,$$

and a subclass $L^p(\omega)_\beta$ of L^p-class as

$$L^p(\omega)_\beta = \{f\in L^p\colon \omega_\beta f(\delta)_{L^p}\le\omega(\delta)\},$$

where ω is a function of modulus of continuity type on $[0,2\pi]$, i.e., ω is a nondecreasing continuous function having the properties

$$\omega(0)=0,\ \ \omega(\delta_1+\delta_2)\le\omega(\delta_1)+\omega(\delta_2), \quad 0\le\delta_1\le\delta_2\le\delta_1+\delta_2\le 2\pi.$$

We write $K_1 \ll K_2$ if there exists a positive constant C (it may depend on some parameters) such that $K_1\le CK_2$.

2. Known results

The product summability means of Fourier series have been considered in various directions, for example, Mittal [5, Theorem 1] has estimated the deviation $t_n^{B\cdot A}(f;.)-f(.)$ pointwise with lower triangular infinite matrix B defined by

$$b_{n,m} = \begin{cases} \frac{1}{n+1}, & 0\le m\le n,\\ 0, & m>n.\end{cases}$$

This matrix corresponds to the Cesàro summability of order 1 and is denoted by C^1. He also discussed the (F_1)-effectiveness of $C^1\cdot A$ method. Lenski and Szal [4, Theorem 2] have extended the results of Mittal [5] to more general means $B\cdot A$, and proved their results in terms of moduli of continuity. They proved the following result:

$$\left|t_n^{B\cdot A}f(x)-f(x)\right| \ll \sum_{m=0}^{n} b_{n,m}\left[\frac{1}{m+1}\sum_{k=0}^{m}\omega_x f\left(\frac{\pi}{k+1}\right)\right]$$

for every natural number n and all real x, where

$$\omega_x f(\delta) = \sup_{0\le t\le\delta}\left|\frac{1}{t}\int_0^t \phi(x,u)du\right|,$$

known as the integral modulus of continuity of f.

Lenski and Szal [3] defined the class $L^p(\omega)_\beta$ and proved their results by using the sequences $\alpha_n=(a_{n,k})_{k=0}^n$ of rest bounded variation $(RBVS)$ or head bounded variation $(HBVS)$. They estimated the pointwise deviation as follows (see [3, Theorem 3]):

$$|T_{n,A}f(x)-f(x)| = O_x\left((n+1)^{\beta+\frac{1}{p}+1}\, a_n\, \omega\left(\frac{\pi}{n+1}\right)\right),$$

where

$$a_n = \begin{cases} a_{n,0} & \text{if } \{a_{n,k}\} \in RBVS, \\ a_{n,n} & \text{if } \{a_{n,k}\} \in HBVS. \end{cases}$$

Recently, Krasniqi [2, Theorem 10] used the lower triangular infinite matrix $A \equiv (a_{n,k})$ with $a_{n,m} \leq \sum_{k=m}^{n} |\triangle a_{n,k}|$, and proved his result in the same class $L^p(\omega)_\beta$ as follows:

$$|T_{n,A}f(x) - f(x)| = O_x\left((n+1)^{\beta+\frac{1}{p}+1} \sum_{k=0}^{n} |\Delta a_{n,k}|\, \omega\left(\frac{\pi}{n+1}\right)\right).$$

Clearly, in these results, the error of approximation depends on p. Further, very recently, Singh and Srivastava [6, Theorem 2.2] obtained the degree of approximation of functions belonging to weighted Lipschitz class $W(L^p, \xi(t), \beta)$ by $C^1 \cdot A$ means of its Fourier series; their result is given as

$$\|t_n^{C^1 \cdot A}(f;x) - f(x)\|_p = O((n+1)^\beta \omega(1/(n+1))),$$

where $\omega(t)$ is a positive increasing function. We note that deviation in this result is free from p.

3. Main results

In this paper, we extend the results of Krasniqi [2] to the product means defined in (2). More precisely, we prove the following theorem.

Theorem 3.1. *Let $f \in L^p(\omega)_\beta$ with $0 < \beta < 1 - 1/p$, $p > 1$, and the entries of the lower triangular matrices $A \equiv (a_{n,k})$ and $B \equiv (b_{n,k})$ satisfy the conditions*

$$b_{n,n} \ll \frac{1}{n+1}, \; n \in \mathbb{N}_0, \tag{3}$$

$$|b_{n,m}a_{m,0} - b_{n,m+1}a_{m+1,1}| \ll \frac{b_{n,m}}{(m+1)^2}, \quad 0 \leq m \leq n-1, \tag{4}$$

and

$$\begin{aligned} \sum_{k=0}^{m-1} |(b_{n,m}a_{m,m-k} - b_{n,m+1}a_{m+1,m+1-k}) \\ -(b_{n,m}a_{m,m-k-1} - b_{n,m+1}a_{m+1,m-k})| \\ \ll \frac{b_{n,m}}{(m+1)^2}, \quad 0 \leq m \leq n-1, \end{aligned} \tag{5}$$

with $A_{n,n} = B_{n,n} = 1$ for $n = 0, 1, 2, \ldots$. Then the degree of approximation of f by $B \cdot A$ means of its Fourier series is given by

$$\left|t_n^{B\cdot A}(f;x) - f(x)\right| = O_x\left(\sum_{m=0}^{n} \frac{b_{n,m}}{m+1}(n+1)^{\beta+1}\omega(1/(n+1))\right),$$

provided that the positive nondecreasing function ω *satisfies the conditions*

$$\omega(t)/t \text{ is a decreasing function,} \tag{6}$$

$$\left\{\int_0^{\pi/(n+1)} \left(\frac{|\phi(t)|\sin^\beta(t/2)}{\omega(t)}\right)^p dt\right\}^{1/p} = O_x\left((n+1)^{-1/p}\right), \tag{7}$$

$$\left\{\int_{\pi/(n+1)}^{\pi} \left(t^{-\gamma}\frac{|\phi(t)|\sin^\beta(t/2)}{\omega(t)}\right)^p dt\right\}^{1/p} = O_x\left((n+1)^{\gamma-1/p}\right), \tag{8}$$

where γ *is an arbitrary number such that* $1/p < \gamma < \beta + 1/p$, $p^{-1} + q^{-1} = 1$.

Note 3.2. *The condition* (6) *implies that*

$$\frac{\omega(\pi/(n+1))}{\pi/(n+1)} \le \frac{\omega(1/(n+1))}{1/(n+1)}, \quad i.e., \, \omega\left(\frac{\pi}{n+1}\right) = O\left(\omega\left(\frac{1}{n+1}\right)\right).$$

In the proof of above theorem given in Section 5 we use Hölder's inequality for $p > 1$. Therefore, the proof is not applicable for $p = 1$. Moreover, for $p = 1$, the number β becomes negative. Thus, for $p = 1$, we have the following theorem.

Theorem 3.3. *Let* $f \in L^1(\omega)_\beta$ *with* $0 < \beta < 1$ *and the entries of the lower triangular matrices* A *and* B *satisfy the conditions* (3)$-$(5) *with* $A_{n,n} = B_{n,n} = 1$ *for* $n = 0, 1, 2, \ldots$. *Then the degree of approximation of* f *by* $B \cdot A$ *means of its Fourier series is given by*

$$\left|t_n^{B\cdot A}(f;x) - f(x)\right| = O_x\left(\sum_{m=0}^{n} \frac{b_{n,m}}{m+1}(n+1)^{\beta+1}\omega(1/(n+1))\right),$$

provided that the positive nondecreasing function ω *satisfies* (6) *and the conditions*

$$\omega(t)/t^\beta \text{ is a non-decreasing function,} \tag{9}$$

$$\int_0^{\pi/(n+1)} \frac{|\phi(t)|\sin^\beta(t/2)}{\omega(t)} dt = O_x\left((n+1)^{-1}\right), \tag{10}$$

$$\int_{\pi/(n+1)}^{\pi} t^{-\gamma}\frac{|\phi(t)|}{\omega(t)} dt = O_x\left((n+1)^{\gamma-1}\right), \tag{11}$$

where γ *is an arbitrary number such that* $1 < \gamma < \beta + 1$ *and* $p^{-1} + q^{-1} = 1$.

4. Lemmas

We need the following lemmas for proving our theorems.

Lemma 4.1 (see [4])**.** *If the conditions* (4) *and* (5) *hold, then*

$$|b_{n,r}a_{r,r-l} - b_{n,r+1}a_{r+1,r+1-l}| \ll \frac{b_{n,r}}{(r+1)^2}, \quad 0 \le l \le r-1 \le n-2.$$

For the proof, we refer to [4, Lemma 3.2].

Lemma 4.2. *If the matrices A and B satisfy the conditions of Theorem 3.1, then*

$$|(B\cdot A)_n(t)| = O\left(n+1\right), \quad 0<t\le \pi/(n+1).$$

Proof. Since $1/\sin(t/2) = O\left(\pi/t\right)$ and $0 \le sin(nt) \le nt$, for $0 < t \le \pi/(n+1)$, we have

$$\begin{aligned}
|(B\cdot A)_n(t)| &= \left|\frac{1}{2\pi}\sum_{m=0}^{n}\sum_{k=0}^{m} b_{n,m}a_{m,k}\frac{\sin\left(k+1/2\right)t}{\sin(t/2)}\right| \\
&\le \frac{1}{2\pi}\sum_{m=0}^{n}\sum_{k=0}^{m} b_{n,m}a_{m,k}\left|\frac{\sin\left(k+1/2\right)t}{\sin(t/2)}\right| \\
&= O\left(\sum_{m=0}^{n}\sum_{k=0}^{m} b_{n,m}a_{m,k}\frac{(k+1)t}{t}\right) \\
&= O\left((n+1)\sum_{m=0}^{n} b_{n,m}\left(\sum_{k=0}^{m} a_{m,k}\right)\right) \\
&= O\left((n+1)\sum_{m=0}^{n} b_{n,m}A_{m,m}\right) \\
&= O\left((n+1)B_{n,n}\right) = O(n+1),
\end{aligned}$$

because $A_{n,n} = B_{n,n} = 1$. □

Lemma 4.3. *If the matrices A and B satisfy the conditions of Theorem 3.1, then*

$$|(B\cdot A)_n(t)| = O\left(\frac{1}{t^2}\left(\sum_{m=0}^{n}\frac{b_{n,m}}{m+1}+\frac{1}{n+1}\right)\right), \quad \pi/(n+1)<t\le\pi.$$

Proof. Since $1/\sin(t/2) = O\left(\pi/t\right)$ for $\pi/(n+1) < t \le \pi$, we have

$$\begin{aligned}
|(B\cdot A)_n(t)| &= \left|\frac{1}{2\pi}\sum_{m=0}^{n}\sum_{k=0}^{m} b_{n,m}a_{m,k}\frac{\sin\left(k+1/2\right)t}{\sin(t/2)}\right| \\
&= O\left(\frac{1}{t}\right)\left|\sum_{m=0}^{n}\sum_{k=0}^{m} b_{n,m}a_{m,k}\sin\left(k+1/2\right)t\right|.
\end{aligned}$$

Now, using Abel's transformation after changing the order of summation, we have

$$\begin{aligned}
&\left|\sum_{m=0}^{n}\sum_{k=0}^{m} b_{n,m}a_{m,k}\sin(k+1/2)t\right| \\
= \ & \left|\sum_{m=0}^{n}\sum_{k=0}^{m} b_{n,m}a_{m,m-k}\sin(m-k+1/2)t\right| \\
= \ & \left|\sum_{k=0}^{n}\left[\sum_{m=k}^{n-1}(b_{n,m}a_{m,m-k}-b_{n,m+1}a_{m+1,m+1-k})\sum_{l=m}^{k}\sin(l-k+1/2)t\right.\right. \\
& \left.\left. +b_{n,n}a_{n,n-k}\sum_{l=k}^{n}\sin(l-k+1/2)t\right]\right| \\
= \ & O\left(\frac{1}{t}\right)\left(\sum_{m=0}^{n-1}\left[\sum_{k=0}^{m}|b_{n,m}a_{m,m-k}-b_{n,m+1}a_{m+1,m+1-k}|\right]\right. \\
& \left. +\sum_{k=0}^{n} b_{n,n}a_{n,n-k}\right) \\
= \ & O\left(\frac{1}{t}\right)\left[\sum_{m=0}^{n-1}\sum_{k=0}^{m-1}|b_{n,m}a_{m,m-k}-b_{n,m+1}a_{m+1,m+1-k}|+b_{n,n}\right. \\
& \left. +\sum_{m=0}^{n-1}|b_{n,m}a_{m,0}-b_{n,m+1}a_{m+1,1}|\right] \\
= \ & O\left(\frac{1}{t}\right)\left[\sum_{m=0}^{n-1} m.\frac{b_{n,m}}{(m+1)^2}+b_{n,n}+\sum_{m=0}^{n-1}\frac{b_{n,m}}{(m+1)^2}\right] \\
= \ & O\left(\frac{1}{t}\right)\left[\sum_{m=0}^{n}\frac{b_{n,m}}{(m+1)}+\frac{1}{(n+1)}\right],
\end{aligned}$$

in view of Lemma 4.1, conditions (3) and (4), and $A_{n,n}=1$.

Hence

$$|(B\cdot A)_n(t)| = O\left(\frac{1}{t^2}\left(\sum_{m=0}^{n}\frac{b_{n,m}}{m+1}+\frac{1}{n+1}\right)\right).$$

□

5. Proofs of main results

Proof of Theorem 3.1. By using the integral representation of $s_k(f;x)$, we have

$$s_k(f;x)-f(x)=\frac{1}{2\pi}\int_0^{\pi}\phi(t)\frac{\sin(k+1/2)t}{\sin(t/2)}dt.$$

From (2), we get

$$\begin{aligned}
t_n^{B\cdot A}(f;x)-f(x) &= \sum_{m=0}^{n}\sum_{k=0}^{m} b_{n,m}a_{m,k}(s_k(f;x)-f(x))\\
&= \sum_{m=0}^{n}\sum_{k=0}^{m} b_{n,m}a_{m,k}\left(\frac{1}{2\pi}\int_0^{\pi}\phi(t)\frac{\sin{(k+1/2)t}}{\sin(t/2)}dt\right)\\
&= \frac{1}{2\pi}\int_0^{\pi}\phi(t)\sum_{m=0}^{n}\sum_{k=0}^{m} b_{n,m}a_{m,k}\frac{\sin{(k+1/2)t}}{\sin(t/2)}dt\\
&= \int_0^{\pi}\phi(t)(B\cdot A)_n(t)dt\\
&= \int_0^{\pi/(n+1)}\phi(t)(B\cdot A)_n(t)dt+\int_{\pi/(n+1)}^{\pi}\phi(t)(B\cdot A)_n(t)dt\\
&= I_1+I_2. \qquad (12)
\end{aligned}$$

Now, using Lemma 4.2, the equality $1/\sin(t/2)=O\left(\pi/t\right)$ for $0<t\leq\pi/(n+1)$, and Hölder's inequality, we have

$$\begin{aligned}
|I_1| &\leq \int_0^{\pi/(n+1)}|\phi(t)(B\cdot A)_n(t)|\,dt=\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\pi/(n+1)}|\phi(t)||(B\cdot A)_n(t)|dt\\
&= O\left(\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\pi/(n+1)}\frac{|\phi(t)|\sin^{\beta}(t/2)}{\omega(t)}.\frac{(n+1)\omega(t)}{\sin^{\beta}(t/2)}dt\right)\\
&= O\left[(n+1)\left\{\int_0^{\pi/(n+1)}\left(\frac{|\phi(t)|\sin^{\beta}(t/2)}{\omega(t)}\right)^p dt\right\}^{1/p}\right.\\
&\qquad \left.\times\left\{\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\pi/(n+1)}\left(\frac{\omega(t)}{\sin^{\beta}(t/2)}\right)^q dt\right\}^{1/q}\right]\\
&= O_x\left[(n+1)^{1-1/p}\omega(\pi/(n+1))\left\{\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\pi/(n+1)}t^{-q\beta}dt\right\}^{1/q}\right]\\
&= O_x\left[(n+1)^{1-1/p}\omega(\pi/(n+1))(n+1)^{\beta-1/q}\right]\\
&= O_x\left(\omega(\pi/(n+1))(n+1)^{\beta+1-1/p-1/q}\right)\\
&= O_x\left(\omega(\pi/(n+1))(n+1)^{\beta}\right),
\end{aligned}$$

in view of condition (7), mean value theorem for integrals, $0<\beta<1-1/p$, and $p^{-1}+q^{-1}=1$.

Again, by Lemma 4.3, the equality $1/\sin(t/2) = O\left(\pi/t\right)$ for $\pi/(n+1) < t \leq \pi$, and Hölder's inequality, we have

$$\begin{aligned} |I_2| &\leq \int_{\pi/(n+1)}^{\pi} |\phi(t)(B\cdot A)_n(t)|\,dt \\ &= \int_{\pi/(n+1)}^{\pi} \frac{|\phi(t)|}{t^2(n+1)}dt + \int_{\pi/(n+1)}^{\pi} \frac{|\phi(t)|}{t^2}\sum_{m=0}^{n-1}\frac{b_{n,m}}{(m+1)}dt \\ &= I_{21} + I_{22}. \end{aligned}$$

Here

$$\begin{aligned} I_{21} &= O\left[\int_{\pi/(n+1)}^{\pi} \frac{|\phi(t)|}{t^2(n+1)}dt\right] \\ &= O\left[(n+1)^{-1}\int_{\pi/(n+1)}^{\pi} \frac{t^{-\gamma}|\phi(t)|\sin^{\beta}(t/2)}{\omega(t)}.\frac{\omega(t)}{t^{-\gamma+2}\sin^{\beta}(t/2)}dt\right] \\ &= O\left[(n+1)^{-1}\left\{\int_{\pi/(n+1)}^{\pi}\left(\frac{t^{-\gamma}|\phi(t)|\sin^{\beta}(t/2)}{\omega(t)}\right)^{p}dt\right\}^{1/p}\right. \\ &\qquad \left.\times\left\{\int_{\pi/(n+1)}^{\pi}\left(\frac{\omega(t)}{t^{-\gamma+2}\sin^{\beta}(t/2)}\right)^{q}dt\right\}^{1/q}\right] \\ &= O_x\left[(n+1)^{\gamma-1/p-1}\left\{\int_{\pi/(n+1)}^{\pi}\left(\frac{\omega(t)}{t}t^{-(-\gamma+\beta+1)}\right)^{q}dt\right\}^{1/q}\right] \\ &= O_x\left[(n+1)^{\gamma-1/p-1}\omega(\pi/(n+1))(n+1)\left\{\int_{\pi/(n+1)}^{\pi}t^{-q(-\gamma+\beta+1)}dt\right\}^{1/q}\right] \\ &= O_x\left[(n+1)^{\gamma-1/p}\omega(\pi/(n+1))(n+1)^{-\gamma+\beta+1-1/q}\right] \\ &= O_x\left(\omega(\pi/(n+1))(n+1)^{\beta}\right), \end{aligned}$$

and, similarly,

$$\begin{aligned} I_{22} &= O\left[\sum_{m=0}^{n}\frac{b_{n,m}}{(m+1)}\left\{\int_{\pi/(n+1)}^{\pi}\left(\frac{t^{-\gamma}|\phi(t)|\sin^{\beta}(t/2)}{\omega(t)}\right)^{p}dt\right\}^{1/p}\right. \\ &\qquad \left.\times\left\{\int_{\pi/(n+1)}^{\pi}\left(\frac{\omega(t)}{t^{-\gamma+2}\sin^{\beta}(t/2)}\right)^{q}dt\right\}^{1/q}\right] \end{aligned}$$

$$= O_x\left[\sum_{m=0}^{n}\frac{b_{n,m}}{(m+1)}(n+1)^{\gamma-1/p}\left\{\int_{\pi/(n+1)}^{\pi}\left(\frac{\omega(t)}{t}t^{-(-\gamma+\beta+1)}\right)^{q}dt\right\}^{1/q}\right]$$
$$= O_x\left[\sum_{m=0}^{n}\frac{b_{n,m}}{(m+1)}(n+1)^{\gamma-1/p}\omega(\pi/(n+1))(n+1)(n+1)^{-\gamma+\beta+1-1/q}\right]$$
$$= O_x\left(\sum_{m=0}^{n}\frac{b_{n,m}}{(m+1)}\omega(\pi/(n+1))(n+1)^{\beta+1}\right),$$

in view of conditions (6) and (8), mean value theorem for integrals, $1/p < \gamma < \beta + 1/p$, and $p^{-1} + q^{-1} = 1$.

Further, we have
$(n+1)^{\beta}\omega(\pi/(n+1)) + \sum_{m=0}^{n}\frac{b_{n,m}}{m+1}(n+1)^{\beta+1}\omega(\pi/(n+1)$

$$= \omega(\pi/(n+1))(n+1)^{\beta}\left[1+\sum_{m=0}^{n}\frac{b_{n,m}}{m+1}(n+1)\right]$$
$$\geq 2\omega(\pi/(n+1))(n+1)^{\beta},$$

thus

$$(n+1)^{\beta}\omega(\pi/(n+1)) = O\left(\sum_{m=0}^{n}\frac{b_{n,m}}{m+1}(n+1)^{\beta+1}\omega(\pi/(n+1)\right). \tag{13}$$

Finally, from (12)–(13) we get

$$|t_n^{B\cdot A}(f;x) - f(x)| = O_x\left(\sum_{m=0}^{n}\frac{b_{n,m}}{m+1}(n+1)^{\beta+1}\omega(1/(n+1))\right),$$

in view of Note 3.2. This completes the proof of Theorem 3.1.

Proof of Theorem 3.3. Following the proof of Theorem 3.1, by Hölder's inequality for $p = 1, q = \infty$, we have

$$|I_1| = O\left[(n+1)\int_0^{\pi/(n+1)}\frac{|\phi(t)|\sin^{\beta}(t/2)}{\omega(t)}dt\right.$$
$$\left.\times \operatorname*{ess\,sup}_{0<t\leq\pi/(n+1)}\left|\frac{\omega(t)}{\sin^{\beta}(t/2)}\right|\right]$$
$$= O_x\left[(n+1)(n+1)^{-1}\operatorname*{ess\,sup}_{0<t\leq\pi/(n+1)}\left|\frac{\omega(t)}{t^{\beta}}\right|\right]$$
$$= O_x\left(\omega(\pi/(n+1))(n+1)^{\beta}\right), \tag{14}$$

in view of conditions (9) and (10).

Similarly,

$$\begin{aligned} I_{21} &= O\left[(n+1)^{-1}\int_{\pi/(n+1)}^{\pi}\frac{t^{-\gamma}|\phi(t)|\sin^{\beta}(t/2)}{\omega(t)}dt \right. \\ &\quad \left. \times \operatorname*{ess\,sup}_{\pi/(n+1)<t\le\pi}\left|\frac{\omega(t)}{t^{-\gamma+2}\sin^{\beta}(t/2)}\right|\right] \\ &= O_x\left[(n+1)^{\gamma-1-1}\omega\left(\frac{\pi}{n+1}\right)\left(\frac{(n+1)^{2+\beta-\gamma}}{\pi^{2+\beta-\gamma}}\right)\right] \\ &= O_x\left(\omega(\pi/(n+1))(n+1)^{\beta}\right), \qquad (15) \end{aligned}$$

and

$$\begin{aligned} I_{22} &= O\left[\sum_{m=0}^{n}\frac{b_{n,m}}{(m+1)}\int_{\pi/(n+1)}^{\pi}\frac{t^{-\gamma}|\phi(t)|\sin^{\beta}(t/2)}{\omega(t)}dt \right. \\ &\quad \left. \times \operatorname*{ess\,sup}_{\pi/(n+1)<t\le\pi}\left|\frac{\omega(t)}{t^{-\gamma+2}\sin^{\beta}(t/2)}\right|\right] \\ &= O_x\left[\sum_{m=0}^{n}\frac{b_{n,m}}{(m+1)}(n+1)^{\gamma-1}\omega\left(\frac{\pi}{n+1}\right)\left(\frac{(n+1)^{2+\beta-\gamma}}{\pi^{2+\beta-\gamma}}\right)\right] \\ &= O_x\left(\sum_{m=0}^{n}\frac{b_{n,m}}{(m+1)}\omega(\pi/(n+1))(n+1)^{\beta+1}\right), \qquad (16) \end{aligned}$$

in view of the decreasing nature of $\omega(t)/t^{\beta+2-\gamma}$, and condition (11).

Using (14)–(16), we get

$$\left|t_n^{B\cdot A}(f;x)-f(x)\right| = O_x\left(\sum_{m=0}^{n}\frac{b_{n,m}}{m+1}(n+1)^{\beta+1}\omega(1/(n+1))\right),$$

in view of Note 3.2. This completes the proof of Theorem 3.3.

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