

## Fixed points of $\alpha - \psi$ multivalued contractive mappings in cone metric spaces

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ABSTRACT. Some results on fixed points of  $\alpha - \psi$  multivalued mappings of different contractive nature over a cone metric space with a normal constant equal to 1 have been established.

### 1. Introduction

Starting from the fundamental results of fixed point theory, namely the Banach contraction principle, many results on fixed points of mappings with their applications in different branches of mathematics are available in the current literature. The fixed point of multifunctions is a generalization of the fixed point of single valued mappings. In some works of non convex analysis, especially in ordered normed vector spaces, one defined an order by using a cone in a vector space (see [16]). In the recent past years Huang and Zhang [6] initiated a new notion of cone metric that generalizes the concept of the usual metric, by replacing its values (real numbers) with ordered elements in a Banach space. Subsequently many results in this direction can also be found in [1]–[3], [4], [5], [7], [8], [9]–[11], [15].

Samet et al. [19] introduced a new idea of an  $\alpha - \psi$  contractive mapping and obtained some fixed point theorems in a complete metric space. Then Karapinar [12] and Karapinar et al. [13], [14] succeeded in showing many results on fixed points of  $(\alpha - \psi)$ -contractive mappings over complete metric spaces. In 2009 Rezapour and Haghi [17] proved some results on fixed points of multifunctions over cone metric spaces with a normal constant equal to 1. In this paper we prove some fixed point theorems for a class of  $\alpha - \psi$  multivalued mappings of different contractive characters over a cone metric space having normal constant equal to 1. Some suitable examples are given in support of our theorems.

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## 2. Preliminaries

Let  $E$  be a real Banach space. A non-empty subset  $P$  of  $E$  is said to be a cone whenever

- (i)  $P$  is closed and  $P \neq \{0\}$ ,
- (ii)  $ax + by \in P$  for all  $x, y \in P$  and for all real numbers  $a, b \geq 0$ ,
- (iii)  $P \cap (-P) = \{0\}$ .

For a given cone  $P \subseteq E$ , one can define the partial ordering  $\leq$  with respect to  $P$  by  $x \leq y$  if and only if  $y - x \in P$ . In addition,  $a < b$  stands for  $a \leq b$  but  $a \neq b$ , while  $a \ll b$  stands for  $b - a \in \text{int } P$ , where  $\text{int } P$  denotes the interior of  $P$ .

A cone  $P$  is said to be a normal cone if there exists a real number  $k > 0$  such that for all  $x, y \in E$ ,

$$0 \leq x \leq y \Rightarrow \|x\| \leq k\|y\|.$$

The least positive number  $k$  satisfying the above condition is called the normal constant of  $P$ . Clearly  $k \geq 1$ .

In the following we always suppose that  $E$  is a real Banach space and  $P$  is a cone in  $E$  with normal constant  $k = 1$ ,  $\text{int } P \neq \emptyset$  and  $\leq$  is the partial ordering with respect to  $P$ .

**Definition 2.1** (see [6]). Let  $X$  be a non-empty set. Suppose that the mapping  $d : X \times X \rightarrow E$  satisfies

- (i)  $0 \leq d(x, y)$  for all  $x, y \in X$  and  $d(x, y) = 0$  if and only if  $x = y$ ,
- (ii)  $d(x, y) = d(y, x)$ ,  $x, y \in X$ ,
- (iii)  $d(x, y) \leq d(x, z) + d(z, y)$ ,  $x, y, z \in X$ .

Then one says that  $d$  is a cone metric on  $X$  and  $(X, d)$  is a cone metric space.

**Example 2.2** (see [6] or [17]). Let  $E = \mathbb{R}^2$ ,  $P = \{(x, y) \in E : x, y \geq 0\}$ ,  $X = \mathbb{R}$  and  $d : X \times X \rightarrow E$  defined by  $d(x, y) = (|x - y|, \alpha|x - y|)$ , where  $\alpha \geq 0$  is a constant. Then  $(X, d)$  is a cone metric space and the normal constant of  $P$  is  $k = 1$ .

**Definition 2.3** (see [6]). Let  $(X, d)$  be a cone metric space,  $x \in X$  and let  $\{x_n\}$  be a sequence in  $X$ . Then

- (A)  $\{x_n\}$  converges to  $x \in X$  ( $\lim_{n \rightarrow \infty} x_n = x$ ) whenever for every  $c \in E$  with  $0 \ll c$  there is a natural number  $N$  such that  $d(x_n, x) \ll c$  for all  $n \geq N$ ;
- (B)  $\{x_n\}$  is a Cauchy sequence whenever for every  $c \in E$  with  $0 \ll c$  there is a natural number  $N$  such that  $d(x_n, x_m) \ll c$  for all  $n, m \geq N$ ;
- (C)  $(X, d)$  is a complete cone metric space if every Cauchy sequence in  $X$  is convergent in  $X$ .

**Definition 2.4** (see [17]). Let  $(X, d)$  be a cone metric space and  $B \subset X$ .

- (1) An element  $b \in B$  is called an interior point of  $B$  whenever there is  $0 \ll p$  such that  $N(b, p) \subset B$ , where

$$N(b, p) = \{y \in X : d(y, b) \ll p\}.$$

- (2) A subset  $A \subset X$  is called open if each element of  $A$  is an interior point of  $A$ .

The family  $\mathfrak{B} = \{N(x, e) : x \in X, 0 \ll e\}$  is a sub-basis for a topology  $\tau_c$ , called cone topology on  $X$ . The topology  $\tau_c$  is Hausdorff and first countable (see [17]).

We mainly deal with normal cones having normal constant equal to 1. But for each integer  $k > 1$ , there are cones with normal constant  $k$ . Also there are non-normal cones (see [18]). Before introducing our main results we first state some known lemmas and give some more definitions.

**Lemma 2.5** (see [17]). *Let  $(X, d)$  be a cone metric space,  $P$  be a normal cone with normal constant 1, and let  $A$  be a compact set in  $(X, \tau_c)$ . Then for every  $x \in X$ , there exists  $a_0 \in A$  such that  $\|d(x, a_0)\| = \inf_{a \in A} \|d(x, a)\|$ .*

**Lemma 2.6** (see [17]). *Let  $(X, d)$  be a cone metric space,  $P$  be a normal cone with normal constant 1, and let  $A, B$  be two compact sets in  $(X, \tau_c)$ . Then  $\sup_{x \in B} d'(x, A) < \infty$ , where,  $d'(x, A) = \inf_{a \in A} \|d(x, a)\|$ .*

**Definition 2.7** (see [17]). Let  $(X, d)$  be a cone metric space,  $P$  be a normal cone with normal constant 1, let  $\mathcal{H}_c(X)$  be the set of all compact subsets of  $(X, \tau_c)$ , and  $A \in \mathcal{H}_c(X)$ . Define

$$h_A : \mathcal{H}_c(X) \rightarrow [0, \infty) \text{ by } h_A(B) = \sup_{x \in A} d'(x, B)$$

and

$$d_H : \mathcal{H}_c(X) \times \mathcal{H}_c(X) \rightarrow [0, \infty) \text{ by } d_H(A, B) = \max\{h_A(B), h_B(A)\}.$$

*Remark 2.8* (see [17]). Let  $(X, d)$  be a cone metric space with normal constant equal to 1. Define  $\rho : X \times X \rightarrow [0, \infty)$  by  $\rho(x, y) = \|d(x, y)\|$ . Then  $(X, \rho)$  is a metric space.

*Remark 2.9* (see [17]). For each  $A, B \in \mathcal{H}_c(X)$  and  $x, y \in X$ , we have the following relations:

- (a)  $d'(x, A) \leq \|d(x, y)\| + d'(y, A)$ ,
- (b)  $d'(x, A) \leq d'(x, B) + h_B(A)$ ,
- (c)  $d'(x, A) \leq \|d(x, y)\| + d'(y, B) + h_B(A)$ .

**Definition 2.10** (see [19]). Let  $T : X \rightarrow X$  be a map and  $\alpha : X \times X \rightarrow \mathbb{R}$  be a function. Then  $T$  is said to be  $\alpha$ -admissible if  $\alpha(x, y) \geq 1$  implies  $\alpha(Tx, Ty) \geq 1$ , for all  $x, y \in X$ . An  $\alpha$ -admissible map  $T$  is said to be

triangular  $\alpha$ -admissible if  $\alpha(x, y) \geq 1$  and  $\alpha(y, z) \geq 1$  imply  $\alpha(x, z) \geq 1$  for all  $x, y, z \in X$ .

**Lemma 2.11** (see [13]). *Let  $T : X \rightarrow X$  be a triangular  $\alpha$ -admissible map. Assume that there exists  $x_1 \in X$  such that  $\alpha(x_1, Tx_1) \geq 1$ . Define a sequence  $\{x_n\} \in X$  by  $x_{n+1} = Tx_n$ . Then we have  $\alpha(x_n, x_m) \geq 1$  for all  $m, n \in \mathbb{N}$  with  $n < m$ .*

**Definition 2.12** (see [12]). Let  $\Psi$  be the class of all functions  $\psi : [0, \infty) \rightarrow [0, \infty)$  satisfying the following conditions:

- (i)  $\psi$  is non-decreasing,
- (ii)  $\psi$  is sub-additive, i.e.,  $\psi(s + t) \leq \psi(s) + \psi(t)$  for all  $s, t \in [0, \infty)$ ,
- (iii)  $\psi$  is continuous,
- (iv)  $\psi(t) = 0$  if and only if  $t = 0$ .

### 3. Main results

**Definition 3.1.** Let  $T : X \rightarrow \mathcal{H}_c(X)$  be a map and  $\alpha : \mathcal{H}_c(X) \times \mathcal{H}_c(X) \rightarrow \mathbb{R}$  be a function. Then  $T$  is said to be Hausdorff  $\alpha$ -admissible if  $\alpha(A, B) \geq 1$  implies  $\alpha(TA, TB) \geq 1$ , for all  $A, B \in \mathcal{H}_c(X)$ .

**Definition 3.2.** A Hausdorff  $\alpha$ -admissible map  $T$  is said to be a triangular  $\alpha$ -orbital contraction if  $\alpha(A, B) \geq 1$  and  $\alpha(B, C) \geq 1$  imply  $\alpha(A, C) \geq 1$ , for all  $A, B, C \in \mathcal{H}_c(X)$

**Lemma 3.3.** *Let  $T : X \rightarrow \mathcal{H}_c(X)$  be a triangular  $\alpha$ -orbital contraction. Assume that*

$$\alpha(x_1, Tx_1) \geq 1 \text{ for some } x_1 \in X. \quad (3.1)$$

*Define a sequence  $\{x_n\}$  by  $x_{n+1} \in Tx_n$ . Then we have  $\alpha(x_n, x_m) \geq 1$  for all  $m, n \in \mathbb{N}$  with  $n < m$ .*

*Proof.* The proof is straightforward. □

**Definition 3.4.** Let  $(X, d)$  be a cone metric space, let  $\alpha : \mathcal{H}_c(X) \times \mathcal{H}_c(X) \rightarrow \mathbb{R}$  be a function and let  $\psi \in \Psi$ . A map  $T : X \rightarrow \mathcal{H}_c(X)$  is called an  $(\alpha - \psi)$ -Banach type contraction if there is a constant  $c$ ,  $0 < c < 1$ , such that

$$\alpha(x, y)\psi(d_H(Tx, Ty)) \leq c\psi(d'(x, y)), \quad x, y \in X. \quad (3.2)$$

**Theorem 3.5.** *Let  $(X, d)$  be a complete cone metric space with normal constant equal to 1, let  $\alpha : \mathcal{H}_c(X) \times \mathcal{H}_c(X) \rightarrow \mathbb{R}$  be a function and let  $\psi \in \Psi$ . Assume that  $T : X \rightarrow \mathcal{H}_c(X)$  is an  $(\alpha - \psi)$ -Banach type triangular  $\alpha$ -orbital contraction such that (3.1) holds. Then  $T$  has a fixed point.*

*Proof.* For some  $x_1 \in X$  we have  $\alpha(x_1, Tx_1) \geq 1$ . Then we can define a sequence  $\{x_n\} \subset X$  by  $x_{n+1} \in Tx_n$  for all  $n \in \mathbb{N}$ . If  $x_{n_0} = x_{n_0+1}$  for some  $n_0 \in \mathbb{N}$ , then  $x_{n_0}$  is a fixed point. So the proof is done in this case.

Let  $x_n \neq x_{n+1}$  for all  $n \in \mathbb{N}$ . Then by Lemma 3.3 we get that  $\alpha(x_n, x_{n+1}) \geq 1$  for all  $n \in \mathbb{N}$ . As  $x_1 \in X$ , again by Lemma 2.6, there exists  $x_2 \in Tx_1$  such that  $d'(x_1, Tx_1) = \|d(x_1, x_2)\|$ . In this way, if  $x_n \in Tx_{n-1}$ , then there exists  $x_{n+1} \in Tx_n$  such that  $d'(x_n, Tx_n) = \|d(x_n, x_{n+1})\|$ . Now, since

$$\begin{aligned} \psi(\|d(x_n, x_{n+1})\|) &= \psi(d'(x_n, Tx_n)) \\ &\leq \psi(h_{Tx_{n-1}}(Tx_n)) \leq \psi(d_H(Tx_{n-1}, Tx_n)) \\ &\leq \alpha(x_{n-1}, x_n)\psi(d_H(Tx_{n-1}, Tx_n)), \end{aligned} \quad (3.3)$$

by (3.2) and  $c < 1$  we have

$$\begin{aligned} \psi(\|d(x_n, x_{n+1})\|) &\leq c(\psi(d'(x_{n-1}, x_n))) = c\psi(\|d(x_{n-1}, x_n)\|) \\ &< \psi(\|d(x_{n-1}, x_n)\|). \end{aligned}$$

As  $\psi$  is non-decreasing we get,

$$\|d(x_n, x_{n+1})\| < \|d(x_{n-1}, x_n)\|.$$

Thus  $\{\|d(x_n, x_{n+1})\|\}$  is decreasing sequence of real numbers which is also bounded below. Hence the sequence is convergent to a number  $r \geq 0$ . We claim that  $r = 0$ . Otherwise, if  $r > 0$ , then in view of the continuity of  $\psi$  we have that  $\psi(\|d(x_n, x_{n+1})\|) \rightarrow \psi(r)$ , as  $n \rightarrow \infty$ . So from (3.3) we see that  $\psi(r) \leq c\psi(r)$ , i.e.,  $c \geq 1$ , a contradiction. Therefore,  $r = 0$ . Hence the sequence  $\{\|d(x_n, x_{n+1})\|\}$  converges to 0. But

$$\begin{aligned} \psi(\|d(x_n, x_{n+1})\|) &\leq c(\psi(\|d(x_{n-1}, x_n)\|)) \\ &\leq \dots \leq c^{n-1}(\psi(\|d(x_1, x_2)\|)). \end{aligned} \quad (3.4)$$

Hence

$$\begin{aligned} \psi(\|d(x_n, x_m)\|) &\leq \psi\left(\sum_{i=m}^{n-1} \|d(x_i, x_{i+1})\|\right) \\ &\leq \sum_{i=m}^{n-1} \psi(\|d(x_i, x_{i+1})\|) \\ &\leq (c^{n-2} + \dots + c^{m-1})\psi(\|d(x_1, x_2)\|) \\ &\leq \frac{c^{m-1}}{1-c}\psi(\|d(x_1, x_2)\|), \end{aligned} \quad (3.5)$$

which shows that

$$\lim_{m, n \rightarrow \infty} \psi(\|d(x_n, x_m)\|) = 0.$$

Consequently,

$$\lim_{m, n \rightarrow \infty} \|d(x_n, x_m)\| = 0.$$

So  $\{x_n\}$  is a Cauchy sequence in  $X$ . As  $X$  is complete,  $\lim_{n \rightarrow \infty} x_n = \xi \in X$ . As  $\alpha(x_n, x_m) \geq 1$  for  $n < m$ , we see that  $\alpha(x_n, \xi) \geq 1$  for all  $n$ . Since

$$d'(\xi, T\xi) \leq d'(\xi, Tx_n) + h_{Tx_n}(T\xi) \leq d'(\xi, Tx_n) + d_H(Tx_n, T\xi),$$

so

$$\begin{aligned} \psi(d'(\xi, T\xi)) &\leq \psi(d'(\xi, Tx_n)) + \psi(d_H(Tx_n, T\xi)) \\ &\leq \psi(d'(\xi, Tx_n)) + \alpha(x_n, \xi)\psi(d_H(Tx_n, T\xi)), \end{aligned} \quad (3.6)$$

which, because of (3.2), gives

$$\psi(d'(\xi, T\xi)) \leq \psi(d'(\xi, Tx_n)) + c(\psi(d'(x_n, \xi))).$$

If  $n \rightarrow \infty$ , then  $\psi(d'(\xi, T\xi)) = 0$  implying that  $d'(\xi, T\xi) = 0$ . Hence  $\xi \in T\xi$  showing that  $\xi$  is a fixed point of  $T$ .  $\square$

**Definition 3.6.** Let  $(X, d)$  be a cone metric space, let  $\alpha : \mathcal{H}_c(X) \times \mathcal{H}_c(X) \rightarrow \mathbb{R}$  be a function and let  $\psi \in \Psi$ . A map  $T : X \rightarrow \mathcal{H}_c(X)$  is called an  $(\alpha - \psi)$ -Kannan type contraction if there is a constant  $c$ ,  $0 < c < 1/2$ , such that

$$\alpha(x, y)\psi(d_H(Tx, Ty)) \leq c\psi(d'(Tx, x) + d'(y, Ty)), \quad x, y \in X. \quad (3.7)$$

**Theorem 3.7.** Let  $(X, d)$  be a complete cone metric space with normal constant equal to 1. Let  $\alpha : \mathcal{H}_c(X) \times \mathcal{H}_c(X) \rightarrow \mathbb{R}$  be a function,  $\psi \in \Psi$  and let  $T : X \rightarrow \mathcal{H}_c(X)$  be an  $(\alpha - \psi)$ -Kannan type triangular  $\alpha$ -orbital contraction. If (3.1) holds, then  $T$  has a fixed point.

*Proof.* For  $x_1 \in X$  we have  $\alpha(x_1, Tx_1) \geq 1$ . Then we can define a sequence  $\{x_n\} \subset X$  by  $x_{n+1} \in Tx_n$  for all  $n \in \mathbb{N}$ . If  $x_{n_0} = x_{n_0+1}$  for some  $n_0 \in \mathbb{N}$ , then  $x_{n_0}$  is a fixed point and the proof is done. As in the proof of Theorem 3.5, for  $x_n \neq x_{n+1}$  ( $n \in \mathbb{N}$ ) we determine a sequence  $\{x_n\}$  such that (3.3) holds. Then by (3.7) we have

$$\begin{aligned} \psi(\|d(x_n, x_{n+1})\|) &\leq c\psi(d'(Tx_{n-1}, x_{n-1}) + d'(Tx_n, x_n)) \\ &\leq c(\psi(\|d(x_{n-1}, x_n)\|) + \psi(\|d(x_{n+1}, x_n)\|)), \end{aligned}$$

which shows that

$$\psi(\|d(x_n, x_{n+1})\|) \leq \frac{c}{1-c}\psi(\|d(x_{n-1}, x_n)\|) = p\psi(\|d(x_{n-1}, x_n)\|), \quad (3.8)$$

where  $p = \frac{c}{1-c} < 1$ . So

$$\psi(\|d(x_n, x_{n+1})\|) < \psi(\|d(x_{n-1}, x_n)\|),$$

and by a similar argument as in Theorem 3.5, using only (3.4) and (3.5) with  $p$  instead of  $c$ , we see that the sequence  $\{x_n\}$  converges to an element  $\xi \in X$  and  $\alpha(x_n, \xi) \geq 1$  for all  $n$ .

To prove that  $\xi$  is a fixed point of  $T$ , we observe that (3.6) together with (3.7) gives that

$$\psi(d'(\xi, T\xi)) \leq \psi(d'(\xi, Tx_n)) + c(\psi(d'(Tx_n, x_n) + d'(T\xi, \xi)))$$

and so

$$\psi(d'(\xi, T\xi)) \leq \frac{1}{1-c}\psi(d'(\xi, Tx_n)) + \frac{c}{1-c}\psi(d'(Tx_n, x_n)). \quad (3.9)$$

Thus, as  $n \rightarrow \infty$ , we get  $\psi(d'(\xi, T\xi)) = 0$  which implies  $d'(\xi, T\xi) = 0$ . Hence  $\xi \in T\xi$ , i.e.,  $\xi$  is a fixed point of  $T$ .  $\square$

**Definition 3.8.** Let  $(X, d)$  be a cone metric space, let  $\alpha : \mathcal{H}_c(X) \times \mathcal{H}_c(X) \rightarrow \mathbb{R}$  be a function and let  $\psi \in \Psi$ . A map  $T : X \rightarrow \mathcal{H}_c(X)$  is called an  $(\alpha - \psi)$ -Chatterjea type contraction if there is a constant  $c$ ,  $0 < c < 1/2$ , such that

$$\alpha(x, y)\psi(d_H(Tx, Ty)) \leq c\psi(d'(Tx, y) + d'(Ty, x)), \quad x, y \in X. \quad (3.10)$$

**Theorem 3.9.** Let  $(X, d)$  be a complete cone metric space with normal constant equal to 1. Let  $\alpha : \mathcal{H}_c(X) \times \mathcal{H}_c(X) \rightarrow \mathbb{R}$  be a function,  $\psi \in \Psi$  and let  $T : X \rightarrow \mathcal{H}_c(X)$  be an  $(\alpha - \psi)$ -Chatterjea type triangular  $\alpha$ -orbital contraction. If (3.1) is satisfied, then  $T$  has a fixed point.

*Proof.* In the same way as in the proof of Theorem 3.7, in the case  $x_n \neq x_{n+1}$  ( $n \in \mathbb{N}$ ) we determine a sequence  $\{x_n\}$  such that (3.3) holds. Then by (3.10) we get that

$$\begin{aligned} \psi(\|d(x_n, x_{n+1})\|) &\leq c\psi(d'(Tx_{n-1}, x_n) + d'(Tx_n, x_{n-1})) \\ &\leq c(\psi(\|d(x_{n+1}, x_{n-1})\|)) \\ &\leq c(\psi(\|d(x_{n+1}, x_n)\| + \|d(x_n, x_{n-1})\|)) \\ &\leq c(\psi(\|d(x_{n+1}, x_n)\|) + \psi(\|d(x_n, x_{n-1})\|)). \end{aligned}$$

Thus (3.8) holds and the sequence  $\{x_n\}$  converges to an element  $\xi \in X$  satisfying  $\alpha(x_n, \xi) \geq 1$  for all  $n$ .

To find a fixed point of  $T$ , we observe that (3.6) together with (3.10) gives that

$$\psi(d'(\xi, T\xi)) \leq \psi(d'(\xi, Tx_n)) + c(\psi(d'(Tx_n, \xi) + d'(T\xi, x_n))).$$

Therefore,

$$\psi(d'(\xi, T\xi)) \leq \frac{1+c}{1-c}\psi(d'(\xi, Tx_n)) + \frac{c}{1-c}\psi(d'(x_n, \xi)),$$

which, similarly to (3.9), shows that  $\xi$  is a fixed point of  $T$ .  $\square$

**Example 3.10.** Let  $X = [0, 1]$ ,  $E = \mathbb{R}^2$  and  $P = \{(x, y) \in E : x, y \geq 0\}$ . Define  $d : X \times X \rightarrow E$  by  $d(x, y) = (\frac{1}{2}|x - y|, \frac{1}{2}|x - y|)$ ,  $x, y \in X$ . Then

$(X, d)$  is a complete cone metric space with normal constant of  $P$  equal to 1. Define  $T : X \rightarrow \mathcal{H}_c(X)$  by

$$T(x) = \begin{cases} \{0\} & \text{if } x \in [0, \frac{1}{2}]; \\ [0, \frac{1}{2}(x - \frac{1}{2})^2] & \text{if } x \in (\frac{1}{2}, 1]. \end{cases}$$

define  $\psi : [0, \infty) \rightarrow [0, \infty)$  by  $\psi(t) = t$  for all  $t \in [0, \infty)$  and let  $\alpha : \mathcal{H}_c(X) \times \mathcal{H}_c(X) \rightarrow \mathbb{R}$  be defined by  $\alpha(A, B) = 1$  for all  $A, B \in \mathcal{H}_c(X)$ . Clearly all the conditions of Theorem 3.5 are satisfied by assuming  $c = \frac{3}{4}$ . Then  $T$  has a fixed point 0.

**Example 3.11.** Let  $X = [0, 1]$ ,  $E = \mathbb{R}^2$  and  $P = \{(x, y) \in E : x, y \geq 0\}$ . Define  $d : X \times X \rightarrow E$  by  $d(x, y) = (|x - y|, \frac{1}{2}|x - y|)$ ,  $x, y \in X$ . Then  $(X, d)$  is a complete cone metric space with normal constant of  $P$  equal to 1. Define  $T : X \rightarrow \mathcal{H}_c(X)$  by

$$T(x) = \begin{cases} \{0\} & \text{if } x \in [0, \frac{1}{2}]; \\ [0, \frac{1}{2}(x - \frac{1}{2})^2] & \text{if } x \in (\frac{1}{2}, 1]. \end{cases}$$

define  $\psi : [0, \infty) \rightarrow [0, \infty)$  by  $\psi(t) = t$  for all  $t \in [0, \infty)$  and let  $\alpha : \mathcal{H}_c(X) \times \mathcal{H}_c(X) \rightarrow \mathbb{R}$  be defined by  $\alpha(A, B) = 1$  for all  $A, B \in \mathcal{H}_c(X)$ . Clearly all the conditions of Theorem 3.7 are satisfied by assuming  $c = 0.49$ . Then  $T$  has a fixed point 0.

**Example 3.12.** Let  $X = \{a_1, a_2, a_3, \dots\}$  be a countable set,  $E = (l^2, \|\cdot\|_2)$  and  $P = \{\{x_n\}_{n \geq 1} \in l^2 : x_n \geq 0 \text{ for all } n \geq 1\}$ . Let  $x_i = \{\frac{3^i}{n}\}_{n \geq 1}$  for all  $i \geq 1$  and note that  $x_i \in l^2 (i \geq 1)$ . Define the map  $d : X \times X \rightarrow P$  by

$$d(a_i, a_j) = |x_i - x_j| = \left\{ \frac{|3^i - 3^j|}{n} \right\}_{n \geq 1}.$$

Then we can easily see that  $(X, d)$  is a complete cone metric space with the normal constant of  $P$  is equal to 1. We define the multifunction

$$T : X \rightarrow \mathcal{H}_c(X)$$

by  $Ta_1 = \{a_1\}$  and  $Ta_i = \{a_1, a_2, \dots, a_{i-1}\}$  for all  $i > 1$ . Again we define  $\psi : [0, \infty) \rightarrow [0, \infty)$  by  $\psi(t) = t$  for all  $t \in [0, \infty)$  and let  $\alpha : \mathcal{H}_c(X) \times \mathcal{H}_c(X) \rightarrow \mathbb{R}$  be defined by  $\alpha(A, B) = 1$  for all  $A, B \in \mathcal{H}_c(X)$ . Then we see that  $T$  is a triangular  $\alpha$ -admissible map satisfying all the conditions of Theorem 3.9 by assuming  $c = \frac{1}{3}$  with a fixed point  $Ta_1 = a_1$ .

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## References

- [1] T. Abdeljawad and E. Karapinar, *A common fixed point theorem of a Greguš type on convex cone metric spaces*, J. Comput. Anal. Appl, **13** (2011), 609–621.
- [2] T. Abdeljawad, E. Karapinar, and K. Taş, *Common fixed point theorems in cone Banach spaces*, Hacet. J. Math. Stat. **40** (2011), 211–217.
- [3] A. Amini-Harandi and M. Fakhar, *Fixed point theory in cone metric spaces obtained via the scalarization method*, Comput. Math. Appl. **59** (2010), 3529–3534.
- [4] W.-S. Du, *A note on cone metric fixed point theory and its equivalence*, Nonlinear Anal. **72** (2010), 2259–2261.
- [5] Y. Feng and N. Mao, *The equivalence of cone metric spaces and metric spaces fixed point theory*, Fixed Point Theory **11** (2011), 259–264.
- [6] L. G. Huang and X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl. **332** (2007), 1468–1476.
- [7] S. Janković, Z. Kadelburg, and S. Radenović, *On cone metric spaces: a survey*, Nonlinear Anal. **74** (2011), 2591–2601.
- [8] S. Janković, Z. Kadelburg, S. Radenović, and V. Rakočević, *A note on the equivalence of some metric and cone metric fixed point results*, Appl. Math. Lett. **24** (2011), 370–374.
- [9] E. Karapinar, *Fixed point theorems in cone Banach spaces*, Fixed Point Theory Appl. **2009**, Art. ID 609281, 9 pp.
- [10] E. Karapinar, *Couple fixed point theorems for nonlinear contractions in cone metric spaces*, Comput. Math. Appl. **59** (2010), 3656–3668.
- [11] E. Karapinar, *Some nonunique fixed point theorems of Ćirić type on cone metric spaces*, Abstr. Appl. Anal. **2010**, Art. ID 123094, 14 pp.
- [12] E. Karapinar,  *$\alpha - \psi$ -Geraghty contraction type mappings and some related fixed point results*, Filomat, **28** (2014), 37–48.
- [13] E. Karapinar, P. Kumam, and P. Salimi, *On  $\alpha - \psi$ -Meir-Keeler contractive mappings*, Fixed Point Theory Appl. **2013**, 2013:94, 12 pp.
- [14] E. Karapinar and B. Samet, *Generalized  $(\alpha - \psi)$ -contractive type mappings and related fixed point theorems with applications*, Abstr. Appl. Anal. **2012**, Art. ID 793486, 17 pp.
- [15] M. A. Khamsi, *Remarks on cone metric spaces and fixed point theorems of contractive mappings*, Fixed Point Theory Appl. **2010**, Art. ID 315398, 7 pp.
- [16] H. Mohebi, *Topical functions and their properties in a class of ordered Banach spaces*, in: Appl. Optim. **99**, Springer, New York, 2005, pp. 343–361.
- [17] Sh. Rezapour and R. H. Haghi, *Fixed point of multifunctions on cone metric spaces*, Numer. Funct. Anal. Optim. **30** (2009), 825–832.
- [18] Sh. Rezapour and R. Hambarani, *Some notes on the paper “Cone metric spaces and fixed point theorems of contractive mappings”*, J. Math. Anal. Appl. **345** (2008), 719–724.
- [19] B. Samet, C. Vetro, and P. Vetro, *Fixed point theorems for  $\alpha - \psi$ -contractive type mappings*, Nonlinear Anal. **75** (2012), 2154–2165.

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