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Nonhomogeneous linear differential polynomials generated by solutions of complex differential equations in the unit disc

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Abstract. We consider the complex oscillation of nonhomogeneous linear differential polynomials $g_k = \sum_{j=0}^{k} d_j f^{(j)} + b$, where d_j $(j = 0, 1, \ldots, k)$ and b are meromorphic functions of finite [p,q]-order in the unit disc Δ, generated by meromorphic solutions of linear differential equations with meromorphic coefficients of finite [p,q]-order in Δ.

1. Introduction

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of Nevanlinna's value distribution theory on the complex plane and in the unit disc $\Delta = \{z\colon\, |z| < 1\}$ (see [11], [12], [16], [17], [25]).

First, let us recall some notations about the finite iterated order and the growth index to classify generally meromorphic functions of fast growth in Δ as those in $\mathbb{C}$ (see [6], [15], [16]). Let us define inductively, for $r \in (0, +\infty)$, $\exp_1 r = e^r$ and $\exp_{p+1} r = \exp\left(\exp_p r\right)$, $p \in \mathbb{N}$. We also define, for all r sufficiently large in $(0, +\infty)$, $\log_1 r = \log r$ and $\log_{p+1} r = \log\left(\log_p r\right)$, $p \in \mathbb{N}$. Moreover, we use the notations $\exp_0 r = r$, $\log_0 r = r$, $\exp_{-1} r = \log_1 r$, and $\log_{-1} r = \exp_1 r$.

Definition 1.1 (see [8]). The iterated p-order of a meromorphic function f in Δ is defined by

$$\rho_p(f) = \limsup_{r \to 1^-} \frac{\log_p^+ T(r, f)}{\log \frac{1}{1-r}} \quad (p \geq 1).$$

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For an analytic function f in Δ, we also define

$$\rho_{M,p}(f) = \limsup_{r \to 1^-} \frac{\log_{p+1}^{+} M(r,f)}{\log \frac{1}{1-r}} \quad (p \geq 1).$$

Remark 1.1. It follows by M. Tsuji [25] that if f is an analytic function in Δ, then $\rho_1(f) \leq \rho_{M,1}(f) \leq \rho_1(f) + 1$. However, by Proposition 2.2.2 in [16], we have $\rho_{M,p}(f) = \rho_p(f)$ $(p \geq 2)$.

Definition 1.2 (see [8])**.** The growth index of the iterated order of a meromorphic function $f(z)$ in Δ is defined by

$$i(f) = \begin{cases} 0, & \text{if } f \text{ is non-admissible,} \\ \min\{\rho_j(f) < \infty : j \in \mathbb{N}\}, & \text{if } f \text{ is admissible,} \\ +\infty, & \text{if } \rho_j(f) = \infty \text{ for all } j \in \mathbb{N}. \end{cases}$$

For an analytic function f in Δ, we also define

$$i_M(f) = \begin{cases} 0, & \text{if } f \text{ is non-admissible,} \\ \min\{\rho_{M,j}(f) < \infty : j \in \mathbb{N}\}, & \text{if } f \text{ is admissible,} \\ +\infty, & \text{if } \rho_{M,j}(f) = \infty \text{ for all } j \in \mathbb{N}. \end{cases}$$

Definition 1.3 (see [7])**.** Let f be a meromorphic function. Then the iterated p-convergence exponent of the sequence of zeros of $f(z)$ is defined by

$$\lambda_p(f) = \limsup_{r \to 1^-} \frac{\log_p^{+} N\left(r, \frac{1}{f}\right)}{\log \frac{1}{1-r}},$$

where $N\left(r, \frac{1}{f}\right)$ is the integrated counting function of zeros of $f(z)$ in $\{z \in \mathbb{C} \colon |z| \leq r\}$. Similarly, the iterated p-convergence exponent of the sequence of distinct zeros of $f(z)$ is defined by

$$\overline{\lambda}_p(f) = \limsup_{r \to 1^-} \frac{\log_p^{+} \overline{N}\left(r, \frac{1}{f}\right)}{\log \frac{1}{1-r}},$$

where $\overline{N}\left(r, \frac{1}{f}\right)$ is the integrated counting function of distinct zeros of $f(z)$ in $\{z \in \mathbb{C} \colon |z| \leq r\}$.

Definition 1.4 (see [7])**.** The growth index of the convergence exponent of the sequence of the zeros of $f(z)$ in Δ is defined by

$$i_\lambda(f) = \begin{cases} 0, & \text{if } N\left(r, \frac{1}{f}\right) = O\left(\log \frac{1}{1-r}\right), \\ \min\{\lambda_j(f) < \infty \colon j \in \mathbb{N}\}, & \text{if } \lambda_j(f) < \infty \text{ for some } j \in \mathbb{N}, \\ +\infty, & \text{if } \lambda_j(f) = \infty \text{ for all } j \in \mathbb{N}. \end{cases}$$

Remark 1.2. Similarly, we can define the growth index $i_{\overline{\lambda}}(f)$ of $\overline{\lambda}_p(f)$.

Definition 1.5 (see [11])**.** For $a \in \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the deficiency of a with respect to a meromorphic function f in Δ is defined as

$$\delta(a,f) = \liminf_{r \to 1^-} \frac{m\left(r, \frac{1}{f-a}\right)}{T(r,f)} = 1 - \limsup_{r \to 1^-} \frac{N\left(r, \frac{1}{f-a}\right)}{T(r,f)}$$

provided that f has unbounded characteristic.

Consider the complex differential equation

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0 \tag{1.1}$$

and the k^{th} order nonhomogeneous linear differential polynomial

$$g_k = d_k f^{(k)} + d_{k-1} f^{(k-1)} + \cdots + d_0 f + b, \tag{1.2}$$

where A_j $(j = 0, 1, \ldots, k-1)$, d_i $(i = 0, 1, \ldots, k)$, and b are meromorphic functions in Δ. Let $\mathcal{L}(\mathbf{G})$ denote a differential subfield of the field $\mathcal{M}(\mathbf{G})$ of meromorphic functions in a domain $\mathbf{G} \subset \mathbb{C}$. If $\mathbf{G} = \boldsymbol{\Delta}$, we simply write $\mathcal{L}$ instead of $\mathcal{L}(\boldsymbol{\Delta})$. A special case of such a differential subfield is

$$\mathcal{L}_{p+1,\rho} = \{g \text{ meromorphic: } \rho_{p+1}(g) < \rho\},$$

where ρ is a positive constant. In [18], Laine and Rieppo considered value distribution theory of differential polynomials generated by solutions of linear differential equations in the complex plane. After that, Cao et al. [7] studied the complex oscillation of differential polynomial generated meromorphic solutions of second order linear differential equations with meromorphic coefficient in Δ, and obtained the following result.

Theorem A (see [7])**.** *Let A be an admissible meromorphic function of finite iterated order $\rho_p(A) = \rho > 0$ $(1 \leq p < \infty)$ in the unit disc Δ such that*

$$\delta(\infty, A) := \liminf_{r \to 1^-} \frac{m(r,A)}{T(r,A)} = \delta > 0,$$

and let f be a non-zero meromorphic solution of the differential equation

$$f'' + A(z) f = 0$$

such that $\delta(\infty, f) > 0$. Moreover, let

$$P[f] = \sum_{j=0}^{k} p_j f^{(j)}$$

be a linear differential polynomial with coefficients $p_j \in \mathcal{L}_{p+1,\rho}$, assuming that at least one of the coefficients p_j does not vanish identically. If $\varphi \in \mathcal{L}_{p+1,\rho}$

is a non-zero meromorphic function in Δ*, and neither* $P[f]$ *nor* $P[f]-\varphi$ *vanishes identically, then*

$$i(f) = i_{\overline{\lambda}}(P[f]-\varphi) = p+1$$

and

$$\overline{\lambda}_{p+1}(P[f]-\varphi) = \rho_{p+1}(f) = \rho_p(A) = \rho$$

if $p>1$*, while*

$$\rho_p(A) \leq \overline{\lambda}_{p+1}(P[f]-\varphi) \leq \rho_{p+1}(f) \leq \rho_p(A)+1$$

if $p=1$*.*

Recently, the author and Latreuch investigated the growth and oscillation of higher order differential polynomial with meromorphic coefficients in the unit disc Δ generated by solutions of equation (1.1). They obtained the following results.

Theorem B (see [20])**.** *Let* $A_i(z)$ $(i=0,1,\ldots,k-1)$ *be meromorphic functions in* Δ *of finite iterated p-order. Let* $d_j(z)$ $(j=0,1,\ldots,k)$ *be finite iterated p-order meromorphic functions in* Δ *that are not all vanishing identically such that*

$$h_k = \begin{vmatrix} \alpha_{0,0} & \alpha_{1,0} & \cdots & \alpha_{k-1,0} \\ \alpha_{0,1} & \alpha_{1,1} & \cdots & \alpha_{k-1,1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{0,k-1} & \alpha_{1,k-1} & \cdots & \alpha_{k-1,k-1} \end{vmatrix} \not\equiv 0, \tag{1.3}$$

where the meromorphic functions $\alpha_{i,j}$ $(i,j=0,\ldots,k-1)$ *in* Δ *are defined by*

$$\alpha_{i,j} = \begin{cases} \alpha'_{i,j-1}+\alpha_{i-1,j-1}-A_i\alpha_{k-1,j-1}, & \text{if } i,j=1,\ldots,k-1, \\ \alpha'_{0,j-1}-A_0\alpha_{k-1,j-1}, & \text{if } i=0,\ j=1,\ldots,k-1, \\ d_i-d_kA_i, & \text{if } j=0,\ i=0,\ldots,k-1. \end{cases} \tag{1.4}$$

If $f(z)$ *is an infinite iterated p-order meromorphic solution in* Δ *of* (1.1) *with* $\rho_{p+1}(f)=\rho$*, then the differential polynomial* $g_k=\sum_{j=0}^{k} d_j f^{(j)}$ *satisfies* $\rho_p(g_k)=\rho_p(f)=\infty$ *and*

$$\rho_{p+1}(g_k) = \rho_{p+1}(f) = \rho.$$

Furthermore, if f *is a finite iterated p-order meromorphic solution in* Δ *such that*

$$\rho_p(f) > \max_{\substack{i=0,1,\ldots,k-1 \\ j=0,1,\ldots,k}} \{\rho_p(A_i),\ \rho_p(d_j)\},$$

then $\rho_p(g_k)=\rho_p(f)$*.*

Theorem C (see [20]). *Under the hypotheses of Theorem B, let $\varphi(z) \not\equiv 0$ be a meromorphic function in Δ with finite iterated p-order such that*

$$\psi_k(z) = \frac{\begin{vmatrix} \varphi & \alpha_{1,0} & \cdots & \alpha_{k-1,0} \\ \varphi' & \alpha_{1,1} & \cdots & \alpha_{k-1,1} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi^{(k-1)} & \alpha_{1,k-1} & \cdots & \alpha_{k-1,k-1} \end{vmatrix}}{h_k(z)},$$

is not a solution of (1.1)*. If $f(z)$ is an infinite iterated p-order meromorphic solution in Δ of* (1.1) *with $\rho_{p+1}(f) = \rho$, then the differential polynomial $g_k = \sum_{j=0}^{k} d_j f^{(j)}$ satisfies*

$$\overline{\lambda}_p(g_k - \varphi) = \lambda_p(g_k - \varphi) = \rho_p(f) = \infty$$

and

$$\overline{\lambda}_{p+1}(g_k - \varphi) = \lambda_{p+1}(g_k - \varphi) = \rho.$$

Furthermore, if f is a finite iterated p-order meromorphic solution in Δ such that

$$\rho_p(f) > \max_{\substack{i=0,1,\ldots,k-1 \\ j=0,1,\ldots,k}} \{\rho_p(A_i),\ \rho_p(d_j),\ \rho_p(\varphi)\},$$

then

$$\overline{\lambda}_p(g_k - \varphi) = \lambda_p(g_k - \varphi) = \rho_p(f).$$

Juneja et al. [13, 14] investigated some properties of entire functions of $[p,q]$-order, and obtained some results concerning their growth. In 2010, Liu et al. [22] firstly studied the growth of solutions of equation (1.1) with entire coefficients of $[p,q]$-order in the complex plane. After that, many authors applied the concepts of entire (meromorphic) functions in the complex plane and analytic functions in the unit disc $\Delta = \{z \in \mathbb{C}\colon |z| < 1\}$ of $[p,q]$-order to investigate complex differential equations (see [2]–[5], [19], [21], [23], [24], [26]). In this paper, we use the concept of $[p,q]$-order to study the growth and zeros of differential polynomial (1.2) generated by meromorphic solutions of $[p,q]$-order in the unit disc to equation (1.1).

In the following, we will give similar definitions as in [13, 14] for analytic and meromorphic functions of $[p,q]$-order, $[p,q]$-type and $[p,q]$-exponent of convergence of the zero-sequence in the unit disc.

Definition 1.6 (see [2]). Let $p \geq q \geq 1$ be integers, and let f be a meromorphic function in Δ. The $[p,q]$-order of $f(z)$ is defined by

$$\rho_{[p,q]}(f) = \limsup_{r \to 1^-} \frac{\log_p^+ T(r,f)}{\log_q \frac{1}{1-r}}.$$

For an analytic function f in Δ, we also define

$$\rho_{M,[p,q]}(f)=\limsup_{r\to 1^-}\frac{\log_{p+1}^{+}M(r,f)}{\log_q\frac{1}{1-r}}.$$

Remark 1.3. It is easy to see that $0\leq\rho_{[p,q]}(f)\leq+\infty$ $(0\leq\rho_{M,[p,q]}(f)\leq+\infty$) whenever $p\geq q\geq 1$. By Definition 1.6, we have that $\rho_{[1,1]}=\rho(f)$ $(\rho_{M,[1,1]}=\rho_M(f))$ and $\rho_{[2,1]}=\rho_2(f)$ $\left(\rho_{M,[2,1]}=\rho_{M,2}(f)\right)$.

In [23], Tu and Huang extended Proposition 1.1 in [2] with more details, as follows.

Proposition 1.1 (see [23])**.** *Let f be an analytic function of $[p,q]$-order in Δ. Then the following five statements hold.*

(i) *If $p=q=1$, then $\rho(f)\leq\rho_M(f)\leq\rho(f)+1$.*
(ii) *If $p=q\geq 2$ and $\rho_{[p,q]}(f)<1$, then $\rho_{[p,q]}(f)\leq\rho_{M,[p,q]}(f)\leq 1$.*
(iii) *If $p=q\geq 2$ and $\rho_{[p,q]}(f)\geq 1$, or $p>q\geq 1$, then $\rho_{[p,q]}(f)=\rho_{M,[p,q]}(f)$.*
(iv) *If $p\geq 1$ and $\rho_{[p,p+1]}(f)>1$, then $D(f)=\limsup_{r\to 1^-}\frac{T(r,f)}{\log\frac{1}{1-r}}=\infty$; if $\rho_{[p,p+1]}(f)<1$, then $D(f)=0$.*
(v) *If $p\geq 1$ and $\rho_{M,[p,p+1]}(f)>1$, then $D_M(f)=\limsup_{r\to 1^-}\frac{\log^+ M(r,f)}{\log\frac{1}{1-r}}=\infty$; if $\rho_{M,[p,p+1]}(f)<1$, then $D_M(f)=0$.*

Definition 1.7 (see [19])**.** Let $p\geq q\geq 1$ be integers. The $[p,q]$-type of a meromorphic function $f(z)$ in Δ of $[p,q]$-order $\rho_{[p,q]}(f)$ $\left(0<\rho_{[p,q]}(f)<+\infty\right)$ is defined by

$$\tau_{[p,q]}(f)=\limsup_{r\to 1^-}\frac{\log_{p-1}^{+}T(r,f)}{\left(\log_{q-1}\frac{1}{1-r}\right)^{\rho_{[p,q]}(f)}}.$$

Definition 1.8 (see [19])**.** Let $p\geq q\geq 1$ be integers. The $[p,q]$-exponent of convergence of the zero-sequence of $f(z)$ in Δ is defined by

$$\lambda_{[p,q]}(f)=\limsup_{r\to 1^-}\frac{\log_p^{+}N\left(r,\frac{1}{f}\right)}{\log_q\frac{1}{1-r}}.$$

Similarly, the $[p,q]$-exponent of convergence of the sequence of distinct zeros of $f(z)$ is defined by

$$\overline{\lambda}_{[p,q]}(f)=\limsup_{r\to 1^-}\frac{\log_p^{+}\overline{N}\left(r,\frac{1}{f}\right)}{\log_q\frac{1}{1-r}}.$$

There exists a natural question: how about the growth and oscillation of the differential polynomial (1.2) with meromorphic coefficients of finite

$[p,q]$-order generated by solutions of equation (1.1) in the unit disc? The main purpose of this paper is to consider the above question.

2. Main results

Before we state our results, assuming that b and $\varphi(z)$ are meromorphic functions in Δ with $\rho_{[p,q]}(\varphi) < \infty$, we define the functions $\psi_k(z)$ by

$$\psi_k(z) = \frac{\begin{vmatrix} \varphi - b & \alpha_{1,0} & \cdots & \alpha_{k-1,0} \\ \varphi' - b' & \alpha_{1,1} & \cdots & \alpha_{k-1,1} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi^{(k-1)} - b^{(k-1)} & \alpha_{1,k-1} & \cdots & \alpha_{k-1,k-1} \end{vmatrix}}{h_k(z)},$$

where $h_k \not\equiv 0$ and $\alpha_{i,j}$ $(i,j = 0,\dots,k-1)$ are determined, respectively, in (1.3) and (1.4).

The main results state as follows.

Theorem 2.1. *Let $A_i(z)$ $(i = 0,1,\dots,k-1)$ be meromorphic functions in Δ of finite $[p,q]$-order. Let $d_j(z)$ $(j = 0,1,\dots,k)$ and b be finite $[p,q]$-order meromorphic functions in Δ that are not all vanishing identically such that $h_k \not\equiv 0$. If $f(z)$ is an infinite $[p,q]$-order meromorphic solution in Δ of* (1.1) *with $\rho_{[p+1,q]}(f) = \rho$, then the differential polynomial* (1.2) *satisfies*

$$\rho_{[p,q]}(g_k) = \rho_{[p,q]}(f) = \infty$$

and

$$\rho_{[p+,q]}(g_k) = \rho_{[p+1,q]}(f) = \rho.$$

Furthermore, if f is a finite $[p,q]$-order meromorphic solution in Δ such that

$$\rho_{[p,q]}(f) > \max_{\substack{i=0,1,\dots,k-1 \\ j=0,1,\dots,k}} \left\{ \rho_{[p,q]}(A_i),\ \rho_{[p,q]}(d_j),\ \rho_{[p,q]}(b) \right\}, \tag{2.1}$$

then

$$\rho_{[p,q]}(g_k) = \rho_{[p,q]}(f).$$

Remark 2.1. In Theorem 2.1, if we do not have the condition $h_k \not\equiv 0$, then the conclusions of Theorem 2.1 cannot hold. For example, if we take $d_i = d_k A_i$ $(i = 0,\dots,k-1)$, then $h_k \equiv 0$. It follows that $g_k \equiv b$ and $\rho_{[p,q]}(g_k) = \rho_{[p,q]}(b)$. So, if $f(z)$ is an infinite $[p,q]$-order meromorphic solution of (1.1), then $\rho_{[p,q]}(g_k) = \rho_{[p,q]}(b) < \rho_{[p,q]}(f) = \infty$, and if f is a finite $[p,q]$-order meromorphic solution of (1.1) such that (2.1) holds, then $\rho_{[p,q]}(g_k) = \rho_{[p,q]}(b) < \rho_{[p,q]}(f)$.

Theorem 2.2. *Under the hypotheses of Theorem 2.1, let $\varphi(z)$ be a meromorphic function in Δ with finite $[p,q]$-order such that $\psi_k(z)$ is not a solution of* (1.1). *If $f(z)$ is an infinite $[p,q]$-order meromorphic solution in Δ of* (1.1) *with $\rho_{[p+1,q]}(f)=\rho$, then the differential polynomial* (1.2) *satisfies*

$$\overline{\lambda}_{[p,q]}(g_k-\varphi)=\lambda_{[p,q]}(g_k-\varphi)=\rho_{[p,q]}(f)=\infty$$

and

$$\overline{\lambda}_{[p+1,q]}(g_k-\varphi)=\lambda_{[p+1,q]}(g_k-\varphi)=\rho_{[p+1,q]}(f)=\rho.$$

Furthermore, if f is a finite $[p,q]$-order meromorphic solution in Δ such that

$$\rho_{[p,q]}(f)>\max_{\substack{i=0,1,\ldots,k-1\\ j=0,1,\ldots,k}}\left\{\rho_{[p,q]}(A_i),\ \rho_{[p,q]}(d_j),\ \rho_{[p,q]}(b),\ \rho_{[p,q]}(\varphi)\right\},\qquad(2.2)$$

then

$$\overline{\lambda}_p(g_k-\varphi)=\lambda_p(g_k-\varphi)=\rho_{[p,q]}(f).$$

Remark 2.2. Obviously, Theorems 2.1 and 2.2 are generalizations of Theorems A, B and C.

From Theorems 2.1 and 2.2, and Lemmas 3.4 and 3.5 below, we easily obtain the following corollaries.

Corollary 2.1. *Let $p\geq q\geq 1$ be integers. Let H be a set of complex numbers with $\overline{dens}_{\Delta}\{|z|\colon z\in H\subseteq\Delta\}>0$, and let $A_0(z),\ldots,A_{k-1}(z)$ be analytic functions in the unit disc Δ satisfying*

$$\max_{i=1,\ldots,k-1}\rho_{[p,q]}(A_i)\leq\rho_{[p,q]}(A_0)=\rho.$$

Suppose that there exists a real number μ satisfying $0\leq\mu<\rho$ such that for any given ε $(0<\varepsilon<\rho-\mu)$ sufficiently small, we have

$$T(r,A_0)\geq\exp_p\left\{(\rho-\varepsilon)\log_q\left(\frac{1}{1-|z|}\right)\right\}$$

and

$$T(r,A_i)\leq\exp_p\left\{\mu\log_q\left(\frac{1}{1-|z|}\right)\right\}\quad(i=1,\ldots,k-1)$$

as $|z|\to1^-$ for $z\in H$. Let $d_j(z)$ $(j=0,1,\ldots,k)$ and b be finite $[p,q]$-order analytic functions in Δ that are not all vanishing identically such that $h_k\not\equiv0$. If $f\not\equiv0$ is a solution of (1.1), *then the differential polynomial* (1.2) *satisfies $\rho_{[p,q]}(g_k)=\rho_{[p,q]}(f)=\rho_{M,[p,q]}(f)=\infty$ and*

$$\rho_{[p,q]}(A_0)\leq\rho_{[p+1,q]}(g_k)=\rho_{[p+1,q]}(f)=\rho_{M,[p+1,q]}(f)\\ \leq\max\{\rho_{M,[p,q]}(A_i)\colon i=0,1,\ldots,k-1\}.$$

Furthermore, if $p>q$, then

$$\rho_{[p+1,q]}(g_k)=\rho_{[p+1,q]}(f)=\rho_{M,[p+1,q]}(f)=\rho_{[p,q]}(A_0).$$

Corollary 2.2. *Let p, q and H be as in Corollary 2.1, and let $A_0(z), \ldots, A_{k-1}(z)$ be analytic functions in the unit disc Δ satisfying*

$$\max_{i=1,\ldots,k-1} \rho_{M,[p,q]}(A_i) \leq \rho_{M,[p,q]}(A_0) = \rho.$$

Suppose that there exists a real number μ satisfying $0 \leq \mu < \rho$ such that for any given ε $(0 < \varepsilon < \rho - \mu)$ sufficiently small, we have

$$|A_0(z)| \geq \exp_{p+1}\left\{(\rho - \varepsilon)\log_q\left(\frac{1}{1-|z|}\right)\right\}$$

and

$$|A_i(z)| \leq \exp_{p+1}\left\{\mu \log_q\left(\frac{1}{1-|z|}\right)\right\} \quad (i = 1, \ldots, k-1)$$

as $|z| \to 1^-$ for $z \in H$. Let $d_j(z)$ $(j = 0, 1, \ldots, k)$ and b be finite $[p,q]$-order analytic functions in Δ that are not all vanishing identically such that $h_k \not\equiv 0$. If $f \not\equiv 0$ is a solution of (1.1)*, then the differential polynomial* (1.2) *satisfies $\rho_{[p,q]}(g_k) = \rho_{[p,q]}(f) = \rho_{M,[p,q]}(f) = \infty$ and*

$$\rho_{[p+1,q]}(g_k) = \rho_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f) = \rho_{M,[p,q]}(A_0) = \rho.$$

Corollary 2.3. *Under the hypotheses of Corollary 2.1, let $\varphi(z)$ be an analytic function in Δ with finite $[p,q]$-order such that $\psi_k(z)$ is not a solution of* (1.1)*. If $f \not\equiv 0$ is a solution of* (1.1)*, then the differential polynomial* (1.2) *satisfies*

$$\overline{\lambda}_{[p,q]}(g_k - \varphi) = \lambda_{[p,q]}(g_k - \varphi) = \rho_{[p,q]}(f) = \rho_{M,[p,q]}(f) = \infty$$

and

$$\begin{aligned}\rho_{[p,q]}(A_0) \leq \overline{\lambda}_{[p+1,q]}(g_k - \varphi) &= \lambda_{[p+1,q]}(g_k - \varphi) \\ &= \rho_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f) \\ &\leq \max\{\rho_{M,[p,q]}(A_j) : j = 0, 1, \ldots, k-1\}.\end{aligned}$$

Furthermore, if $p > q$, then

$$\overline{\lambda}_{[p+1,q]}(g_k - \varphi) = \lambda_{[p+1,q]}(g_k - \varphi) = \rho_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f) = \rho_{[p,q]}(A_0).$$

Corollary 2.4. *Under the hypotheses of Corollary 2.2, let $\varphi(z)$ be an analytic function in Δ with finite $[p,q]$-order such that $\psi_k(z)$ is not a solution of* (1.1)*. If $f \not\equiv 0$ is a solution of* (1.1)*, then the differential polynomial* (1.2) *satisfies*

$$\overline{\lambda}_{[p,q]}(g_k - \varphi) = \lambda_{[p,q]}(g_k - \varphi) = \rho_{[p,q]}(f) = \rho_{M,[p,q]}(f) = \infty$$

and

$$\begin{aligned}\overline{\lambda}_{[p+1,q]}(g_k - \varphi) = \lambda_{[p+1,q]}(g_k - \varphi) &= \rho_{[p+1,q]}(f) \\ &= \rho_{M,[p+1,q]}(f) = \rho_{M,[p,q]}(A_0) = \rho.\end{aligned}$$

We now consider the differential equation

$$f'' + A(z) f = 0, \tag{2.3}$$

where $A(z)$ is a meromorphic function of finite $[p,q]$-order in the unit disc Δ. In the following, we will give sufficient conditions on A which satisfied the results of Theorem 2.1 and Theorem 2.2 without the conditions "$h_k \not\equiv 0$" and "$\psi_k(z)$ is not a solution of (1.1)", where $k=2$.

Corollary 2.5. *Let $p \geq q \geq 1$ be integers, and let $A(z)$ be a meromorphic function in Δ with $0 < \rho_{[p,q]}(A) = \rho < \infty$ such that $\delta(\infty, A) > 0$. Let d_0, d_1, d_2, b be meromorphic functions in Δ that are not all vanishing identically such that*

$$\max_{j=0,1,2} \left\{ \rho_{[p,q]}(d_j),\ \rho_{[p,q]}(b) \right\} < \rho_{[p,q]}(A).$$

If $f \not\equiv 0$ is a meromorphic solution of (2.3) such that $\delta(\infty, f) > 0$, then the differential polynomial $g_2 = d_2 f'' + d_1 f' + d_0 f + b$ satisfies $\rho_{[p,q]}(g_2) = \rho_{[p,q]}(f) = \infty$ and

$$\rho_{[p,q]}(A) \leq \rho_{[p+1,q]}(g_2) = \rho_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f) \leq \rho_{[p,q]}(A) + 1.$$

Furthermore, if $p > q$, then

$$\rho_{[p+1,q]}(g_2) = \rho_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f) = \rho_{[p,q]}(A).$$

Corollary 2.6. *Under the hypotheses of Corollary 2.5, suppose that $0 < \tau_{[p,q]}(A) < +\infty$, and let φ be a meromorphic function in Δ such that $\varphi - b \not\equiv 0$ with $\rho_{[p,q]}(\varphi) < \infty$. If $f \not\equiv 0$ is a meromorphic solution of (2.3) such that $\delta(\infty, f) > 0$, then the differential polynomial $g_2 = d_2 f'' + d_1 f' + d_0 f + b$ with $d_2 \not\equiv 0$ satisfies*

$$\overline{\lambda}_{[p,q]}(g_2 - \varphi) = \lambda_{[p,q]}(g_2 - \varphi) = \rho_{[p,q]}(f) = \infty$$

and

$$\begin{aligned} \rho_{[p,q]}(A) &\leq \overline{\lambda}_{[p+1,q]}(g_2 - \varphi) = \lambda_{[p+1,q]}(g_2 - \varphi) \\ &= \rho_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f) \\ &\leq \rho_{[p,q]}(A) + 1. \end{aligned}$$

Furthermore, if $p > q$, then

$$\overline{\lambda}_{[p+1,q]}(g_2 - \varphi) = \lambda_{[p+1,q]}(g_2 - \varphi) = \rho_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f) = \rho_{[p,q]}(A).$$

3. Auxiliary lemmas

Lemma 3.1 (see [3]). *Let $p \geq q \geq 1$ be integers, and let f be a meromorphic function of $[p,q]-$order in Δ. Then $\rho_{[p,q]}(f') = \rho_{[p,q]}(f)$.*

Lemma 3.2 (see [3]). *Let $p \geq q \geq 1$ be integers, and let f and g be non-constant meromorphic functions of $[p,q]$-order in Δ. Then*

$$\rho_{[p,q]}(f+g) \leq \max\left\{\rho_{[p,q]}(f), \rho_{[p,q]}(g)\right\}$$

and

$$\rho_{[p,q]}(fg) \leq \max\left\{\rho_{[p,q]}(f), \rho_{[p,q]}(g)\right\}.$$

Furthermore, if $\rho_{[p,q]}(f) > \rho_{[p,q]}(g)$, then

$$\rho_{[p,q]}(f+g) = \rho_{[p,q]}(fg) = \rho_{[p,q]}(f).$$

By using similar proof of Lemma 2.6 in [3] or Lemma 2.6 in [19], we easily obtain the following lemma.

Lemma 3.3 (see [3], [19]). *Let $p \geq q \geq 1$ be integers. Let A_i $(i = 0, \ldots, k-1)$ and $F \not\equiv 0$ be meromorphic functions in Δ, and let $f(z)$ be a solution of the differential equation*

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = F$$

satisfying

$$\max_{i=0,\ldots,k-1} \left\{\rho_{[p,q]}(A_i),\ \rho_{[p,q]}(F)\right\} < \rho_{[p,q]}(f) = \rho \leq +\infty.$$

Then

$$\overline{\lambda}_{[p,q]}(f) = \lambda_{[p,q]}(f) = \rho_{[p,q]}(f)$$

and

$$\overline{\lambda}_{[p+1,q]}(f) = \lambda_{[p+1,q]}(f) = \rho_{[p+1,q]}(f).$$

Lemma 3.4 (see [4]). *Let $p \geq q \geq 1$ be integers. Let H be a set of complex numbers satisfying $\overline{dens}_\Delta \{|z| \colon z \in H \subseteq \Delta\} > 0$, and let $A_0(z), \ldots, A_{k-1}(z)$ be analytic functions in the unit disc Δ satisfying*

$$\max\{\rho_{[p,q]}(A_i) \colon i = 1, \ldots, k-1\} \leq \rho_{[p,q]}(A_0) = \rho.$$

Suppose that there exists a real number μ satisfying $0 \leq \mu < \rho$ such that for any given ε $(0 < \varepsilon < \rho - \mu)$ sufficiently small, we have

$$T(r, A_0) \geq \exp_p\left\{(\rho - \varepsilon)\log_q\left(\frac{1}{1-|z|}\right)\right\}$$

and

$$T(r, A_i) \leq \exp_p\left\{\mu \log_q\left(\frac{1}{1-|z|}\right)\right\} \quad (i = 1, \ldots, k-1)$$

as $|z| \to 1^-$ *for* $z \in H$. *Then every solution* $f \not\equiv 0$ *of* (1.1) *satisfies* $\rho_{[p,q]}(f) = \rho_{M,[p,q]}(f) = \infty$ *and*

$$\rho_{[p,q]}(A_0) \leq \rho_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f) \leq \max_{i=0,1,\ldots,k-1} \rho_{M,[p,q]}(A_i).$$

Furthermore, if $p > q$, *then*

$$\rho_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f) = \rho_{[p,q]}(A_0).$$

Lemma 3.5 (see [3]). *Let* $p \geq q \geq 1$ *be integers. Let* H *be a set of complex numbers with* $\overline{dens}_\Delta \{|z| \colon z \in H \subseteq \Delta\} > 0$, *and let* $A_0(z), \ldots, A_{k-1}(z)$ *be analytic functions in the unit disc* Δ *satisfying*

$$\max\{\rho_{M,[p,q]}(A_i) \colon i = 1, \ldots, k-1\} \leq \rho_{M,[p,q]}(A_0) = \rho.$$

Suppose that there exists a real number μ *satisfying* $0 \leq \mu < \rho$ *such that for any given* ε $(0 < \varepsilon < \rho - \mu)$ *sufficiently small, we have*

$$|A_0(z)| \geq \exp_{p+1}\left\{(\rho - \varepsilon)\log_q\left(\frac{1}{1-|z|}\right)\right\}$$

and

$$|A_i(z)| \leq \exp_{p+1}\left\{\mu \log_q\left(\frac{1}{1-|z|}\right)\right\} \quad (i = 1, \ldots, k-1)$$

as $|z| \to 1^-$ *for* $z \in H$. *Then every solution* $f \not\equiv 0$ *of* (1.1) *satisfies* $\rho_{[p,q]}(f) = \rho_{M,[p,q]}(f) = \infty$ *and*

$$\rho_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f) = \rho_{M,[p,q]}(A_0) = \rho.$$

Lemma 3.6 (see [11], [12], [25]). *Let* f *be a meromorphic function in the unit disc and let* $k \in \mathbb{N}$. *Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r, f),$$

where

$$S(r, f) = O\left(\log^+ T(r, f) + \log\left(\frac{1}{1-r}\right)\right)$$

possibly outside a set $E_1 \subset [0, 1)$ *with* $\int_{E_1} \frac{dr}{1-r} < \infty$.

Lemma 3.7 (see [2]). *Let* $p \geq q \geq 1$ *be integers. Let* f *be a meromorphic function in the unit disc* Δ *such that* $\rho_{[p,q]}(f) = \rho < \infty$, *and let* $k \geq 1$ *be an integer. Then for any* $\varepsilon > 0$,

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\exp_{p-1}\left\{(\rho + \varepsilon)\log_q\left(\frac{1}{1-r}\right)\right\}\right)$$

holds for all r *outside a set* $E_2 \subset [0, 1)$ *with* $\int_{E_2} \frac{dr}{1-r} < \infty$.

Lemma 3.8 (see [1], [12]). *Let $g\colon (0,1) \to \mathbb{R}$ and $h\colon (0,1) \to \mathbb{R}$ be monotone increasing functions such that $g(r) \le h(r)$ holds outside of an exceptional set $E_3 \subset [0,1)$ for which $\int_{E_3} \frac{dr}{1-r} < \infty$. Then there exists a constant $d \in (0,1)$ such that if $s(r) = 1 - d(1-r)$, then $g(r) \le h(s(r))$ for all $r \in [0,1)$.*

Lemma 3.9 (see [10], Corollary 2.5). *Suppose that $0 < \rho < r < t < R < \infty$ and the path $\Gamma = \Gamma(\theta_0, \rho, t)$ is given by the segment*

$$\Gamma_1\colon \quad z = \tau e^{i\theta_0}, \ \rho \le \tau \le t < \frac{1}{4}(3r + R),$$

followed by the circle

$$\Gamma_2\colon \quad z = te^{i\theta}, \quad \theta_0 \le \theta \le \theta_0 + 2\pi.$$

We suppose that f is a meromorphic solution of the equation

$$f^{(k)} + a_{k-1}(z) f^{(k-1)} + \cdots + a_1(z) f' + a_0(z) f = 0,$$

where the coefficients $a_0(z), a_1(z), \ldots, a_{k-1}(z)$ are meromorphic in the disc $|z| \le R$. We also define

$$C = C(a_n, \rho, r, R) = (k+2)\exp\left[\frac{20R}{R-r}\sum_{n=0}^{k-1} T(R, a_n) + \left(\sum_{n=0}^{k-1} p_n\right)\log\left(\frac{R}{\rho}\right)\right],$$

where p_n is the multiplicity of the pole of a_n at the origin if $a_n(0) = \infty$, and $p_n = 0$ otherwise. If $\delta = \delta(\infty, f) > 0$ and $0 \le \varepsilon < \delta$, then

$$T(r, f) \le \left(\frac{1}{\delta - \varepsilon}\right)(2\pi + 1) RC, \quad r_1(\varepsilon) < r < R.$$

Lemma 3.10. *Let $p \ge q \ge 1$ be integers, and let $A(z)$ be a meromorphic function with $0 < \rho_{[p,q]}(A) = \rho < \infty$ such that $\delta(\infty, A) > 0$. If $f \not\equiv 0$ is a meromorphic solution of*

$$f^{(k)} + A(z) f = 0 \tag{3.1}$$

such that $\delta(\infty, f) > 0$, then $\rho_{[p,q]}(f) = \infty$ and

$$\rho_{[p,q]}(A) \le \rho_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f) \le \rho_{[p,q]}(A) + 1.$$

Furthermore, if $p > q$, then

$$\rho_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f) = \rho_{[p,q]}(A).$$

Proof. First, we prove that $\rho_{[p,q]}(f) = \infty$. We suppose that $\rho_{[p,q]}(f) = \beta < +\infty$ and then we obtain a contradiction. It follows from the definition of deficiency (see Theorem A) $\delta(\infty, A)$ that, for $r \to 1^-$, we have

$$m(r, A) \ge \frac{\delta}{2} T(r, A).$$

So, when $r \to 1^-$, we get by (3.1) and Lemma 3.7 that

$$T(r,A) \le \frac{2}{\delta} m(r,A) = \frac{2}{\delta} m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\exp_{p-1}\left\{\beta \log_q\left(\frac{1}{1-r}\right)\right\}\right)$$

holds for all r outside a set $E_2 \subset [0,1)$ with $\int_{E_2} \frac{dr}{1-r} < \infty$. Therefore, by Lemma 3.8 we obtain $\rho_{[p-1,q]}(A) < \infty$ which is a contradiction since A is a meromorphic function with $\rho_{[p,q]}(A) = \rho > 0$. Hence $\rho_{[p,q]}(f) = \infty$.

Now, we prove that

$$\rho_{[p,q]}(A) \le \rho_{[p+1,q]}(f) \le \rho_{[p,q]}(A) + 1$$

and

$$\rho_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f) = \rho_{[p,q]}(A)$$

if $p > q$. Since $\rho_{[p,q]}(f) = \infty$, by (3.1) and Lemma 3.6 it follows from the definition of deficiency that, for $r \to 1^-$, we have

$$\begin{aligned} T(r,A) &\le \frac{2}{\delta} m(r,A) = \frac{2}{\delta} m\left(r, \frac{f^{(k)}}{f}\right) \\ &= O\left(\log^+ T(r,f) + \log\left(\frac{1}{1-r}\right)\right). \end{aligned}$$

By Lemma 3.8, there exists a constant $d \in (0,1)$ such that if $s(r) = 1 - d(1-r)$, then

$$T(r,A) \le O\left(\log^+ T(1-d(1-r), f) + \log\left(\frac{1}{d(1-r)}\right)\right)$$

for $r \to 1^-$. Hence, by the definition of $[p,q]$-order,

$$\rho_{[p+1,q]}(f) = \limsup_{r\to 1^-} \frac{\log_{p+1}^+ T(r,f)}{\log_q \frac{1}{1-r}} \ge \rho_{[p,q]}(A) = \rho.$$

On the other hand, if $\delta(\infty,f) > 0$, then by Lemma 3.9, for any fixed ε, $0 \le \varepsilon < \delta_1 := \delta(\infty,f)$, and $r_1(\varepsilon) < r < t < R := \frac{1+r}{2} < 1$,

$$T(r,f) \le \left(\frac{1}{\delta_1 - \varepsilon}\right)(2\pi+1) RC \tag{3.2}$$

holds on the path $\Gamma = \Gamma(\theta_0, \rho, t)$ chosen in accordance with Lemma 3.9, where

$$C = (k+2)\exp\left[\frac{20R}{R-r} T(R,A) + p_0 \log\left(\frac{R}{\rho}\right)\right],$$

p_0 is the multiplicity of the pole of A at the origin if $A(0) = \infty$, and $p_0 = 0$ otherwise. By (3.2), we immediately get that

$$\rho_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f) \le \rho_{[p,q]}(A) + 1$$

and

$$\rho_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f) \leq \rho_{[p,q]}(A)$$

if $p > q$. Therefore,

$$\rho_{[p,q]}(A) \leq \rho_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f) \leq \rho_{[p,q]}(A) + 1$$

and

$$\rho_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f) = \rho_{[p,q]}(A)$$

if $p > q$. □

Remark 3.1. Lemma 3.10 was proved for $q = 1$ by Cao et al. [9].

Lemma 3.11 (see [19]). *Let $p \geq q \geq 1$ be integers, and let f and g be meromorphic functions of $[p,q]$-order in Δ such that $0 < \rho_{[p,q]}(f), \rho_{[p,q]}(g) < \infty$ and $0 < \tau_{[p,q]}(f), \tau_{[p,q]}(g) < \infty$. The following statements hold.*

(i) *If $\rho_{[p,q]}(f) > \rho_{[p,q]}(g)$, then*

$$\tau_{[p,q]}(f+g) = \tau_{[p,q]}(fg) = \tau_{[p,q]}(f).$$

(ii) *If $\rho_{[p,q]}(f) = \rho_{[p,q]}(g)$ and $\tau_{[p,q]}(f) \neq \tau_{[p,q]}(g)$, then*

$$\rho_{[p,q]}(f+g) = \rho_{[p,q]}(fg) = \rho_{[p,q]}(f) = \rho_{[p,q]}(g).$$

4. Proofs of main results

Proof of Theorem 2.1. Suppose that f is an infinite $[p,q]$-order meromorphic solution of (1.1). By (1.1) we have

$$f^{(k)} = -\sum_{i=0}^{k-1} A_i f^{(i)} \tag{4.1}$$

which implies

$$g_k - b = d_k f^{(k)} + d_{k-1} f^{(k-1)} + \cdots + d_0 f = \sum_{i=0}^{k-1} (d_i - d_k A_i) f^{(i)}. \tag{4.2}$$

We can rewrite (4.2) as

$$g_k - b = \sum_{i=0}^{k-1} \alpha_{i,0} f^{(i)}, \tag{4.3}$$

where $\alpha_{i,0}$ is defined in (1.4). Differentiating both sides of equation (4.3) and using (4.1), we obtain

$$\begin{aligned} g_k' - b' &= \sum_{i=0}^{k-1} \alpha_{i,0}' f^{(i)} + \sum_{i=0}^{k-1} \alpha_{i,0} f^{(i+1)} = \sum_{i=0}^{k-1} \alpha_{i,0}' f^{(i)} + \sum_{i=1}^{k} \alpha_{i-1,0} f^{(i)} \\ &= \alpha_{0,0}' f + \sum_{i=1}^{k-1} \alpha_{i,0}' f^{(i)} + \sum_{i=1}^{k-1} \alpha_{i-1,0} f^{(i)} + \alpha_{k-1,0} f^{(k)} \\ &= \alpha_{0,0}' f + \sum_{i=1}^{k-1} \alpha_{i,0}' f^{(i)} + \sum_{i=1}^{k-1} \alpha_{i-1,0} f^{(i)} - \sum_{i=0}^{k-1} \alpha_{k-1,0} A_i f^{(i)} \\ &= \left(\alpha_{0,0}' - \alpha_{k-1,0} A_0\right) f + \sum_{i=1}^{k-1} \left(\alpha_{i,0}' + \alpha_{i-1,0} - \alpha_{k-1,0} A_i\right) f^{(i)}. \end{aligned} \tag{4.4}$$

We can rewrite (4.4) as

$$g_k' - b' = \sum_{i=0}^{k-1} \alpha_{i,1} f^{(i)}, \tag{4.5}$$

where

$$\alpha_{i,1} = \begin{cases} \alpha_{i,0}' + \alpha_{i-1,0} - \alpha_{k-1,0} A_i, & \text{if } i = 1, \ldots, k-1, \\ \alpha_{0,0}' - A_0 \alpha_{k-1,0}, & \text{if } i = 0. \end{cases} \tag{4.6}$$

Differentiating both sides of equation (4.5) and using (4.1), we obtain

$$\begin{aligned} g_k'' - b'' &= \sum_{i=0}^{k-1} \alpha_{i,1}' f^{(i)} + \sum_{i=0}^{k-1} \alpha_{i,1} f^{(i+1)} = \sum_{i=0}^{k-1} \alpha_{i,1}' f^{(i)} + \sum_{i=1}^{k} \alpha_{i-1,1} f^{(i)} \\ &= \alpha_{0,1}' f + \sum_{i=1}^{k-1} \alpha_{i,1}' f^{(i)} + \sum_{i=1}^{k-1} \alpha_{i-1,1} f^{(i)} + \alpha_{k-1,1} f^{(k)} \\ &= \alpha_{0,1}' f + \sum_{i=1}^{k-1} \alpha_{i,1}' f^{(i)} + \sum_{i=1}^{k-1} \alpha_{i-1,1} f^{(i)} - \sum_{i=0}^{k-1} A_i \alpha_{k-1,1} f^{(i)} \\ &= \left(\alpha_{0,1}' - \alpha_{k-1,1} A_0\right) f + \sum_{i=1}^{k-1} \left(\alpha_{i,1}' + \alpha_{i-1,1} - A_i \alpha_{k-1,1}\right) f^{(i)}. \end{aligned} \tag{4.7}$$

This implies

$$g_k'' - b'' = \sum_{i=0}^{k-1} \alpha_{i,2} f^{(i)}, \tag{4.8}$$

where

$$\alpha_{i,2} = \begin{cases} \alpha_{i,1}' + \alpha_{i-1,1} - A_i \alpha_{k-1,1}, & \text{if } i = 1, \ldots, k-1, \\ \alpha_{0,1}' - A_0 \alpha_{k-1,1}, & \text{if } i = 0. \end{cases} \tag{4.9}$$

By using the same method as above we can easily deduce that

$$g_k^{(j)} - b^{(j)} = \sum_{i=0}^{k-1} \alpha_{i,j} f^{(i)}, \; j = 0, 1, \dots, k-1, \tag{4.10}$$

where the coefficients $a_{i,j}$ are determined by (1.4). By (4.3) – (4.10) we obtain the system of equations

$$\begin{cases} g_k - b = \alpha_{0,0} f + \alpha_{1,0} f' + \cdots + \alpha_{k-1,0} f^{(k-1)}, \\ g_k' - b' = \alpha_{0,1} f + \alpha_{1,1} f' + \cdots + \alpha_{k-1,1} f^{(k-1)}, \\ g_k'' - b'' = \alpha_{0,2} f + \alpha_{1,2} f' + \cdots + \alpha_{k-1,2} f^{(k-1)}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ g_k^{(k-1)} - b^{(k-1)} = \alpha_{0,k-1} f + \alpha_{1,k-1} f' + \cdots + \alpha_{k-1,k-1} f^{(k-1)}. \end{cases}$$

By Cramer's rule, since $h_k \not\equiv 0$, we have

$$f = \frac{\begin{vmatrix} g_k - b & \alpha_{1,0} & \cdots & \alpha_{k-1,0} \\ g_k' - b' & \alpha_{1,1} & \cdots & \alpha_{k-1,1} \\ \vdots & \vdots & \ddots & \vdots \\ g_k^{(k-1)} - b^{(k-1)} & \alpha_{1,k-1} & \cdots & \alpha_{k-1,k-1} \end{vmatrix}}{h_k}.$$

Then

$$f = C_0 \left(g_k - b\right) + C_1 \left(g_k' - b'\right) + \cdots + C_{k-1} \left(g_k^{(k-1)} - b^{(k-1)}\right), \tag{4.11}$$

where C_j are finite $[p,q]$-order meromorphic functions in Δ depending on $\alpha_{i,j}$, where $\alpha_{i,j}$ are defined in (1.4).

If $\rho_{[p,q]}(g_k) < +\infty$, then by (4.11), and Lemmas 3.1 and 3.2, we obtain $\rho_{[p,q]}(f) < +\infty$, and this is a contradiction. Hence $\rho_{[p,q]}(g_k) = \rho_{[p,q]}(f) = +\infty$.

Now, we prove that $\rho_{[p+1,q]}(g_k) = \rho_{[p+1,q]}(f) = \rho$. By (4.2), Lemma 3.1 and Lemma 3.2, we get $\rho_{[p+1,q]}(g_k) \le \rho_{[p+1,q]}(f)$, and by (4.11) we have $\rho_{[p+1,q]}(f) \le \rho_{[p+1,q]}(g_k)$. This yields $\rho_{[p+1,q]}(g_k) = \rho_{[p+1,q]}(f) = \rho$.

Furthermore, if f is a finite $[p,q]$-order meromorphic solution in Δ of equation (1.1) such that (2.1) holds, then

$$\rho_{[p,q]}(f) > \max\left\{\rho_{[p,q]}(\alpha_{i,j}) \colon i = 0, \dots, k-1, j = 0, \dots, k-1\right\}. \tag{4.12}$$

So by (4.2) we have $\rho_{[p,q]}(g_k) \le \rho_{[p,q]}(f)$. To prove the equality $\rho_{[p,q]}(g_k) = \rho_{[p,q]}(f)$, we suppose that $\rho_{[p,q]}(g_k) < \rho_{[p,q]}(f)$. Then, by (4.11) and (4.12),

$$\rho_{[p,q]}(f) \le \max\left\{\rho_{[p,q]}(C_j) \quad (j = 0, \dots, k-1), \rho_{[p,q]}(g_k)\right\} < \rho_{[p,q]}(f)$$

which is a contradiction. Hence $\rho_{[p,q]}(g_k) = \rho_{[p,q]}(f)$. □

Remark 4.1. From (4.11) it follows that the condition $h_k \not\equiv 0$ is equivalent to the condition that $g_k - b, g_k' - b', \ldots, g_k^{(k-1)} - b^{(k-1)}$ are linearly independent over the field of meromorphic functions of finite $[p,q]$-order. As it was noted in the paper by Laine and Rieppo [18], one may assume that $d_k \equiv 0$. Note that the linear dependence of $g_k - b, g_k' - b', \ldots, g_k^{(k-1)} - b^{(k-1)}$ implies that f satisfies a linear differential equation of order smaller than k with appropriate coefficients, and vise versa (e.g., Theorem 2.3 in the paper of Laine and Rieppo [18]).

Proof of Theorem 2.2. Suppose that f is an infinite $[p,q]$-order meromorphic solution of equation (1.1) with $\rho_{[p+1,q]}(f) = \rho$. Set $w(z) = g_k - \varphi$. Since $\rho_{[p,q]}(\varphi) < \infty$, by Lemma 3.2 and Theorem 2.1 we have $\rho_{[p,q]}(w) = \rho_{[p,q]}(g_k) = \infty$ and $\rho_{[p+1,q]}(w) = \rho_{[p+1,q]}(g_k) = \rho$. To prove $\overline{\lambda}_{[p,q]}(g_k - \varphi) = \lambda_{[p,q]}(g_k - \varphi) = \infty$ and $\overline{\lambda}_{[p+1,q]}(g_k - \varphi) = \lambda_{[p+1,q]}(g_k - \varphi) = \rho$, we must show that $\overline{\lambda}_{[p,q]}(w) = \lambda_{[p,q]}(w) = \infty$ and $\overline{\lambda}_{[p+1,q]}(w) = \lambda_{[p+1,q]}(w) = \rho$. By the equalities $g_k = w + \varphi$ and (4.11),

$$f = C_0 w + C_1 w' + \cdots + C_{k-1} w^{(k-1)} + \psi_k(z), \tag{4.13}$$

where

$$\psi_k(z) = C_0(\varphi - b) + C_1(\varphi' - b') + \cdots + C_{k-1}\left(\varphi^{(k-1)} - b^{(k-1)}\right).$$

Substituting (4.13) into (1.1), we obtain

$$\begin{aligned} C_{k-1} w^{(2k-1)} + \sum_{j=0}^{2k-2} \phi_j w^{(j)} &= -\left(\psi_k^{(k)} + A_{k-1}(z)\psi_k^{(k-1)} + \cdots + A_0(z)\psi_k\right) \\ &= H, \end{aligned}$$

where C_{k-1} and ϕ_j $(j = 0, \ldots, 2k-2)$ are meromorphic functions in Δ with finite $[p,q]$-order. Since $\psi_k(z)$ is not a solution of (1.1), it follows that $H \not\equiv 0$. Thus by Lemma 3.3, we obtain $\overline{\lambda}_{[p,q]}(w) = \lambda_{[p,q]}(w) = \infty$ and $\overline{\lambda}_{[p+1,q]}(w) = \lambda_{[p+1,q]}(w) = \rho$, i.e., $\overline{\lambda}_{[p,q]}(g_k - \varphi) = \lambda_{[p,q]}(g_k - \varphi) = \infty$ and $\overline{\lambda}_{[p+1,q]}(g_k - \varphi) = \lambda_{[p+1,q]}(g_k - \varphi) = \rho$.

Suppose that f is a finite $[p,q]$-order meromorphic solution in Δ of equation (1.1) such that (2.2) holds. Set $w(z) = g_k - \varphi$. Since $\rho_{[p,q]}(\varphi) < \rho_{[p,q]}(f)$, by Lemma 3.2 and Theorem 2.1, we have $\rho_{[p,q]}(w) = \rho_{[p,q]}(g_k) = \rho_{[p,q]}(f)$. In order to prove that $\overline{\lambda}_{[p,q]}(g_k - \varphi) = \lambda_{[p,q]}(g_k - \varphi) = \rho_{[p,q]}(f)$, we must show that $\overline{\lambda}_{[p,q]}(w) = \lambda_{[p,q]}(w) = \rho_{[p,q]}(f)$. Using the same reasoning as above, we get

$$C_{k-1} w^{(2k-1)} + \sum_{j=0}^{2k-2} \phi_j w^{(j)} = -\left(\psi_k^{(k)} + A_{k-1}(z)\psi_k^{(k-1)} + \cdots + A_0(z)\psi_k\right)$$

$$= F,$$

where C_{k-1} and ϕ_j $(j = 0, \ldots, 2k-2)$ are meromorphic functions in Δ with finite $[p,q]$-order $\rho_{[p,q]}(C_{k-1}) < \rho_{[p,q]}(f)$,

$$\rho_{[p,q]}(\phi_j) < \rho_{[p,q]}(f) \quad (j = 0, \ldots, 2k-2),$$

and

$$\psi_k(z) = C_0(\varphi - b) + C_1(\varphi' - b') + \cdots + C_{k-1}\left(\varphi^{(k-1)} - b^{(k-1)}\right),$$

$$\rho_{[p,q]}(F) < \rho_{[p,q]}(f).$$

Since $\psi_k(z)$ is not a solution of (1.1), it follows that $F \not\equiv 0$. Then by Lemma 3.3, we obtain $\overline{\lambda}_{[p,q]}(w) = \lambda_{[p,q]}(w) = \rho_{[p,q]}(f)$, i.e., $\overline{\lambda}_{[p,q]}(g_k - \varphi) = \lambda_{[p,q]}(g_k - \varphi) = \rho_{[p,q]}(f)$. □

Proof of Corollary 2.5. Suppose that f is a nontrivial meromorphic solution of (2.3). Then, by Lemma 3.10, we have $\rho_{[p,q]}(f) = \infty$ and

$$\rho_{[p,q]}(A) \le \rho_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f) \le \rho_{[p,q]}(A) + 1.$$

Furthermore, if $p > q$, then

$$\rho_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f) = \rho_{[p,q]}(A).$$

By the same reasoning as before we obtain that

$$\begin{cases} g_2 - b = \alpha_{0,0} f + \alpha_{1,0} f', \\ g_2' - b' = \alpha_{0,1} f + \alpha_{1,1} f', \end{cases} \tag{4.14}$$

where

$$\alpha_{0,0} = d_0 - d_2 A, \quad \alpha_{1,1} = -d_2 A + d_0 + d_1'$$

and

$$\alpha_{0,1} = -(d_2 A)' - d_1 A + d_0', \ \alpha_{1,0} = d_1.$$

First, we suppose that $d_2 \not\equiv 0$. We have

$$h_2 = \begin{vmatrix} \alpha_{0,0} & \alpha_{1,0} \\ \alpha_{0,1} & \alpha_{1,1} \end{vmatrix} = d_2^2 A^2 - \left(-d_2' d_1 + d_1' d_2 + 2 d_0 d_2 - d_1^2\right) A$$
$$+ d_1 d_2 A' - d_0' d_1 + d_0 d_1' + d_0^2.$$

Since $d_2 \not\equiv 0$, $A \not\equiv 0$, by Lemma 3.11 we have $\rho_{[p,q]}(h_2) = \rho_{[p,q]}(A) > 0$. Hence $h_2 \not\equiv 0$. Now suppose that $d_2 \equiv 0$, $d_1 \not\equiv 0$; then

$$h_2 = d_1^2 A - d_0' d_1 + d_0 d_1' + d_0^2,$$

and, by Lemma 3.2, we have $\rho_{[p,q]}(h_2) = \rho_{[p,q]}(A) > 0$. Hence $h_2 \not\equiv 0$. Finally, if $d_2 \equiv 0$, $d_1 \equiv 0$ and $d_0 \not\equiv 0$, then $h_2 = d_0^2 \not\equiv 0$. By (4.14), since $h_2 \not\equiv 0$, we obtain

$$f = \frac{-\alpha_{1,0}(g_2' - b') + \alpha_{1,1}(g_2 - b)}{h_2}. \tag{4.15}$$

It is clear that $\rho_{[p,q]}(g_2) \leq \rho_{[p,q]}(f)$ $(\rho_{[p+1,q]}(g_2) \leq \rho_{[p+1,q]}(f))$ and, by (4.15), we have $\rho_{[p,q]}(f) \leq \rho_{[p,q]}(g_2)$ $(\rho_{[p+1,q]}(f) \leq \rho_{[p+1,q]}(g_2))$. Hence $\rho_{[p,q]}(g_2) = \rho_{[p,q]}(f)$ $(\rho_{[p+1,q]}(g_2) = \rho_{[p+1,q]}(f))$. □

Proof of Corollary 2.6. Set $w(z) = d_2 f'' + d_1 f' + d_0 f + b - \varphi$. Then, since $\rho_{[p,q]}(\varphi) < \infty$, we have $\rho_{[p,q]}(w) = \rho_{[p,q]}(g_2) = \rho_{[p,q]}(f)$ and $\rho_{[p+1,q]}(w) = \rho_{[p+1,q]}(g_2) = \rho_{[p+1,q]}(f)$. To prove that $\overline{\lambda}_{[p,q]}(g_2 - \varphi) = \lambda_{[p,q]}(g_2 - \varphi) = \rho_{[p,q]}(f)$ and $\overline{\lambda}_{[p+1,q]}(g_2 - \varphi) = \lambda_{[p+1,q]}(g_2 - \varphi) = \rho_{[p+1,q]}(f)$, we only need to prove that $\overline{\lambda}_{[p,q]}(w) = \lambda_{[p,q]}(w) = \rho_{[p,q]}(f)$ and $\overline{\lambda}_{[p+1,q]}(w) = \lambda_{[p+1,q]}(w) = \rho_{[p+1,q]}(f)$. Since $g_2 = w + \varphi$, from (4.15) it follows that

$$f = \frac{-\alpha_{1,0} w' + \alpha_{1,1} w}{h_2} + \psi_2, \tag{4.16}$$

where

$$\psi_2(z) = \frac{-\alpha_{1,0}(\varphi' - b') + \alpha_{1,1}(\varphi - b)}{h_2}. \tag{4.17}$$

Substituting (4.16) into equation (2.3), we obtain

$$\frac{-\alpha_{1,0}}{h_2} w''' + \phi_2 w'' + \phi_1 w' + \phi_0 w = -\left(\psi_2'' + A(z)\psi_2\right) = F,$$

where ϕ_j $(j = 0,1,2)$ are meromorphic functions in Δ with $\rho_{[p,q]}(\phi_j) < \infty$ $(j = 0,1,2)$. First, we prove that $\psi_2 \not\equiv 0$. Suppose that $\psi_2 \equiv 0$; then by (4.17), since $\varphi - b \not\equiv 0$, we obtain that

$$\alpha_{1,1} = \alpha_{1,0} \frac{\varphi' - b'}{\varphi - b}.$$

Since $\rho_{[p,q]}(\varphi - b) \leq \max\left\{\rho_{[p,q]}(\varphi), \rho_{[p,q]}(b)\right\} = \eta < \infty$, by Lemma 3.7, it follows that

$$\begin{aligned} m(r,A) \leq{}& m(r, \frac{1}{d_2}) + m(r,d_0) + m(r,d_1') + m(r,d_1) \\ &+ O\left(\exp_{p-1}\left\{(\eta+\varepsilon)\log_q \frac{1}{1-r}\right\}\right) + O(1) \end{aligned}$$

for all r outside a set $E_2 \subset [0,1)$ with $\int_{E_2} \frac{dr}{1-r} < \infty$. Thus

$$\begin{aligned} \frac{\delta}{2} T(r,A) \leq m(r,A) \leq{}& T(r,d_2) + T(r,d_0) + T(r,d_1') + T(r,d_1) \\ &+ O\left(\exp_{p-1}\left\{(\eta+\varepsilon)\log_q \frac{1}{1-r}\right\}\right) + O(1) \quad (r \notin E_2). \end{aligned}$$

By $d_2 \not\equiv 0$ and Lemma 3.8 we obtain the contradiction

$$\rho_{[p,q]}(A) \leq \max_{j=0,1,2} \rho_{[p,q]}(d_j).$$

Hence $\psi_2 \not\equiv 0$. It is clear now that $\psi_2 \not\equiv 0$ cannot be a solution of (2.3), because $\rho_{[p,q]}(\psi_2) < \infty$. Thus, by Lemma 3.3,

$$\overline{\lambda}_{[p,q]}(g_2 - \varphi) = \lambda_{[p,q]}(g_2 - \varphi) = \rho_{[p,q]}(f) = \infty$$

and

$$\begin{aligned}\rho_{[p,q]}(A) \leq \overline{\lambda}_{[p+1,q]}(g_2 - \varphi) &= \lambda_{[p+1,q]}(g_2 - \varphi) = \rho_{[p+1,q]}(f) \\ &= \rho_{M,[p+1,q]}(f) \leq \rho_{[p,q]}(A) + 1.\end{aligned}$$

Furthermore, if $p > q$, then

$$\overline{\lambda}_{[p+1,q]}(g_2 - \varphi) = \lambda_{[p+1,q]}(g_2 - \varphi) = \rho_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f) = \rho_{[p,q]}(A).$$

□

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References

[1] S. Bank, *General theorem concerning the growth of solutions of first-order algebraic differential equations*, Compos. Math. **25** (1972), 61–70.

[2] B. Belaïdi, *Growth of solutions to linear differential equations with analytic coefficients of* $[p,q]$-order in the unit disc, Electron. J. Differential Equations **2011**, No. 156, 1–11.

[3] B. Belaïdi, *Growth and oscillation theory of* $[p,q]$-order analytic solutions of linear differential equations in the unit disc, J. Math. Anal. **3**(1) (2012), 1–11.

[4] B. Belaïdi, *On the* $[p,q]$-order of analytic solutions of linear differential equations in the unit disc, Novi Sad J. Math. **42**(1) (2012), 117–129.

[5] B. Belaïdi, *Differential polynomials generated by meromorphic solutions of* $[p,q]$-order to complex linear differential equations, Rom. J. Math. Comput. Sci. **5**(1) (2015), 46–62.

[6] L. G. Bernal, *On growth* k-order of solutions of a complex homogeneous linear differential equation, Proc. Amer. Math. Soc. **101**(2) (1987), 317–322.

[7] T.-B. Cao, H.-Y. Xu, and C.-X. Zhu, *On the complex oscillation of differential polynomials generated by meromorphic solutions of differential equations in the unit disc*, Proc. Indian Acad. Sci. Math. Sci. **120**(4) (2010), 481–493.

[8] T.-B. Cao and H.-X. Yi, *The growth of solutions of linear differential equations with coefficients of iterated order in the unit disc*, J. Math. Anal. Appl. **319**(1) (2006), 278–294.

[9] T.-B. Cao, C.-X. Zhu, and K. Liu, *On the complex oscillation of meromorphic solutions of second order linear differential equations in the unit disc*, J. Math. Anal. Appl. **374**(1) (2011), 272–281.

[10] Y.-M. Chiang and H. K. Hayman, *Estimates on the growth of meromorphic solutions of linear differential equations*, Comment. Math. Helv. **79**(3) (2004), 451–470.

[11] W. K. Hayman, *Meromorphic Functions*, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1964.

[12] J. Heittokangas, *On complex differential equations in the unit disc*, Ann. Acad. Sci. Fenn. Math. Diss. **122** (2000), 1–54.

[13] O. P. Juneja, G. P. Kapoor, and S. K. Bajpai, *On the (p,q)-order and lower (p,q)-order of an entire function*, J. Reine Angew. Math. **282** (1976), 53–67.

[14] O. P. Juneja, G. P. Kapoor, and S. K. Bajpai, *On the (p,q)-type and lower (p,q)-type of an entire function*, J. Reine Angew. Math. **290** (1977), 385–405.

[15] L. Kinnunen, *Linear differential equations with solutions of finite iterated order*, Southeast Asian Bull. Math. **22**(4) (1998), 385–405.

[16] I. Laine, *Nevanlinna Theory and Complex Differential Equations*, de Gruyter Studies in Mathematics **15**. Walter de Gruyter & Co., Berlin – New York, 1993.

[17] I. Laine, *Complex differential equations*, in: Handbook of Differential Equations: Ordinary Differential Equations. Vol. IV, Elsevier/North-Holland, Amsterdam, 2008, pp. 269–363.

[18] I. Laine and J. Rieppo, *Differential polynomials generated by linear differential equations*, Complex Var. Theory Appl. **49**(12) (2004), 897–911.

[19] Z. Latreuch and B. Belaïdi, *Linear differential equations with analytic coefficients of* [p,q]-order in the unit disc, Sarajevo J. Math. **9(21)** (2013), 71–84.

[20] Z. Latreuch and B. Belaïdi, *Properties of higher order differential polynomials generated by solutions of complex differential equations in the unit disc*, J. Math. Appl. **37**, (2014), 67–84.

[21] L.-M. Li and T.-B. Cao, *Solutions for linear differential equations with meromorphic coefficients of [p, q]-order in the plane*, Electron. J. Differential Equations **2012**, No. 195, 1–15.

[22] J. Liu, J. Tu, and L.-Z. Shi, *Linear differential equations with entire coefficients of* [p, q]-order in the complex plane, J. Math. Anal. Appl. **372** (2010), 55–67.

[23] J. Tu and H.-X. Huang, *Complex oscillation of linear differential equations with analytic coefficients of [p, q]-order in the unit disc*, Comput. Methods Funct. Theory **15**(2) (2015), 225–246.

[24] J. Tu and Z.-X. Xuan, *Complex linear differential equations with certain analytic coefficients of* [p, q]-order in the unit disc, Adv. Difference Equ. **2014**, 2014:167, 12 pp.

[25] M. Tsuji, *Potential Theory in Modern Function Theory*, Chelsea, New York, 1975.

[26] H.-Y. Xu, J. Tu, and Z.-X. Xuan, *The oscillation on solutions of some classes of linear differential equations with meromorphic coefficients of finite* [p, q]-order, Sci. World J. **2013**, Art. ID 243873, 8 pp.

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